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# Differential Equations Which are Substitutable to Bessel Differential Equation

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**Abstract:** *Differential equations* $x^2y''+xy'+\left(x^2-p^2\right)y=0$ *in which p is an integer is known as p order of Bessel equation. This differential equation is one of the most important differential equations in applied mathematics. The answers or solutions of this equation are known as Bessel equation. The first solution to this equation is:* $J_p\left(x\right)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{k!\Gamma\left(k+p+1\right)}\left(\frac{x}{2}\right)^{2k+p}$ *Which is called Bessel function of the first kind. If p is non-integer, the second solution to the above mentioned equation is:* $J_{-p}\left(x\right)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{k!\Gamma\left(k-p+1\right)}\left(\frac{x}{2}\right)^{2k-p}$ *So, the general solution to the above equation is:* $y=c_1J_p\left(x\right)+c_2J_{-p}\left(x\right)$ *However, if p is an integer, then the second solution with the first solution is not linearly independent but linearly dependent; thus, the second solution is determined by the following relation which is called Bessel function of the second-order.*

$$Y_n\left(x\right)=\begin{cases}\dfrac{J_n\left(x\right)\cos n\pi-J_{-n}\left(x\right)}{\sin n\pi} & n\neq 0,1,2,3,\cdots\\ \lim\limits_{p\to n}\dfrac{J_p\left(x\right)\cos p\pi-J_{-p}\left(x\right)}{\sin p\pi} & n=0,1,2,3,\cdots\end{cases}$$

*And with the first solution, it is linearly independent, so the general solution in both cases is:* $y\left(x\right)=c_1J_p\left(x\right)+c_2Y_n\left(x\right)$ *All the differential equation of the second-order that can be substituted into Bessel equation due to some appropriate replacements, we can easily find out their solution by Bessel equation.*

**Keywords:** Bessel equation, Bessel functions of the first and second kind, differential equations of the second-order, differential equations of the second-order solution

**Method:** The article is based on library research method and used reliable books in the ground.

## 1. Introduction

Bessel equation is considered one of the most important differential equations in applied mathematics. It is first defined by Daniel Bernoulli Irish Swedish mathematician and physics scholar. Then the German astronomer Friedrich Wilhelm Bessel investigated the general form of it and applied his studies on the movement of the planet. After that Bessel differential equation solution as series function which is called Bessel function is used to solve the theorems of vibration of stretched membrane, fluid movement, static potential, wave propagation and etc. Hence, Bessel equation is defined finitely, so differential equations which are substitutable to Bessel equation, we can easily obtain their solution by Bessel functions.

## 2. Bessel Differential Equation

We see the following differential equation (Bessel equation) in the solution of heat, wave, potential with variables separation method in polar coordinates system, cylindrical or spherical.

$$x^2y''+xy'+(x^2-p^2)y=0$$

Bessel equation is a differential equation which finds out with its series of solution. $x=0$ is the only unusual order point of it, so we solve it with Frobenius series method. With assume

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$

And to let it and its derivatives in Bessel equation, to two values $r_1=p$ and $r_2=-p$ obtain for $r$. Thus, when we equal the coefficients $x$ of approximation exponent to zero we conclude that

$$a_1=0\quad,\quad a_n=-\frac{a_{n-2}}{\left(r+n\right)^2-p^2}\quad,\quad n\geq 2$$

With letting of the first root $r_1=p$ in this recurrence equation and coefficients determination $\cdots,a_3,a_2$ at $a_0$ and putting in series, the first solution as

$$y_1(x)=a_0x^p\left(1-\frac{1}{4(p+1)}x^2+\frac{1}{4^2(2!)(p+1)(p+2)}x^4-\cdots\right)$$

Obtains. It is usual that $a_0 = \frac{1}{2^p p!}$ considers, in this case it is determined with term $J_p(x)$ and it is called Bessel function of the first kind and $p$ order , so

$$J_p(x) = \left(\frac{x}{2}\right)^p \left(\frac{1}{p!} - \frac{1}{1!(p+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(p+2)!}\left(\frac{x}{2}\right)^4 - \cdots\right)$$

or

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!}\left(\frac{x}{2}\right)^{2k+p}$$

Since $\Gamma(n+1) = n!$ therefore $\Gamma(k+p+1) = (k+p)!$ thus, the last function is also written as follow:

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k+p}$$

For determination of second solution it is enough to substitute $p$ to $-p$ we specify this with $J_{-p}(x)$

$$J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-p+1)}\left(\frac{x}{2}\right)^{2k-p}$$

For second solution $p = 0$ the first solution is satisfied. On the other hand the expression $J_{-p}(x)$ for $p = 1,2,3,\cdots$ that means it is not computable for integer $p$ . In other word the Bessel generalized solution for non-integer $p$ is not equal to (Jamshedi, 2016)

$$y = c_1 J_p(x) + c_2 J_{-p}(x)$$

## 3. Bessel Function of the Second Kind

Bessel function of the second kind which is denoted with $Y_n(x)$ is defined as the following series (Ghaniyari, 2007):

$$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi} & n \neq 0,1,2,3,\cdots \\ \lim\limits_{p \to n} \dfrac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} & n = 0,1,2,3,\cdots \end{cases}$$

Generally, the general solution to differential equation is as follow:

$$y(x) = \begin{cases} c_1 J_p(x) + c_2 J_{-p}(x) & p \notin \mathbb{Z} \\ c_1 J_n(x) + c_2 Y_n(x) & n \in \mathbb{Z} \end{cases}$$

### Bessel function properties

1) If we make $p = m \in \mathbb{Z}^+$, then $J_p(x)$ and $J_{-p}(x)$ is linearly dependent with each other because

a. $J_{-m}(x) = (-1)^m J_m(x)$

2) $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$

3) $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$

4) $J'_p(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$

5) $J'_p(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$

6) $2J'_p(x) = J_{p-1}(x) - J_{p+1}(x)$

7) $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$

8) $\frac{d}{dx}\left[x J_p(x) J_{p+1}(x)\right] = x\left[J_p^2(x) - J_{p+1}^2(x)\right]$

(Zada, 2006)

9) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ , $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

(Faryabi, 2009)

### Bessel function specific states

For $p = 0$ Bessel function obtains zero order.

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

For $p = 1$ Bessel function obtains one order.

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!}\left(\frac{x}{2}\right)^{2k+1} = \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 - \frac{1}{3!4!}\left(\frac{x}{2}\right)^7 + \cdots$$

In Bessel function $J_0(x)$ and $J_1(x)$ more apply in which $J_0(x)$ is even function and $J_1(x)$ is odd function (Ferozkohi & Sarhang, 2009).

### Differential equations which are substitutable to Bessel differential equation

There are some differential equations which substitute to Bessel differential equation with some appropriate replacements, with that solution the equation is found out. The subject of this article is to introduce this kind of equations that describe as follow:

**Differential equations** $x^2 y'' + xy' + \left(\lambda^2 x^2 - p^2\right) y = 0$

Differential equations which have the above form due to replacement of $\lambda x = t$ , it is substitutable to Bessel equation.

**Proof**

$t = \lambda x$ , $dt = \lambda dx$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \lambda \frac{dy}{dt}$$

$$y'' = \lambda \frac{d^2 y}{dt^2} \cdot \frac{dt}{dx} = \lambda^2 \frac{d^2 y}{dt^2}$$

With replacement $y'', y', x$ values to the above equation, we get to

$$\left(\frac{t}{\lambda}\right)^2\left(\frac{d^2y}{dt^2}\right)+\frac{t}{\lambda}\left(\lambda\frac{dy}{dt}\right)+\left(t^2-p^2\right)y=0$$

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+\left(t^2-p^2\right)y=0$$

The last equation denotes Bessel equation with function $y$ and variable $t$ so, its general solution is

$$y(x)=c_1J_p(x)+c_2Y_n(x)$$

or

$$y=c_1J_p(\lambda x)+c_2Y_n(\lambda x)$$

(Paryab, 2010)

**Differential equations** $4x^2y''+4xy'+\left(x-p^2\right)y=0$

Differential equations which have the above form due to replacement of $\sqrt{x}=t$ it is substitutable to Bessel equation.

