# Integrating Factors

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**Abstract:** ***The title of this article is, finding factors of integrating or integrating factors for partial equation of*** $P(x,y)dx+Q(x,y)dy=0$ ***. In general case, finding of these factors seem difficult, but in particular cases these factors exist which are called factor of the first-order, factor of the second-order ……, here seven orders of factors with relevant proofs and problems are studied. Finding of integrating factors with expressions classification method also discussed, it is mentioned into two problems and in one table.*** $f=e^{\int\frac{-\Delta}{Q}dx}$ ***is factor of the first-order. If*** $-\dfrac{\Delta}{Q}$ ***is a function of x alone.*** $f=e^{\int\frac{\Delta}{P}dy}$ ***is factor of the second-order. It is used when*** $\dfrac{\Delta}{P}$ ***is a function of y alone.*** $\Delta=\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}$. $f=e^{\int g(z)dz}$ ***is factor of the third-order. It is used when*** $\dfrac{\Delta}{xN-yM}=g(z)$ ***is a function of z alone. Thus*** $z=xy$. $f(x,y)=x^k y^\lambda$ ***is factor of the fourth-order, if the differential equation be as follow:*** $y\left(Ax^m y^n+Bx^P y^q\right)dx+x\left(Cx^m y^n+Dx^P y^q\right)dy=0$ $\dfrac{1}{xM+yN}$ ***is factor of the fifth-order, if*** $xM+yN\neq 0$. $h(u)=e^{\int\frac{\Delta}{2(yM-xN)}du}$ ***is factor of the sixth-order if*** $\dfrac{\Delta}{2(yM-xN)}$ ***is a function of*** $x^2+y^2$. $h(u)=e^{-\int\frac{y^2\Delta}{xM+yN}du}$ ***is factor of the seventh-order if*** $\dfrac{-\Delta}{xM+yN}$ ***is a function of*** $\dfrac{x}{y}$***. For the factors of the third and fourth, …. orders, partial equation of*** $Mdx+Ndy=0$ ***is considered. The number of factors are more than seven, so it is prevented for not increasing the pages and it is not written.***

**Keywords:** integrating factors, Euler's formula, exact differential equation, partial equation, homogenous equation, Delta $(\Delta)$, exact term, expressions classification, general solution, implicit linear differential function, partial derivative

**Method**

The research method is based on the books which are more foreign and published at recent decades.

## 1. Introduction

Integrating factor is one of the most important issue of ordinary differential equations. In general case, finding integrating factors seem difficult, but in particular cases it can be obtained. Integrating factor is a function if it multiplies to partial differential of $P(x,y)dx+Q(x,y)dy=0....(1)$, it gets exact. In exact differential equation Euler's formula satisfies ( $\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y}$). If Euler's formula does not satisfy, then it is not exact or integrating equation.

In this article seven kinds of integrating factors with their conditions and proofs are studied.

The other part of this article is the determination of integrating factor by expressions classification method. The guideline is described and drew in a table. Also Factors of integrating is computed with several problems.

## 2. Integrating Factor

If the differential equation of $P(x,y)dx+Q(x,y)dy=0....(1)$ is not exact or integrating we find the function which if multiplies to equation (1) substitute it to exact equation , this is called function of integrating factor or factor of integrating. We assume that factor is $\mu(x,y)$, so $\mu P(x,y)dx+\mu Q(x,y)dy=0....(2)$ is exact differential in which Euler's formula is satisfied.

Euler's formula (Dehghani & Mirtalebi, 2009)

$$(173:5)\qquad \frac{\partial \mu Q}{\partial x}=\frac{\partial \mu P}{\partial y}$$

**Theorem:** If the differential equation of (1) is not exact and $u(x,y)=c$ is its general solution, in this case one factor is found for it (even infinity of these factors).

**Proof:** We take differential from both sides of $u(x,y)=c$.

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0....(2)$$

We find the value of $\frac{dy}{dx}$ from the $\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0$ equation.

$$\frac{\partial u}{dy}dy=-\frac{\partial u}{\partial x}dx\rightarrow\frac{dy}{dx}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=-\frac{P}{Q}$$

From the comparison of (1) and (2) it is written as $\frac{\partial u}{\partial y}=Q(x,y),\ \frac{\partial u}{\partial x}=P(x,y)$

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{P}{Q}\rightarrow\frac{\partial u}{\partial x}\cdot Q=\frac{\partial u}{\partial y}\cdot P\rightarrow\frac{\partial u}{\partial x}\cdot\frac{1}{P}=\frac{\partial u}{\partial x}\cdot\frac{1}{Q}$$

If $F(x,y)\neq 0$ is integrating factor, we multiply it to equation (1).

$$\begin{cases}F\cdot P(x,y)dx+F\cdot Q(x,y)dy=0....(3)\\ du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0....(4)\end{cases}\rightarrow\frac{\partial u}{\partial x}=F\cdot P$$

From the comparison of (3) and (4), we conclude that

$$k(u)du=k(u)\cdot F\left(P(x,y)dx+Q(x,y)dy\right)=0....(6)$$

The relation (5) expresses that F is a factor of equation (1). If we multiply both sides of (5) to a function such as $k(u)$, we get

$$k(u)du=k(u)\cdot F\left(P(x,y)dx+Q(x,y)dy\right)=0....(6)$$

Which is an exact equation. Therefore $k(u)\cdot F$ is other integral factor. Since K is arbitrary function of u, so there is infinity integral factor for the equation (1) (Nikoukar, 2011).

**Problem:** Denote that $f(x,y)=\frac{1}{x^2+y^2}$ is an integrating factor for the equation $(x^2+y^2-x)dx-ydy=0...(1)$, then solve the equation.

