Trace of Negative Integer Power of Real 2x2 Matrices

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Abstract: The purpose of this paper is to discuss the theorems for the trace of any negative integer power of 2 × 2 real matrix. We obtain a new formula to compute trace of any negative integer power of 2 × 2 real matrix A, in the terms of Trace of A (Tr A) and Determinant of A (Det A), which are based on definition of trace of matrix and multiplication of the matrix n times, where n is Negative integer and this formula gives some corollary for Tr Aⁿ when TrA or DetA are zero.

Keywords: Trace, Determinant, Matrix Multiplication

1. Introduction

Traces of powers of matrices arise in several fields of mathematics, more specially, Network Analysis, Number Theory, Dynamical systems, Matrix Theory, and Differential Equations [1]

2. Main Result

Theorem 1: For even negative integer n and 2×2 real matrix A,

\[ \text{Tr}(A^{-n}) = \frac{1}{|A|^r} \sum_{r=0}^{n/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]... \]

\[ \text{...[up to r terms]} A^r (\text{Tr} A)^{n-2r} \]

Proof: Consider a matrix A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} where a, b, c, d are real.

\[ A^r = \left[ \begin{array}{cc} a^r + 2a^2bc + 2abcd + bcd^2 + b^2c^2 \\ a^2c + a^2cd + 2abc^2 + acd^2 + 2bc^2d + cd^2 \end{array} \right] \]

\[ \text{Adj}(A^r) = \left[ \begin{array}{cc} a^2bc + 2abed + 3bc^2d + b^2c^2 + c^4 & -(a^2b + 2ab^2c + 2b^2cd + acd^2 + abd^2 + b^2c^2) \\ -(a^2c + a^2cd + 2abc^2 + acd^2 + 2bc^2d + cd^2) & a^2 + 3a^2bc + 2abed + bcd^2 + b^2c^2 \end{array} \right] \]

\[ \text{Tr}(A^r) = \frac{1}{|A|^r} \sum_{r=0}^{n/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]... \]

\[ \text{...[up to r terms]} A^r (\text{Tr} A)^{n-2r} \]

Theorem 2: For odd negative integer n and 2×2 real matrix A,

\[ \text{Tr}(A^{-n}) = \frac{1}{|A|^n} \sum_{r=0}^{(n-1)/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]... \]

\[ \text{...[up to r terms]} A^r (\text{Tr} A)^{n-2r} \]

\[ \text{Tr}(A^{-n}) = \frac{1}{|A|^n} \sum_{r=0}^{(n-1)/2} \frac{(-1)^r}{r!} n[n-(r+1)][n-(r+2)]... \]

\[ \text{...[up to r terms]} A^r (\text{Tr} A)^{n-2r} \]
**Proof:** Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d$ are real.

$$Tr(A) = (a + d) \quad \text{(3.1)}$$

$$\text{Adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(3.2)}$$

$$|A| = (ad - bc) \quad \text{(3.3)}$$

$$A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$Tr(A^{-1}) = \frac{1}{(ad - bc)} (a + d)$$

$$Tr(A^{-1}) = \frac{1}{(ad - bc)} Tr(A)$$

$$Tr(A^{n-1}) = \frac{1}{|A|^n} [Tr(A)^n - 3|A|.Tr(A) \text{ (by f/f)}] \quad \text{(3.8)}$$

Now continuing this process, we get

$$Tr(A^{-3}) = \frac{1}{(ad - bc)^3} \left[ a^3 + 3abc + 3bcd + d^3 \right]$$

$$Tr(A^{-2}) = \frac{1}{|A|^2} \left[ (Tr(A))^2 - 3|A|.Tr(A) \right] \quad \text{(3.7)}$$

$$|A|^2 = (abc + 2bcd + d^2)(abc + 2bcd + d^2) - (a^2c + acd + bc^2 + c^2d)^2 \quad \text{(3.6)}$$

$$|A|^3 = (abc + 2bcd + d^2)(abc + 2bcd + d^3) - (a^2d + 3(ad)^2 (bc) + 3(ad)(bc)^2 - (bc)^3$$

$$|A|^4 = (ad)^2 - (bc)^2$$

**Corollary:** For any negative integer $n$ and 2x2 real matrix $A$,

1. $Tr(A^{-1} - B^{-1}) = Tr(A^{-1}) + Tr(B^{-1})$
2. $Tr(A^{-1} - B^{-1}) = Tr(B^{-1} - A^{-1})$
3. $Tr(A^{-1} + B^{-1}) = Tr(A^{-1}) . Tr(B^{-1})$
4. $Tr(cA^{-1}) = c Tr(A^{-1})$
5. $Tr(A^{1/2}) = Tr(A^{-1/2})$

**3. Conclusion and Future Work**

After to discuss Theorem 1 and Theorem 2, corollaries 1, 2, 3, 4 and 5 we are able to find trace of any negative integer power of a 2x2 real matrix. In future, we can be developed similar results for 3x3 real matrices.

**References**