$$\sqrt{x}=t \quad , \quad dt=\frac{1}{2}x^{-\frac{1}{2}}dx$$

$$y'=\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{1}{2}x^{-\frac{1}{2}}\frac{dy}{dt}$$

$$y''=\frac{d^2y}{dt^2}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{d}{dt}\left(\frac{1}{2}x^{-\frac{1}{2}}\frac{dy}{dt}\right)=$$

$$-\frac{1}{4}x^{-\frac{3}{2}}\frac{dy}{dt}+\frac{1}{2}x^{-\frac{1}{2}}\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}=-\frac{1}{4}x^{-\frac{3}{2}}+\frac{1}{4}x^{-1}\frac{d^2y}{dt^2}$$

With replacement $y'',y',x$ values to the above equation we get to

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+\left(t^2-p^2\right)y=0$$

Which Bessel equation is from function $y$ and variable $t$.

So, the general solution with consideration of the above $p$ value as follow

$$y=c_1J_p(t)+c_2Y_n(t)$$

or

$$y=c_1J_p\left(\sqrt{x}\right)+c_2Y_n\left(\sqrt{x}\right)$$

(Nikokar, 2010)

**Differential equations** $x^2y''+xy'+4\left(x^4-p^2\right)y=0$

The above differential equations due to replacement $t=x^2$, it is substitutable to Bessel equation.

$$t=x^2 \quad , \quad \frac{dt}{dx}=2x$$

$$y'=\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=2x\frac{dy}{dt}$$

$$y''=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(2x\frac{dy}{dt}\right)=$$

$$2\frac{dy}{dt}+2x\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}=2\frac{dy}{dt}+4x^2\frac{d^2y}{dt^2}$$

With replacement of $y'',y',x$ , we get to

$$4x^4\frac{d^2y}{dt^2}+2x^2\frac{dy}{dt}+2x^2\frac{dy}{dt}+4\left(x^4-p^2\right)y=0$$

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+\left(t^2-p^2\right)y=0$$

So, the general solution is as follow:

$$y=c_1J_P(t)+c_2Y_n(t)$$

or

$$y=c_1J_P\left(x^2\right)+c_2Y_n\left(x^2\right)$$

(Nikokar, 2010)

**Differential equations** $xy''+(1+2p)y'+xy=0$

The above differential equations due to replacement of $y=x^{-p}u$ , it is substitutable to Bessel equation (function $u$ from $x$).

$$y=x^{-p}u\Rightarrow y'=-px^{-p-1}u+x^{-p}u'$$

$$y''=p(p+1)x^{-p-2}u-2px^{-p-1}u'+x^{-p}u''$$

With replacement $y'',y',y$ in the given equation, we get to

$$x^{-p-1}\left(x^2u''+xu'+\left(x^2-p^2\right)u\right)=0$$

or

$$x^2u''+xu'+\left(x^2-p^2\right)u=0$$

Which is Bessel equation with function $u$ and its general solution is as follow:

$$u=c_1J_p(x)+c_2Y_n(x)$$

And the general equation solution

$xy''+(1+2p)y'+xy=0$ is:

$$y=x^{-p}\left(c_1J_p(x)+c_2Y_n(x)\right)$$

(Nikokar, 2010)

**Differential equations** $y''+\left(1+\frac{1-4p^2}{4x^2}\right)y=0$

With replacement of $y=x^{\frac{1}{2}}u$, it is substitutable to Bessel equation.

$$y=x^{\frac{1}{2}}u \quad , \quad y'=\frac{1}{2}x^{-\frac{1}{2}}u+x^{\frac{1}{2}}u' \quad ,$$

$$y''=-\frac{1}{4}x^{-\frac{3}{2}}u+x^{-\frac{1}{2}}u'+x^{\frac{1}{2}}u''$$

With replacement of $y'',y$ in the equation, we get to

$$x^{\frac{1}{2}}u''+x^{-\frac{1}{2}}u'-\frac{1}{4}x^{-\frac{3}{2}}u+\left(1+\frac{1-4p^2}{4x^2}\right)x^{\frac{1}{2}}u=0$$

$$x^{\frac{5}{2}}u''+x^{\frac{3}{2}}u'+\left(-\frac{1}{4}+x^2+\frac{1}{4}-p^2\right)x^{\frac{1}{2}}u=0$$

$$x^{\frac{1}{2}}\left[x^2u''+xu'+\left(x^2-p^2\right)u\right]=0$$

$$x^2u''+xu'+\left(x^2-p^2\right)u=0$$

The last equation is Bessel equation with function $u$ and the general solution of last equation is

$$u = c_1 J_p(x) + c_2 Y_n(x)$$

And the general solution of the given equation is

$$y = x^{\frac{1}{2}}\left(c_1 J_p(x) + c_2 Y_n(x)\right)$$

(Ayoubi, 2010)

**Differential equations**

$$x^2 y'' + (1-2p)xy' + p^2\left(x^{2p} + 1 - p^2\right)y = 0$$

With replacement of $x^p = z$ and $y = x^p u$, it is substitutable to Bessel equation.

$$y = x^p u \Rightarrow y' = \frac{dy}{dx} = px^{p-1}u + x^p\frac{du}{dx}$$

$$y'' = p(p-1)x^{p-2}u + 2px^{p-1}\frac{du}{dx} + x^p\frac{d^2u}{dx^2}$$

On the other hand

$$x^p = z \Rightarrow \frac{dz}{dx} = px^{p-1}$$

$$\frac{du}{dx} = \frac{du}{dz}\cdot\frac{dz}{dx} = px^{p-1}\frac{du}{dz}$$

$$\frac{d^2u}{dx^2} = p(p-1)x^{p-2}\frac{du}{dz} + p^2x^{2p-2}\frac{d^2u}{dz^2}$$

and

$$y' = px^{p-1}u + px^{2p-1}\frac{du}{dz}$$

$$y'' = p(p-1)x^{p-2}u + 2p^2x^{2p-2}\frac{du}{dz} + p(p-1)x^{2p-2}\frac{du}{dz} + p^2x^{3p-2}\frac{d^2u}{dz^2}$$

With replacement of $y'', y', y$ values in the equation, we get to

$$p^2x^p\left[x^{2p}\frac{d^2u}{dz^2} + x^p\frac{du}{dz} + \left(x^{2p} - p^2\right)u\right] = 0$$

or

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \left(z^2 - p^2\right)u = 0$$

The last equation is Bessel equation with function $u$ and variable $z$ and the general solution of the last equation is

$$u = c_1 J_p(z) + c_2 Y_n(z)$$

And the general solution of the given equation is

$$y = x^p\left[c_1 J_p\left(x^p\right) + c_2 Y_n\left(x^p\right)\right]$$

(Nikokar, 2010)

**Differential equations** $y'' + ax^b y = 0$

With replacement of $z = \frac{2\sqrt{a}}{b+2}\left(x^{b+2}\right)^{\frac{1}{2}}$, $y = x^{\frac{1}{2}}u$, it is substitutable to Bessel equation.

And the general solution of it is as follow:

$$y = x^{\frac{1}{2}}\left[c_1 J_{\frac{1}{b+2}}\left(\frac{2\sqrt{a}}{b+2}\left(x^{b+2}\right)^{\frac{1}{2}}\right) + c_2 Y_{\frac{1}{b+2}}\left(\frac{2\sqrt{a}}{b+2}\left(x^{b+2}\right)^{\frac{1}{2}}\right)\right]$$

(Nikokar, 2010)

**Differential equations**

$$x^2y'' + (1-2p)xy' + \left(a^2b^2x^{2a} + p^2 - c^2a^2\right)y = 0$$

With replacement of $z = bx^a$, $y = x^p u$, it is substitutable to Bessel equation.

$$y = x^p u \Rightarrow y' = px^{p-1}u + x^p u'$$

$$\Rightarrow y'' = p(p-1)x^{p-2}u + 2px^{p-1}u' + x^p u''$$

With replacement of $y'', y', y$ values in the equation, we get to

$$x^2u'' + xu' + a^2\left(b^2x^{2a} - c^2\right)u = 0 \qquad (*)$$

Now we use from replacement $z = bx^a$

$$z = bx^a \Rightarrow \frac{dz}{dx} = abx^{a-1}, u' = \frac{du}{dx} = \frac{du}{dz}\cdot\frac{dz}{dx} = abx^{a-1}\frac{du}{dz}$$

$$u'' = a(a-1)bx^{a-2}\frac{du}{dz} + a^2b^2x^{2a-2}\frac{d^2u}{dz^2}$$

With replacement of $u'', u', u$ values we have in $*$ relation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \left(z^2 - c^2\right)u = 0$$

The last equation is Bessel equation with function $u$ and variable $z$ so, the general solution of the last equation is

$$u = c_1 J_c(z) + c_2 Y_n(z)$$

And the general solution of the given equation is

$$y = x^p\left[c_1 J_c\left(bx^a\right) + c_2 Y_n\left(bx^a\right)\right]$$

(Nikokar, 2010)

**Differential equations**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(x^2 + p^2\right)y = 0$$

The above equations with replacement of $w = ix$, it substitutes as follow:

$$w^2\frac{d^2y}{dw^2} + w\frac{dy}{dw} + \left(w^2 - p^2\right)y = 0$$

Which is Bessel equation with function $y$ and variable $w$.

So the above general equation solution is

$$y(x) = c_1 J_p(ix) + c_2 Y_n(ix)$$ (Brothers, 2013)

**Differential equations**

$$x\frac{d^2y}{dx^2} + a\frac{dy}{dx} + k^2xy = 0$$

The above equations with replacement of $y = x^n z$, it is substitutable to Bessel equation.

$$y = x^n z \quad , \quad \frac{dy}{dx} = x^n \frac{dz}{dx} + nx^{n-1} z \quad ,$$

$$\frac{d^2 y}{dx^2} = x^n \frac{d^2 z}{dx^2} + nx^{n-1} \frac{dz}{dx} + nx^{n-1} \frac{dz}{dx} + n(n-1)x^{n-2} z$$

We replaced the last values, we get to

$$x^{n+1} \frac{d^2 z}{dx^2} + a\left[ x^n \frac{dz}{dx} + nx^{n-1} z \right] + 2nx^n \frac{dz}{dx} +$$

$$n(n-1)x^{n-2} z + k^2 x^{n+1} z = 0$$

$$x^{n+1} \frac{d^2 z}{dx^2} + (a+2n)x^n \frac{dz}{dx} + \left[ an + n(n-1) + k^2 x^2 \right] x^{n-1} z = 0$$

$$x^2 \frac{d^2 z}{dx^2} + (a+2n)x \frac{dz}{dx} + \left[ n^2 + k^2 x^2 + (a-1)n \right] z = 0$$

We place $a + 2n = 1$ so get to

$$x^2 \frac{d^2 z}{dx^2} + x \frac{dz}{dx} + \left( k^2 - n^2 \right) z = 0$$

The last equation solution is

$$z = c_1 J_n(x) + c_2 Y_n(x)$$

And the general solution of the given equation is

$$y = x^n c_1 J_n(kx) + x^n c_2 Y_n(kx)$$

(Khalili, 2012)

## 4. Results

What we have found out about differential equations which are substitutable to Bessel equation is, the differential equations of the second-order do not have specific method of solution such as method of decay, parameter solution method, Cauchy-Euler equation method, series solution each of them are the method of differential equation solution of the second-order. Because Bessel equation includes certain and proved solution which is known to Bessel function. In this case, each differential equation of the second-order that could be substituted due to some appropriate replacement to Bessel equation, the solution of that equation is found from Bessel equation with ease.

## 5. Discussion

The differential equation substitution of the second-order to Bessel equation which happens due to some appropriate replacement, causes the solution of that equation is found out from Bessel equation solution. However, the other method of differential equations solution of the second-order is not such a thing. Every solution method includes a series of certain rule to obtain equation solution. Therefore, one which includes less conditions and rules, finding solution with that is easy.

## 6. Conclusion

I have found from the article differential equations which are substitutable to Bessel equation; to find out many mathematical terms, we shall pass a series of theorems and hard proofs till to find their solution. If the terms have approximation in form, structure and some other specification, we shall proceed too hard proofs singularly to find resolution for each of them. Through substitution of the same terms we could easily find their solution from the proved term. Such as, Bessel equation, Lagrange equation, Hermite equation, Laguerre equation and many other equations which are obtained their solution with long proofs. Equations which have approximation with one of these, we can find their solution from the equation answer.

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