**Solution:** We multiply integrated factor to the equation (1).

$$\frac{1}{x^2+y^2}(x^2+y^2-x)dx-\frac{y}{x^2+y^2}dy=$$

$$0\rightarrow\underbrace{\left(1-\frac{x}{x^2+y^2}\right)}_{P}dx-\underbrace{\left(\frac{y}{x^2+y^2}\right)}_{Q}dy=0...(2)$$

Now we apply the Euler's formula.

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}\quad\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\left(1-\frac{x}{x^2+y^2}\right)=-\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$=-\frac{-x\cdot 2y}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial y}=\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)=-\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$=-\frac{-y\cdot 2x}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}$$

Since Euler's formula is satisfied, in this case the equation (2) got exact we solve it.

$$dx-\frac{xdx}{x^2+y^2}-\frac{ydy}{x^2+y^2}=0\rightarrow dx-\frac{xdx+ydy}{x^2+y^2}=0...(3)$$

We get integral from the both sides of equation (3)

$$\int dx=\frac{1}{2}\int\frac{2xdx+2ydy}{x^2+y^2}=\int 0dx$$

The general solution is as implicit function (Suhrabi, 2012).

$$(56:6)\qquad x-\frac{1}{2}Ln\left|x^2+y^2\right|-c=0$$

**Problem 2:** Denote that functions $\frac{1}{y^2},\frac{1}{x^2}$ and $\frac{1}{xy}$ each are integrating factor for the partial differential equation of $xdy-ydx=0....(1)$.

**Solution:** We make exact the equation (1) by integrating factor of $\frac{1}{x^2}$. We multiply the factor $\frac{1}{x^2}$ to the equation (1):

$$\frac{1}{x^2}\cdot xdy-\frac{1}{x^2}\cdot ydy=0\rightarrow\frac{1}{x}dy-\frac{1}{x^2}ydx=0....(2)$$

We apply Euler's formula to the equation (2).

$$\text{Euler's formula}\begin{cases}P(x,y)=-\frac{y}{x^2}\\ Q(x,y)=\frac{1}{x}\end{cases}\qquad\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

$$\frac{\partial Q}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$$

$$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\left(-\frac{y}{x^2}\right)=-\frac{1}{x^2}\frac{\partial y}{\partial y}=-\frac{1}{x^2}$$

There equation (2) got exact and the factor $\frac{1}{x^2}$ is integrating (Nikoukar, 2011).

Generally, finding integrating factor is not easy, we can find these integrating factors in particular cases which are called factor of the first, second, third and fourth order (Aqayan, 2008).

## 3. Factor of the first-order

If we denote integrating factor with $f(x,y)$ and assume that the equation $P(x,y)dx+Q(x,y)dy=0...(1)$ is not exact, we multiply both sides of the equation (1) to $f(x,y)$ which assumed integrating factor.

$$f(x,y)\cdot P(x,y)dx+f(x,y)\cdot Q(x,y)dy=0....(2)$$

Since $f(x,y)$ is integrating factor, then the equation (2) is exact, we apply Euler's formula on it.

$$\frac{\partial fQ}{\partial x}=\frac{\partial fP}{\partial y}$$

$$\left.\begin{aligned}\frac{\partial}{\partial x}(f\cdot Q)&=\frac{\partial f}{\partial x}\cdot Q+f\cdot\frac{\partial Q}{\partial x}\\ \frac{\partial}{\partial y}(f\cdot P)&=\frac{\partial f}{\partial y}\cdot P+f\cdot\frac{\partial P}{\partial y}\end{aligned}\right\}\Rightarrow\frac{\partial f}{\partial x}\cdot Q+f\frac{\partial Q}{\partial x}=\frac{\partial f}{\partial y}\cdot P+f\frac{\partial P}{\partial y}$$

$$P\frac{\partial f}{\partial y}-Q\frac{\partial f}{\partial x}=f\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)....(3)$$

We divide both sides of equation (3) by f.

$$P\frac{\partial f}{f\partial y}-Q\frac{\partial f}{f\partial x}=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$$

$$P\frac{\frac{\partial f}{f}}{\partial y}-Q\frac{\frac{\partial f}{f}}{\partial x}=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\to P\frac{\partial Lnf}{\partial y}-Q\frac{\partial Lnf}{\partial x}=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$$

If $\frac{\partial Lnf}{\partial y}=0$, then $P\frac{\partial Lnf}{\partial y}=0$.

We assume that $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=\Delta$ , $-Q\frac{\partial Lnf}{\partial x}=\Delta$

$$\frac{\partial Lnf}{\partial x}=\frac{-\Delta}{Q}\to\partial Lnf=-\frac{\Delta}{Q}dx\to\int dLnf=-\int\frac{\Delta}{Q}dx\to Lnf=-\int\frac{\Delta}{Q}dx\to f=e^{-\int\frac{\Delta}{Q}dx}$$

$f=f(x,y)=e^{-\int\frac{\Delta}{Q}dx}$ is factor of the first-order. The above factor is used in the case which $-\frac{\Delta}{Q}$ is a function of x . Actually $-\frac{\Delta}{Q}$ is the condition of factor of the first-order (Ferozkohi & Hashemi, 2013).

**Problem 1:** Integrate the following partial equation by factor of the first-order, then solve it.

$$(2y^2+3x)dx+2xydy=0......(1)$$

First we apply factor of the first-order condition which is $-\frac{\Delta}{Q}$

$$-\frac{\Delta}{Q}=\frac{-\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)}{Q}=-\frac{\frac{\partial}{\partial x}(2xy)-\frac{\partial}{\partial y}(2y^2+3x)}{2xy}=-\frac{2y-4y}{2xy}=\frac{2y}{2xy}=\frac{1}{x}\qquad Q=2xy,\ P=2y^2+3x$$

Since $-\frac{\Delta}{Q}=\frac{1}{x}$ (is a function of x), we find factor of the first-order.

$$f=e^{-\int\frac{\Delta}{Q}dx}=e^{\int\frac{1}{x}dx}=e^{Lnx}=x\to f(x,y)=x$$

$f(x,y)=x$ is integrating factor. We multiply x to the partial equation (1).

Exact equation $(2xy^2+3x^2)dx+2x^2ydy=0....(2)$

We apply Euler's formula ($\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$) to the equation (2).

$$Q=2x^2y,\quad\frac{\partial Q}{\partial x}=\frac{\partial}{\partial x}(2x^2y)=4xy$$

$$P=2xy^2+3x^2,\quad\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}(2xy^2+3x^2)=4xy$$

Now we find a function which is equal to the left side of the equation (2).

$$\begin{cases}(2xy^2+3x^2)dx+2x^2ydy=0\\ du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0\end{cases}$$

We conclude from the comparison of both equations that: we integrate from equation $\frac{\partial u}{\partial y}=2x^2y$ with respect to y.

Since $\int du=2\int x^2ydy$ is integrated with respect to y hence x assumed the proof of $u=x^2y^2+\varphi(x)....(3)$, and $\varphi(x)$ is constant. We derive from the equation (3) with respect to x.

$$\begin{cases}\frac{\partial u}{\partial x}=2xy^2+\varphi'(x)\\ \frac{\partial u}{\partial x}=2xy^2+3x^2.....\end{cases}\to\varphi'(x)=3x^2\to\frac{d\varphi(x)}{dx}=3x^2\ \text{With the problem consideration}$$

$$d\varphi(x)=3x^2dx\to\int d\varphi(x)=3\int x^2dx+c\to\varphi(x)=x^3+c$$

We let the $\varphi(x)$ value into the equation (3).

$u=u(x,y)=x^2y^2+x^3+c$ is the solution of equation (2) (Aqayan, 2008).

### Factor of the second-order

If we multiply the integrating factor of $f(x,y)$ to linearly partial differential equation of the first-order

$P(x,y)dx+Q(x,y)dy=0....(1)$, it substitutes it to exact equation.

The $f(x,y)\cdot P(x,y)dx+f(x,y)\cdot Q(x,y)dy=0....(2)$ equation got exact, we apply Euler's formula to the equation (2).

$$\frac{\partial f\cdot Q}{\partial x}=\frac{\partial f\cdot P}{\partial y}\quad \left.\begin{aligned}\frac{\partial}{\partial x}(f\cdot Q)&=\frac{\partial f}{\partial x}\cdot Q+f\cdot\frac{\partial Q}{\partial x}\\ \frac{\partial}{\partial y}(f\cdot P)&=\frac{\partial f}{\partial y}\cdot P+f\cdot\frac{\partial P}{\partial y}\end{aligned}\right\}\rightarrow \begin{aligned}&\frac{\partial f}{\partial x}Q+f\frac{\partial Q}{\partial x}=\frac{\partial f}{\partial y}P+f\frac{\partial P}{\partial y}\\ &\frac{\partial f}{\partial x}Q-\frac{\partial f}{\partial y}P=f\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)....(3)\end{aligned}$$

We divide both sides of equation (3) to f.

$$\frac{\partial f}{f\,\partial x}Q-\frac{\partial f}{f\,\partial y}P=-\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\quad\Rightarrow\frac{\frac{\partial f}{f}}{\partial x}Q-\frac{\frac{\partial f}{f}}{\partial y}P=-\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$$

$$\frac{\partial Lnf}{\partial x}Q-\frac{\partial Lnf}{\partial y}P=-\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$$

If $\frac{\partial Lnf}{\partial x}=0$.

(Halim, 2014)

$$P\frac{\partial Lnf}{\partial y}=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\rightarrow\frac{\partial Lnf}{\partial y}=\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{P}=\frac{\Delta}{P}\rightarrow dLnf=\frac{\Delta}{P}dy\rightarrow\int dLnf=\int\frac{\Delta}{P}dy$$

$$\rightarrow Lnf=\int\frac{\Delta}{P}dy$$

$f=e^{\int\frac{\Delta}{P}dy}$ is factor of the second-order. Factor of the second-order is used in the case which $\frac{\Delta}{P}$ is a function of y (Ferozkohi & Hashemi, 2013).

**Problem 2:** Integrate the following partial differential equation with consideration of factor of the second-order, then solve it.

$$(y+xy^2)dx-xdy=0.....(1)$$

**Solution:** First we apply factor of the second-order condition. $\Delta=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$

$$\frac{\partial}{\partial x}(-x)=-\frac{\partial x}{\partial x}=-1$$

$$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}(y+xy^2)=1+2xy$$

$$Q=-x\qquad P=y+xy^2$$

$$\frac{\Delta}{P}=\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}{P}=\frac{-1-(1+2xy)}{y+xy^2}=\frac{-2-2xy}{y(1+xy)}=\frac{-2(1+xy)}{y(1+xy)}=-\frac{2}{y}$$

Since $\frac{\Delta}{P}=-\frac{2}{y}$ ($\frac{\Delta}{P}$ is a function of y), so factor of the second exists.

$$f=e^{\int\frac{\Delta}{P}dy}=e^{-\int\frac{2}{y}dy}=e^{-2Lny}=e^{Ln\frac{1}{y^2}}=\frac{1}{y^2}$$

$f=\frac{1}{y^2}$ is factor of the second-order. We multiply the partial equation (1) by $y^{-2}$.

$$\frac{1}{y^2}(y+xy^2)dx-\frac{1}{y^2}xdy=0\rightarrow\underbrace{\left(\frac{1}{y}+x\right)}_{P}dx-\underbrace{\frac{1}{y^2}x}_{Q}dy=0....(2)\qquad P=x+\frac{1}{y}\ ,\ Q=-\frac{x}{y^2}$$

Since $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, thus equation (2) is exact $\left.\begin{array}{l}\frac{\partial Q}{\partial x}=\frac{\partial}{\partial x}\left(-\frac{1}{y^2}x\right)=-\frac{1}{y^2}\\ \frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\left(x+\frac{1}{y}\right)=-\frac{1}{y^2}\end{array}\right\}\rightarrow$

The exact differential function is $u$. $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0$

For the solution of equation (2) we shall find a function such as $u(x,y)$ which the exact differential of it, is equal to the left side of the equation (2), in this case from the comparison of equation with the exact differential function of $u$ it is written as:

$$\frac{\partial u}{\partial x}=x+\frac{1}{y}....(3)$$

We integrate for finding u function with respect to x.

$$du=\left(x+\frac{1}{y}\right)dx....(4)$$

$$\int du=\int\left(x+\frac{1}{y}\right)dx+\varphi(y)\Rightarrow u=\frac{1}{2}x^2+\frac{1}{y}\cdot x+\varphi(y)....(5)$$

Since it is integrated with respect x, y is constant and $\varphi(y)$ is also constant. For finding $\varphi(y)$ value we derive from equation (5) with respect to y.

$$\left\{\begin{array}{l}\frac{\partial u}{\partial y}=-\frac{x}{y^2}+\varphi'(y)\\ \frac{\partial u}{\partial y}=-\frac{x}{y^2}\end{array}\right.\rightarrow\varphi'(y)=0\rightarrow\frac{d\varphi(y)}{dy}=0\rightarrow d\varphi\rightarrow\int d\varphi(y)=\int 0dy\rightarrow\varphi(y)=c$$

We let the value of $\varphi(y)$ into the equation (5).

$$(87-88:4)\qquad u=u(x,y)=\frac{1}{2}x^2+\frac{1}{y}x+c$$

**Problem 3:** Make exact the partial equation of $2ydx+xdy=0....(1)$ with consideration of factor of the first-order, then solve it.

**Solution:** $f=e^{-\int\frac{\Delta}{Q}dx}$ is factor of the first-order. The factor of the first-order condition $\frac{\Delta}{-Q}$ is a function of the x alone.

$$\Delta=\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=\frac{\partial x}{\partial x}-\frac{\partial}{\partial y}(2y)=1-2=-1\qquad\frac{\Delta}{-Q}=-\frac{-1}{x}=\frac{1}{x}$$

Since $\frac{\Delta}{-Q}=\frac{1}{x}$ is a function of x, thus factor of the first-order exists. $f=e^{-\int\frac{\Delta}{Q}dx}=e^{\int\frac{1}{x}dx}=e^{Lnx}=x$

We multiply partial equation of $2ydx+xdy=0$ by x.

$$2xydx+x^2dy=0....(2)\qquad P(x,y)dx+Q(x,y)dy=0$$

We apply Euler's formula on equation (2) which denotes the equation exactness.

$$P=2xy\ \ ,\ \ Q=x^2\qquad\frac{\partial Q}{\partial x}=\frac{\partial}{\partial x}(x^2)=2x$$

$$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}(2xy)=2x$$

For solving equation (2), we shall find a function such as $u(x,y)$ which its differential equation be equal to the left side of equation (2).

The $u$ exact differential equation formula is.

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0....(3)$$

From the comparison of (2) and (3) it is written as:

(We integrate from this equation with respect to x)

$$\frac{\partial u}{\partial x}=2xy\rightarrow du=2xydx$$

Since it is integrated with respect to x, thus y is constant.

$$\int du=\int 2xydx+\varphi(y)\rightarrow u=x^2y+\varphi(y)...(4)$$

We derive from the equation (4) For finding the $\varphi(y)$ value with respect to y.

$$\left\{\begin{array}{l}\frac{\partial u}{\partial y}=x^2+\varphi'(y)\\ \frac{\partial u}{\partial y}=x^2\end{array}\right.\rightarrow\varphi'(y)=0\rightarrow\varphi(y)=c$$

On the other hand, with consideration of question

We let $\varphi(y)$ value into the equation (4) (Kerayechian, 2015). (33:9) $u=u(x,y)=x^2y+c$

**Factor of the third-order**

If the equation $M(x,y)dx+N(x,y)dy=0$ be partial and $\mu$ integrating factor, then $\mu=e^{\int g(z)dz}$ is the factor of the third-order. The factor of the third-order condition $\frac{\Delta}{x \cdot N - y \cdot M}$ is from z, thus $z = xy$.

**Proof:** $\mu$ is integrating factor, so the following equation is exact.

$$\mu \cdot M(x,y)dx + \mu \cdot N(x,y)dy = 0....(2)$$

Hence Euler's formula satisfies. $\frac{\partial \mu N}{\partial x} = \frac{\partial \mu M}{\partial y}$

We derive partial of both sides.

$$\begin{cases} \frac{\partial \mu N}{\partial x} = \frac{\partial \mu}{\partial x} \cdot N + \mu \cdot \frac{\partial N}{\partial x} \\ \frac{\partial \mu M}{\partial y} = \frac{\partial \mu}{\partial y} \cdot M + \mu \frac{\partial M}{\partial y} \end{cases} \to \frac{\partial \mu}{\partial x} \cdot N + \mu \cdot \frac{\partial N}{\partial x} = \frac{\partial \mu}{\partial y} \cdot M + \mu \cdot \frac{\partial M}{\partial y}$$

$$\to \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} \to \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{\mu}\left(M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x}\right)....(3)$$

Now if $\mu$ is a function of $z = xy$, in this case $z = xy$ is written as $\frac{\partial z}{\partial y} = x$ and $\frac{\partial z}{\partial x} = y$ we let the value of $\frac{\partial \mu}{\partial y}$ and $\frac{\partial \mu}{\partial x}$ into (3).

$$\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial z} \cdot \frac{\partial z}{\partial y} \to \frac{\partial \mu}{\partial y} = x \cdot \frac{\partial \mu}{\partial z}$$

$$\frac{\partial \mu}{\partial x} = \frac{\partial \mu}{\partial z} \cdot \frac{\partial z}{\partial x} \to \frac{\partial \mu}{\partial x} = y \cdot \frac{\partial \mu}{\partial z}$$

We let the value of $\frac{\partial \mu}{\partial y}$ and $\frac{\partial \mu}{\partial x}$ into (3).

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{\mu}\left(M \cdot x \cdot \frac{\partial \mu}{\partial z} - N \cdot y \cdot \frac{\partial \mu}{\partial z}\right) \to \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{\mu} \cdot \frac{\partial \mu}{\partial z}(xM - yN)$$

$$\to \frac{\partial \mu}{\mu} \cdot \frac{xM - yN}{dz} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \to \frac{\partial \mu}{\mu} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} dz$$

If $\frac{\Delta}{xM - yN}$ is a function from z. $\Delta = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ assume as $\frac{\Delta}{xM - yN} = g(z)$, then

$$\frac{\partial \mu}{\mu} = g(z)dz \to \int \frac{d\mu}{\mu} = \int g(z)dz \to Ln\mu = \int g(z)dz \to \mu = e^{\int g(z)dz}$$

In this case factor of the third is proved (Simmons, 2014).

**Problem 4:** Integrate the following partial differential equation with factor of the third-order, then solve.

$$(2yCosx - xySinx)dx + 2xCosxdy = 0....(1)$$

**Solution:** For finding $\frac{\Delta}{xM - yN}$ which is a function of z we go through as follow:

$$\Delta = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad M = 2yCosx - xySinx \qquad N = 2xCosx$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xCosx) = 2(1 \cdot Cosx - x \cdot Sinx) \qquad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2yCosx - xySinx) = 2Cosx - xSinx$$

$$\Delta = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2Cosx - 2xSinx - (2Cosx - xSinx) = -xSinx$$

$$\begin{cases} x \cdot M = x(2yCosx - xySinx) = 2xyCosx - x^2ySinx \\ yN = y \cdot 2xCosx \end{cases}$$

$$xM - yN = 2xyCosx - x^2ySinx - 2xyCosx = -x^2ySinx$$

Now we let the value of $xN, \Delta$ and $yM$ into the $\frac{\Delta}{xM - yN}$

$$\frac{\Delta}{xM - yN} = \frac{-xSinx}{-x^2ySinx} = \frac{1}{xy} = \frac{1}{z} = g(z)$$

Factor of the third-order is

$$\mu = e^{\int g(z)dz} = e^{\int \frac{1}{z}dz} = e^{Lnz} = z = xy$$

We multiply the equation (1) by $xy$.

$$(2xy^2Cosx - x^2y^2Sinx)dx + 2x^2y \cdot Cosxdy = 0....(2)$$

For finding the equation (2) solution we find a function which its exact differential is equal to the left side of equation (2).

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

To find $u$ function we use from $\frac{\partial u}{\partial y} = 2x^2yCosx$. Then

We integrate from $\frac{\partial u}{\partial y} = 2x^2yCosx$ equation.

$$du = 2x^2Cosxydy \to \int du = 2x^2Cosx\int ydy = x^2y^2Cosx + \varphi(x)$$

Since it is integrated with respect to y, so x and $\varphi(x)$ are constants.

**Factor of the fourth-order**

If the differential equation has general denotation as:

$$y\left(Ax^m y^n + Bx^P y^q\right)dx + x\left(Cx^m y^n + Dx^P y^q\right)dy = 0....(1)$$

If $AD - BC \neq 0$. This equation includes integrating factor as $f\left(x,y\right) = x^k y^\lambda$.

The method for finding integrating factor is that we multiply non integrating equation of (1) by $x^k y^\lambda$, then we obtain the value of k and $\lambda$ from the equation $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

**Problem 5:** Determine the integrating factor of this differential equation.

$$\left(x^2 y - xy^4\right)dx + \left(2x^3 - x^2 y^3\right)dy = 0....(1)$$

**Solution:** We write the equation (1) with its general denotation form. $y\left(x^2 - xy^3\right)dx + x\left(2x^2 - xy^3\right)dy = 0$

Now we form the term $AD - BC$, if it is not equal to zero, then it has integrating factor.

$$A = 1,\quad B = -1,\quad C = 2,\quad D = -1$$

$$AD - BC = 1\cdot(-1) - (-1)\cdot 2 = 1 \neq 0$$

Therefore that partial equation (1) includes integrating factor as $f\left(x,y\right) = x^k y^\lambda$. We multiply the equation (1) by $x^k y^\lambda$.

$$x^k y^\lambda\left(x^2 y - xy^4\right)dx + k^k y^\lambda\left(2x^3 - x^2 y^3\right)dy = 0$$

$$\rightarrow\left(x^{k+2} y^{\lambda+1} - x^{k+1} y^{\lambda+4}\right)dx + \left(2x^{k+3} y^\lambda - x^{k+2} y^{\lambda+3}\right)dy = 0$$

For finding the value of k and $\lambda$ we use from the equation $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(2x^{k+3} y^\lambda - x^{k+2} y^{\lambda+3}\right) = 2\left(k+3\right)y^\lambda \cdot x^{k+2} - \left(k+2\right)x^{k+1} y^{\lambda+3} = 0$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(x^{k+2} y^{\lambda+1} - x^{k+1} y^{\lambda+4}\right) = \left(\lambda+1\right)x^{k+2} y^\lambda - \left(\lambda+4\right)x^{k+1} y^{\lambda+3} = 0$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \underbrace{\left(2k - \lambda + 5\right)}_{0} x^{k+2} y^\lambda - \underbrace{\left(\lambda - k + 2\right)}_{0} x^{k+1} y^{\lambda+3} = 0$$

Since $x^{k+2} y^\lambda \neq 0$ and $x^{k+1} y^{\lambda+3} \neq 0$, then $\lambda = -9,\ k = -7$ $\begin{cases} 2k - \lambda + 5 = 0 \\ -k + \lambda + 2 = 0 \end{cases}$ is integrating factor of $f\left(x,y\right) = x^{-7} y^{-9}$. (Aqayan, 2008)

**Factor of the fifth-order**

If the equation $Mdx + Ndy = 0....(1)$ be homogenous, this means the terms M and N are same grade and $xM + yN \neq 0$, in this case the $\frac{1}{xM + yN}$ is an integrating factor of equation (1).

**Problem 6:** Find an integrating factor to the following partial equation.

$$\left(x^2 + y^2\right)dx - 4xydy = 0$$

**Solution:** Since the coefficients of $dx$ and $dy$ are homogenous functions with 2 order, $M = x^2 + y^2$ and $N = -4xy$, thus

$$xM + yN = x\left(x^2 + y^2\right) + y\left(-4xy\right) = x\left(x^2 - 3y^2\right) \neq 0$$

so integrating factor is $\frac{1}{x\left(x^2 - 3y^2\right)}$.

**Factor of the Sixth-order**

If in equation $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{2\left(yM - xN\right)}$, $Mdx + Ndy = 0....(1)$ be a function of $\left(x^2 + y^2\right)$, in this case one integrating factor of equation (1) is as follow:

$$h\left(u\right) = e^{\int \frac{\Delta}{2(yM - xN)}du} \qquad \Delta = \frac{\partial \mathbf{N}}{\partial x} - \frac{\partial M}{\partial y}$$

**Proof:** If integrating factor is $h\left(u\right)$ as $u = x^2 + y^2$, we multiply function $h\left(u\right)$ into the equation (1). The $h\left(u\right)\cdot Mdx + h\left(u\right)\cdot Ndy = 0....(2)$ is integrated ,so we apply Euler's formula.

$$\frac{\partial h\left(u\right)N}{\partial x} = h'_x\left(u\right)\cdot N + h\left(u\right)\cdot\frac{\partial N}{\partial x}$$

$$h'_x\left(x^2 + y^2\right) = h'\left(x\right)\cdot 2x$$

$$\left.\begin{aligned}\frac{\partial h(u)N}{\partial x}&=h'(u)\cdot 2xN+h(u)\frac{\partial N}{\partial x}\\ \frac{\partial h(u)M}{\partial y}&=h'(u)\cdot 2yM+h(u)\frac{\partial M}{\partial y}\end{aligned}\right\} \qquad \frac{\partial h(u)N}{\partial x}=\frac{\partial h(u)M}{\partial y}$$

$$\Rightarrow h'(u)\cdot 2xN+h(u)\frac{\partial N}{\partial x}=h'(u)2yM+h(u)\frac{\partial M}{\partial y}\to h'(u)(2xN-2yM)=h(u)\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$

$$\frac{h'(u)}{h(u)}=\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{2(yM-xN)} \qquad \frac{h'(u)}{h(u)}=\frac{dLnh(u)}{du}\to\frac{dLnh(u)}{du}=\frac{\Delta}{2(yM-xN)}$$

$$\to dLnh(u)=\frac{\Delta}{2(yM-xN)}du$$

We integrate from the both sides of above.

$$\int dLnh(u)=\int\frac{\Delta}{2(yM-xN)}du\Rightarrow Lnh(u)=\frac{1}{2}\int\frac{\Delta}{yM-xN}du$$

Integrating factor $\to h(u)=e^{\frac{1}{2}\int\frac{\Delta}{yM-xN}du}$

**Problem 7:** Find an integrating factor for the equation $xdx+y\left(4x^2y^2+4y^4+1\right)dy=0...(1)$ with factor of the sixth-order, then solve the exact equation.

**Solution:** First we find $\frac{\Delta}{2(yM-xN)}$,

$N=4x^2y^3+4y^5+y,\quad M=x$

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x)=0\ ,\ \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(4x^2y^3+4y^5+y\right)=8xy^3\Rightarrow\Delta=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=8xy^3$$

$$\frac{\Delta}{2(yM-xN)}=\frac{8xy^3}{2\left(yx-4x^3y^3+4xy^5-xy\right)}=-\frac{8xy^3}{2\cdot 4xy^3\left(x^2+y^2\right)}=-\frac{1}{x^2+y^2}=-\frac{1}{u}$$

$$h(u)=e^{-\int\frac{1}{u}du}=e^{-Lnu}=e^{Ln\frac{1}{u}}=\frac{1}{u}=\frac{1}{x^2+y^2}$$

Hence integrating factor is $\frac{1}{x^2+y^2}$. We multiply the equation (1) by $\frac{1}{x^2+y^2}$

$$\frac{xdx}{x^2+y^2}+\frac{4y^3\left(x^2+y^2\right)+y}{x^2+y^2}dy=0\to\frac{x}{x^2+y^2}dx+\left(4y^3+\frac{y}{x^2+y^2}\right)dy=0$$

$$\frac{xdx+ydy}{x^2+y^2}+4y^3dy=0\Rightarrow\frac{1}{2}\int\frac{2xdx+2ydy}{x^2+y^2}+4\int y^3dy=c$$

The equation solution is as implicit function.

$$\frac{1}{2}Ln\left|x^2+y^2\right|+y^4=c$$

**Factor of the seventh-order**

If in partial equation $Mdx+Ndy=0....(1)$ the term $\frac{-\Delta}{Mx+Ny}$ be a function of $\frac{x}{y}$, the integrating factor is as $u=\frac{x}{y},\ \ h(u)=e^{\int H(u)du}$.

**Proof:** We multiply the integrating factor which are $h(u)$ and $u=\frac{x}{y}$ into partial equation (1).

$$h(u)Mdx+h(u)Ndy=0$$

We apply Euler's formula.

$$\frac{\partial h(u)N}{\partial x}=\frac{\partial h(u)M}{\partial y}$$

$$\frac{\partial h(u)N}{\partial x}=\frac{\partial h(u)}{\partial x}N+h(u)\frac{\partial N}{\partial x}=h'(u)\cdot h'_x(u)\cdot N+h(u)\frac{\partial N}{\partial x}$$

$$\frac{\partial h(u)M}{\partial y}=\frac{\partial h(u)}{\partial y}M+h(u)\frac{\partial M}{\partial y}=h'(u)\cdot h'_y\cdot M+h(u)\frac{\partial M}{\partial y}$$

$$\frac{\partial h(u)N}{\partial x}=h'(u)\cdot\frac{N}{y}+h(u)\frac{\partial N}{\partial x}$$

$$\frac{\partial h(u)M}{\partial y}=h'(u)\left(-\frac{x}{y}\right)\cdot M+h(u)\frac{\partial M}{\partial y}$$

$$\Rightarrow h'(u)\frac{1}{y}N+h(u)\frac{\partial N}{\partial x}=h'(u)\left(-\frac{x}{y^2}\right)M+h(u)\frac{\partial M}{\partial y}$$

$$h'(u)\left(\frac{1}{y}N+\frac{x}{y^2}M\right)=h(u)\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \qquad \frac{h'(u)}{h(u)}=\frac{-\Delta}{\frac{1}{y}N+\frac{x}{y^2}M}=\frac{-y^2\Delta}{xM+yN}$$

$$\frac{h'(u)}{h(u)}=\frac{dLnh(u)}{du}\rightarrow\frac{dLnh(u)}{du}=\frac{-y^2\Delta}{xM+yN}\Rightarrow dLnh(u)=\frac{-y^2\Delta}{xM+yN}du$$

$$Lnh(u)=\int\frac{-y^2\Delta}{xM+yN}du\Rightarrow h(u)=e^{-\int\frac{y^2\Delta}{xM+yN}du}$$

**Problem 8:** Integrate the following partial equation with factor of the seventh-order.

$$5y^2dx-3xydy=0....(1)$$

**Solution:** $M=5y^2$ and $N=-3xy$. In factor of the seventh-order the function shall be from $\frac{x}{y}$.

$$\Delta=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=\frac{\partial}{\partial x}(-3xy)-\frac{\partial}{\partial y}(5y^2)=-3y-10y=-13y$$

$$\frac{-y^2\Delta}{xM+yN}=\frac{-13y^3}{5xy^2-3xy^2}=\frac{13y^3}{2xy^2}=\frac{13}{2}\cdot\frac{1}{\frac{x}{y}}\rightarrow h(u)=e^{\int\frac{-y^2\Delta}{xM+yN}du}=e^{\frac{13}{2}\int\frac{1}{u}du}=e^{\frac{13}{2}Lnu}=e^{Ln(u)^{\frac{13}{2}}}=(u)^{\frac{13}{2}}$$

$\left(\frac{x}{y}\right)^{\frac{13}{2}}$ is integrating factor (Paryab, 2010).

**Integrating factors determination with expressions classification**

The integrating factors of some equations can be find with expressions classification, investigation and determination of common integrating factor such as the coefficient of each functions $\frac{1}{xy},\frac{1}{y^2},\frac{1}{x^2}$ or $\frac{1}{x^2+y^2}$ integrates in the term of $xdy-ydx=0$.

$$\frac{1}{x^2}(xdy-ydx)=\frac{xdy-ydx}{x^2}=d\left(\frac{y}{x}\right)$$

$$\frac{1}{y^2}(xdy-ydx)=\frac{xdy-ydx}{y^2}=-d\left(\frac{x}{y}\right)$$

$$\frac{1}{xy}(xdy-ydx)=\frac{dy}{y}-\frac{dx}{x}=dLn\left|\frac{y}{x}\right|$$

$$\frac{1}{x^2+y^2}(xdx-ydx)=\frac{xdy-ydx}{x^2+y^2}=d\left(Tan^{-1}\frac{y}{x}\right)$$

If equation includes the term $xdy-ydx$, one of these function shall be introduced as integrating factor of equation. This determination method of integrating factor is called classification.

**Problem 9:** Obtain the general solution of the following equation.

$$xdy-\left(x^4+x^2y^2+y\right)dx=0$$

**Solution:** We classify the equation expressions as

$$xdy-ydx=x^2\left(x^2+y^2\right)dx$$

Since the division of equation by $x^2+y^2$ integrates both sides, thus $\frac{1}{x^2+y^2}$ is integrating factor.

$$\frac{xdy-ydx}{x^2+y^2}=x^2dx$$

We integrate from both sides of last equation.

The general solution:

$$\int\frac{xdy-ydx}{x^2+y^2}=\int x^2dx+c\Rightarrow Tan^{-1}\frac{y}{x}=\frac{1}{3}x^3+c$$

**Problem 10:** Determine the general solution of the following equation. $\left(y+x^4\right)dx-xdy=0....(1)$

**Solution:** First we classify the expressions $ydx-xdy+x^4dx=0$ with division of both sides by $x^2$, the following term gets. The two left side of each is equal to exact differential of $\frac{ydx-xdy}{x^2}+x^2dx=0....(2)$, in this case the last equation is exact differential.

$$\frac{ydx-xdy}{x^2}=-\frac{xdy-ydx}{x^2}=-d\left(\frac{y}{x}\right),\quad x^2dx=\frac{1}{3}d\left(x^3\right)$$

We integrate from both sides of equation (2).

$$\int\frac{ydx-xdy}{x^2}+\int x^2dx=c\rightarrow-\int d\left(\frac{y}{x}\right)+\frac{1}{3}\int dx^3=$$
$$c\rightarrow-\frac{y}{x}+\frac{1}{3}x^3=c\rightarrow y=\frac{1}{3}x^4-cx$$

In the following table some integrating terms of a differential equation are listed.

| Equation expressions | Integrating factor | Exact terms |
|---|---|---|
| $xdy-ydx$ | $\frac{1}{x^2}$<br>$\frac{1}{y^2}$<br>$\frac{1}{x^2+y^2}$<br>$\frac{1}{xy}$ | $d\left(\frac{y}{x}\right)$<br>$-d\left(\frac{x}{y}\right)$<br>$Tan^{-1}\left(\frac{y}{x}\right)$<br>$Ln\left\|\frac{y}{x}\right\|$ |
| $xdy+ydx$ | $\frac{1}{xy}$<br>$\frac{1}{(xy)^n},\ n>1$ | $Ln\|xy\|$<br>$\frac{-1}{(n-1)(xy)^{n-1}}$ |
| $ydy+xdx$ | $\frac{1}{x^2+y^2}$<br>$\frac{1}{\left(x^2+y^2\right)^n},\ n>1$ | $\frac{1}{2}Ln\left(x^2+y^2\right)$<br>$\frac{-1}{2(n-1)\left(x^2+y^2\right)^{n-1}}$ |
| $aydx+bxdy$ | $x^{a-1}y^{b-1}$ | $x^ay^b$ |

(Jamshidi, 2004)

## 4. Result

In general cases, finding integrating factors seem difficult but there are particular conditions which denote how to obtain these factors. If an equation of the second-order includes solution, it probably has integrating factor, but for the complexity of factor form or being non initial the integrating factor cannot calculate easily.

There are more than seven integrating factors under the particular conditions such as if $\frac{-\Delta}{Q}$ be a function of x alone. So integrating factor exists and it is $f\left(x\right)=e^{\int\frac{-\Delta}{Q}dx}$.

The other method for finding integrating factor is expressions classification, for example if we multiply the partial equation of $xdy-ydx=0$ by $\frac{1}{x^2}$ then gets $\frac{xdy-ydx}{x^2}=d\frac{y}{x}$. By integrating of it unknown function is obtained.

## 5. Discussion

There are nine integrating factors which form under the particular conditions. If $\frac{\Delta}{P}$ is the only function of x. Then $f\left(x\right)=e^{\int\frac{\Delta}{P}dy}$ is integrating factor of the first-order. If $\frac{\Delta}{P}$ is a function of y alone. Then $f\left(x\right)=e^{\int\frac{\Delta}{P}dy}$ is integrating factor of the second-order. If $\frac{\Delta}{xP-yQ}=g\left(z\right)$ is a function of z, thus $z=xy$. $f\left(z\right)=e^{\int g(z)dz}$ is integrating factor of the third-order. If the non-integer differential equation be as $y\left(Ax^my^n+Bx^Py^q\right)dx+x\left(Cx^my^n+Dx^Py^q\right)dy=0$, then $f\left(x,y\right)=x^ky^\lambda$ is integrating factor of it. The other factors are also like this.

Besides the nine forms for integrating factors the other method is expressions classification with the consideration of table guideline after multiplying to a partial differential function, it can be written as an exact expression, by integrating the unknown function of y is found.

For example:

$$\frac{1}{x^2+y^2}\left(xdy-ydx\right)=0\rightarrow\frac{xdy-ydx}{x^2+y^2}=Tan^{-1}\frac{y}{x}$$

## 6. Conclusion

If the Euler's formula does not satisfy to differential equation $Pdx + Qdy = 0....(1)$ the equation is not exact or integer, it means that a function such as $u(x,y)$ cannot be found which the exact differential of it, is not equal to the left side of equation (1). In this case a function shall be found by multiplying it to the equation (1) integrate that. In general conditions there is no theorem which creates integrating factor for all partial differential equations, therefore with respect to the particular conditions of partial equations, functions are created which is integrating factor.

The number of these factors are at least nine. These integrating factors are different with respect to the expressions structure of partial differential equation.

The other method to find integrating factors is expressions classification of partial equation which is written as exact term with respect to the classification guideline and usage of equation table, then by integrating unknown function is found (equation is solved).

In this article I researched about integrating factors from external and internal sources which discussed about differential equations at recent decades. Finally, I explained seven kinds of integrating factors obviously with proof and problems solution. I hope this article be helpful for mathematics learners.

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