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Similarity-Type Solutions for a Power-Law Fluid

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Abstract: ***In this paper, we have discussed the similarity-type solutions for a power-law fluid. We have derived similarity-type transformation that converts the partial differential equations governing the boundary-layer flow of a power-law fluid into an ordinary differential equation. The solution of the two-point boundary value problem was obtained by solving an equivalent initial value problem using a numerical scheme.***

Keywords: Power-law fluid, Boundary-layer

1. Introduction

The equations describing the boundary-layer flow of a power-law fluid along a flat plate are non-linear in character. An exact solution cannot be easily found and numerical methods. However, important insights into the main physical features that may exist within the boundary layer are provided by self-similar solutions to the boundary-layer equations. Similarity-type solutions are known, and have been extensively studied, for flows such as the flat plate, Falkner-Skan, converging channel and Goldstein. Additionally, self-similar solutions often serve as the basis for other methods that are used to study more complex non-similar flows.

Investigations into self-similar solutions of the boundary-layer flow of power-law fluids can be considered to have started with the work of Schowalter[1] and Acrivos *et al*[2]. Both investigators looked at flow along a flat plate and give the form of the similarity transformation as well as the ordinary differential equation from which a self-similar solution is obtained. Acrivos *et al*[2] presented some solutions to the governing Blasius-like differential equation for the case of zero mass transfer through the surface of the plate. The external flow is assumed to be uniform in these investigations.

Lee and Ames[3] consider the form of the similarity transformation for a number of different non-Newtonian fluids. For power-law fluids they considered various external flow regimes and gave the form of the ordinary differential equations governing these flows. A Falkner-Skan-type equation for power-law fluids is derived therein using group-theoretic methods. Self-similar velocity profiles are provided by the solutions of these differential equations.

A self-similar solution for the boundary-layer flow of a power-law fluid with mass transfer through the surface is discussed by Nachman and Taliaferro[4]. They show that similarity is preserved when the function describing mass transfer through the surface is of a specific form that depends on the stream-wise location. They also show that the fluid injection rates need to lie in a critical range to ensure self-similar velocity profiles.

In this paper, we derive a version of the Falkner-Skan-type equation for power-law fluids that is used in subsequent discussions. The derivation is in the style presented by Schlichting [5] rather than the group-theoretic approach used by Lee and Ames[3]. We focus specifically on the case of an external flow with a zero pressure gradient so that the Falkner-Skan-type differential equation reduces to a Blasius-like form. We also look at the asymptotic form of the solution in the far-field. Some techniques for obtaining a numerical solution of the Blasius-like differential equation are discussed and the solutions found are shown. We look at solutions of the Falkner-Skan-type differential equation for various non-zero values of the pressure gradient parameter.

2. Derivation of Governing Equation

The equations governing the boundary-layer flow of a power-law fluid are given as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1.1a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{dp}{dx}+n\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial^2 u}{\partial y^2}, \tag{1.1b}$$

and the corresponding boundary conditions are

$$u=0, v=V(x) \text{ no } y=0, \tag{1.1c}$$

$$u\rightarrow U_e(x) \text{ as } y\rightarrow \infty. \tag{1.1d}$$

These boundary conditions reflect the physical requirement that the fluid flow satisfies full viscous no-slip at the surface and mass transfer rate through the surface that may vary with stream-wise distance. We will take the normal flow through the surface to be constant, possibly zero. The stream-wise velocity within the boundary layer is required to match smoothly onto the free-stream at a large distance from the surface. The mass transfer taking place through the surface may be constant along the entire length being considered or it may vary with stream-wise location along the surface. Furthermore, this mass transfer may be either injection of fluid into, or suction of fluid from, the boundary layer.

By considering the behaviour of the *x*-momentum equation at a large distance from the surface, or alternatively by making use of Bernoulli's equation, it is found that $-\frac{dp}{dx}=U_e(x)\frac{dU_e}{dx}$, where $U_e(x)$ describes the external flow as a function of distance along the surface.

Prescribing a particular form for the external flow, $U_e(x)$, results in a specific type of self-similar solution for the boundary-layer flow. If the external flow is of the form $U_e(x)=Cx^m$ then the self-similar solutions are referred to as being of Falkner-Skan-type. Flows over a flat plate are included in the family of Falkner-Skan solutions and are recovered by letting m=0. For Newtonian fluids the constant C depends only on the parameter m, while for the class of non-Newtonian fluids being considered here C will also depend on the fluid index n.

We proceed to derive the differential equation governing the boundary-layer flow of a power-law fluid for which self-similar solutions are to be found by introducing a similarity variable defined by

$$s\,(x, y) = y\left\{\frac{(2n-1)m+1}{n+1}\frac{U_e^{2-n}}{x}\right\}^{\frac{1}{n+1}} \qquad (1.2)$$

We also make use of the stream

$$\psi(x, y) = f(s)\left\{\frac{n+1}{(2n-1)m+1}\,U_e^{2n-1}x\right\}^{\frac{1}{n+1}} \qquad (1.3)$$

Where *f(s)* is a dimensionless stream function. The stream function ψ identically satisfies the continuity equation (1.1a), while the x-momentum equation (1.1b) is transformed into the Falkner-Skan-like ordinary differential equation.

$$nf'''|f''|^{n-1}+f''f+\beta\,(1-(f')^2)=0, \qquad (1.4)$$

where primes denote differentiation with respect to the similarity variable *s*. The quantity β, referred to as the pressure gradient parameter, is given by

$$\beta = \frac{(n+1)m}{(2n-1)m+1}$$

and the velocity components *u* and *v* are given by

$$u = U_e\, f', \qquad (1.5a)$$

$$v = -\left\{\left(\frac{(2n-1)m+1}{n+1}\right)^n C^{2n-1}x^{(2n-1)m-n}\right\}^{\frac{1}{n+1}}\left[f-\frac{(n-2)m+1}{(2n-1)m+1}sf'\right] \qquad (1.5b)$$

We observe that setting $n = 1$ for the fluid index, which corresponds to a Newtonian fluid, results in the similarity variable *s*, the stream function ψ, as well as equation (1.4) being reduced to the well-known forms associated with Falkner-Skan flows; see Schlichting[5].

We also note that this form of equation (1.5) is equivalent to that given by Lee and Ames[3], the main difference being in the choice of coefficients of the corresponding terms in the Falkner-Skan-like differential equation. The boundary conditions (1.1c) and (1.1d) are transformed into

$$f(0) = -V(x)\left\{\left(\frac{(2n-1)m+1}{n+1}\right)^n C^{2n-1}x^{(2n-1)m-n}\right\}^{-\frac{1}{n+1}} \qquad (1.6a)$$

$$f'(0) = 0, \qquad (1.6b)$$

$$f'(s) \to 1 \text{ as } s \to \infty \qquad (1.6c)$$

To find self-similar solutions we require that equation (1.4) and the accompanying boundary conditions are independent of the original variables *x* and *y*. The boundary condition (1.1c) that permits mass transfer through the surface has become (1.6a) as a result of introducing the similarity variable *s*. The presence of *x* in this boundary condition means that a self-similar solution for equation (1.4) cannot be obtained. However, the form of the function *V(x)* describing the mass transfer through the surface may be chosen so to ensure the existence of a self-similar solution.

Letting $V(x) = V_0 x^{\frac{(2n-1)m-n}{n+1}}$ allows the boundary condition (1.6a) to be expressed as

$$f'(0) = -V_0\hat{C}\,(m, n), \qquad (1.7)$$

where both V_0 and $\hat{C}\,(m, n)$ are constants. This form for $V(x)$ is not unlike that used by Nachman and Taliaferro[4] in their discussion of the boundary-layer flow of a power-law fluid along a flat plate in the presence of similarity-preserving mass transfer. Hence, a self-similar solution can be sought for equation (1.4) subject to boundary conditions (1.6b), (1.6c) and (1.7)

The nature of the flow being considered determines the form that the boundary conditions that accompany equation (1.4) ultimately take. Even though the boundary-layer flow involved fluid injection at the surface, we will focus on flows with zero mass transfer through the surface as this case still provides useful insights into the structure of the boundary layer. Hence, self-similar solutions to equation (1.4) will be sought subject to the following boundary conditions.

$$f = f' = 0 \text{ on } s = 0, \qquad (1.8a)$$

$$f' \to 1 \text{ as } s \to \infty, \qquad (1.8b)$$

Equation (1.4) and the boundary conditions (1.8) constitute a third-order non-linear two-point boundary value problem that has no known analytic solutions (except in the degenerate case when n = 2) and needs to be solved by a numerical scheme. Techniques for obtaining solutions to such boundary value problems are often based on simple shooting, finite-differences or collocation. The method of simple shooting is used to obtain solutions to equation (1.4), these solutions are discussed in the following sections. We proceed by firstly finding solutions to equation (1.4) when $\beta = 0$, which describes the boundary-layer flow along a flat plate. These solutions will, hopefully, provide an intuitive understanding of the structure in the boundary-layer for this simple flow geometry.

Zero Pressure Gradient (β = 0): The choice of β = 0 has the geometric interpretation of corresponding to a potential flow over a flat plate, for which $u \to U_e$ (constant) as s $\to \infty$. Hence, without loss of generality, we set $U_e\,(x)$ = 1. For this choice of β, equation (1.4) simplifies to the following form

$$f''' + \frac{1}{n}f\,(f'')^{2-n} = 0, \qquad (1.9)$$

subject to the boundary conditions (1.8).

We note that, equation (1.9) is essentially of the same form as that given by Acrivos *et al.*[4], where a different coefficient appears due to a slight difference in the choice of the similarity variable. This equation is a third-order non-linear two-point boundary value problem that can only be solved by a suitable numerical method. A number of different numerical methods are available for finding the numerical solution of equation (1.9) and we next provide a description of one such method. After the numerical method has been used to find solutions of equation (1.9), we shall also discuss the asymptotic behaviour of these solutions in the far-field.

3. Numerical Solution

As indicated previously, the non-linear form of equation (1.9) means that a numerical technique needs to be employed to find a solution. Boundary value problems must be solved at all points in the solution domain simultaneously, often using methods based on finite - difference approximations. The presence of the $(f'')^{2-n}$ term in equation (1.9) would make a method based on finite-differences somewhat awkward, quite apart from the matter of the semi-infinite solution domain.

In contrast, initial value problems can be solved by a stepwise, or 'marching', procedure. In this sense, initial value problems are easier to solve. This marching method for solving initial value problems may be adapted and used to solve boundary value problems. The resulting technique is known as the 'shooting' method and we use this method to numerically solve equation (1.9).

The shooting method is based on the idea of converting a boundary value problem into an equivalent initial value problem and integrating by marching from the initial point to the terminal point. As part of the conversion step, it is necessary to specify extra initial conditions and iteratively adjust them until the required conditions at the terminal point are satisfied. To solve equation (1.9) with the shooting method, the asymptotic boundary condition is replaced by an initial condition for $f''(0)$ and integrated from $s = 0$ to a large value of s where $f'(\infty) = 1$ is deemed to be satisfied. The selection of the correct value for $f''(0)$ may be done with a trial-and-error approach or, as is more common, a Newton iteration scheme that converges to the correct value of $f''(0)$ is used. Acrivos *et al.*[2] indicate, but give no details, that a method originally devised by Töepfer for the Blasius equation that requires no guessing for $f''(0)$ can also be used for solving equation (1.9). