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Trace of Negative Integer Power of Real 2x2 Matrices

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Abstract: ***The purpose of this paper is to discuss the theorems for the trace of any negative integer power of 2 × 2 real Matrix. We obtain a new formula to compute trace of any negative integer power of 2 × 2 real matrix A, in the terms of Trace of A (Tr A) and Determinant of A (Det A), which are based on definition of trace of matrix and multiplication of the matrix n times, where n is Negative integer and this formula gives some corollary for Tr A^n when TrA or DetA are zero***.

Keywords: Trace, Determinant, Matrix Multiplication

1. Introduction

Traces of powers of matrices arise in several fields of mathematics, more specially, Network Analysis, Number Theory, Dynamical systems, Matrix Theory, and Differential Equations [1]

2. Main Result

Theorem 1: For even negative integer n and 2×2 real matrix A,

$$Tr(A^{-n}) = \frac{1}{|A|^n}\sum_{r=0}^{n/2}\frac{(-1)^r}{r!}n[n-(r+1)][n-(r+2)]\ldots\ldots\ldots[\text{up to r terms}]|A^r|(\text{Tr A})^{n-2r}$$

Proof: Consider a matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are real.

$$Tr(A) = (a + d) \quad (2.1)$$

$$|A| = (ad - bc) \quad (2.2)$$

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

$$Tr(A^2) = a^2 + 2bc + d^2 \quad (2.3)$$

$$\text{Adj }(A^2) = \begin{bmatrix} bc+d^2 & -(ab+bd) \\ -(ac+cd) & a^2+bc \end{bmatrix}$$

$$|A^2| = (a^2+bc).(bc+d^2) - (ac+cd).(ab+bd)$$

$$|A^2| = a^2d^2 - 2abcd + b^2c^2$$

$$|A^2| = (ad-bc)^2 = |A|^2 \quad (2.4)$$

$$\mathbf{Tr\,(A^{-2})} = \frac{1}{|A|^2}[\mathbf{Tr\,(A^2)}] \quad (2.5)$$

$$\text{Tr }(A^{-2}) = \frac{1}{|A|^2}[(\text{Tr A})^2 - 2|A|] \;\{by\ [1]\} \quad (2.6)$$

$$A^4 = \begin{bmatrix} a^4+3a^2bc+2abcd+bcd^2+b^2c^2 & a^3b+2ab^2c+2b^2cd+a^2bd+abd^2+bd^3 \\ a^3c+a^2cd+2abc^2+acd^2+2bc^2d+cd^3 & a^2bc+2abcd+3bcd^2+b^2c^2+d^4 \end{bmatrix}$$

$$\text{Tr }(A^4) = a^4 + 4a^2bc + 4abcd + 4bcd^2 + 2b^2c^2 + d^4 \quad (2.7)$$

$$\text{Adj }(A^4) = \begin{bmatrix} a^2bc+2abcd+3bcd^2+b^2c^2+d^4 & -(a^3b+2ab^2c+2b^2cd+a^2bd+abd^2+bd^3) \\ -(a^3c+a^2cd+2abc^2+acd^2+2bc^2d+cd^3) & a^4+3a^2bc+2abcd+bcd^2+b^2c^2 \end{bmatrix}$$

$$|A^4| = a^4d^4 + 6a^2b^2c^2d^2 + b^4c^4 - 4ab^3c^3d - 4a^3bcd^3$$

$$|A^4| = (ad)^4 - 4(ad)^3(bc) + 6(ad)^2(bc)^2 - 4(ad)(bc)^3 + (bc)^4$$

$$|A^4| = (ad-bc)^4 = |A|^4 \quad (2.8)$$

$$\text{Tr}(A^{-4}) = \frac{1}{(ad-bc)^4}[a^4 + 4a^2bc + 4abcd + 4bcd^2 + 2b^2c^2 + d^4]$$

$$\mathbf{Tr\,(A^{-4})} = \frac{\mathbf{1}}{\mathbf{|A|^4}}[\mathbf{Tr\,(A^4)}]$$

$$\text{Tr }(A^{-4}) = \frac{1}{|A|^4}[(\text{Tr A})^4 - 4\text{Det A}.(\text{Tr A})^2 + 2|A^2|]\;\{by\ [1]\} \quad (2.9)$$

Continuing this process up to n terms we get

$$\boldsymbol{Tr(A^{-n})} = \frac{\mathbf{1}}{\mathbf{|A|^n}}\sum_{r=0}^{n/2}\frac{(-1)^r}{r!}\mathbf{n[n-(r+1)][n-(r+2)]\ldots\ldots\ldots[up\ to\ r\ terms]|A^r|(Tr\ A)^{n-2r}}$$

Theorem 2: For odd negative integer n and 2×2 real matrix A,

$$Tr(A^{-n}) = \frac{1}{|A|^n}\sum_{r=0}^{(n-1)/2}\frac{(-1)^r}{r!}n[n-(r+1)][n-(r+2)]\ldots\ldots\ldots[\text{up to r terms}]|A^r|(\text{Tr A})^{n-2r}$$

Proof: Consider a matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are real.

$$Tr(A) = (a + d) \qquad (3.1)$$

$$\mathrm{Adj}\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (3.2)$$

$$|A| = (ad - bc) \qquad (3.3)$$

$$A^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\mathrm{Tr}(A^{-1}) = \frac{1}{(ad-bc)}(a+d)$$

$$\mathrm{Tr}(A^{-1}) = \frac{1}{(ad-bc)}\mathrm{Tr}(A)$$

$$\boldsymbol{Tr(A^{-1})} = \frac{1}{|A|}[\boldsymbol{Tr(A)}] \qquad (3.4)$$

$$A^3 = \begin{bmatrix} a^3 + 2abc + bcd & a^2b + b^2c + abd + bd^2 \\ a^2c + acd + bc^2 + cd^2 & abc + 2bcd + d^3 \end{bmatrix}$$

$$\mathrm{Tr}(A^3) = a^3 + 3abc + 3bcd + d^3 \qquad (3.5)$$

$$\mathrm{Adj}(A^3) = \begin{bmatrix} abc + 2bcd + d^3 & -(a^2b + b^2c + abd + bd^2) \\ -(a^2c + acd + bc^2 + cd^2) & a^3 + 2abc + bcd \end{bmatrix} \qquad (3.6)$$

$$|A^3| = (abc + 2bcd + d^3)(abc + 2bcd + d^3) - (a^2c + acd + bc^2 + cd^2).(a^2b + b^2c + abd + bd^2)$$

$$|A^3| = (ad)^3 - 3(ad)^2(bc) + 3(ad)(bc)^2 - (bc)^3$$

$$|A^3| = (ad - bc)^3 = |A|^3 \qquad (3.7)$$

$$\mathrm{Tr}(A^{-3}) = \frac{1}{(ad-bc)^3}[a^3 + 3abc + 3bcd + d^3]$$

$$\boldsymbol{Tr(A^{-3})} = \frac{1}{|A|^3}[(\boldsymbol{Tr}\ (\boldsymbol{A^3})]$$

$$\mathrm{Tr}(A^{-3}) = \frac{1}{|A|^3}[(\mathrm{Tr}\ A)^3 - 3|A|.\mathrm{Tr}\ A]\ \textit{\{by [1]\}} \qquad (3.8)$$

Now continuing this process, we get

$$\boldsymbol{Tr(A^{-n})} = \frac{1}{|A|^n}\sum_{r=0}^{(n-1)/2}\frac{(-1)^r}{r!}\,\mathbf{n[n-(r+1)][n-(r+2)]}$$

$$\dots\dots\dots\ \mathbf{[up\ to\ r\ terms\ ]}|A^r|(\mathbf{Tr\ A})^{n-2r} \qquad (3.9)$$

Corollary: For any negative integer n and 2×2 real matrix A,

1) $\mathrm{Tr}\ (A^{-1} + B^{-1}) = \mathrm{Tr}\ (A^{-1}) + \mathrm{Tr}\ (B^{-1})$
2) $\mathrm{Tr}\ (A^{-1}.B^{-1}) = \mathrm{Tr}\ (B^{-1}.\ A^{-1})$
3) $\mathrm{Tr}\ (A^{-1}.\ B^{-1}) \neq \mathrm{Tr}\ (A^{-1}).\mathrm{Tr}\ (B^{-1})$
4) $\mathrm{Tr}\ (cA^{-1}) = c\ \mathrm{Tr}\ (A^{-1})$
5) $\mathrm{Tr}\ (A^T.B^{-1}) = \mathrm{Tr}\ (A^{-1}.B^T)$

3. Conclusion and Future Work

After to discuss Theorem 1 and Theorem 2, corollaries 1, 2, 3, 4 and 5 we are able to find trace of any negative integer power of a 2× 2 real matrix. In future, we can be developed similar results for 3×3 real matrices.

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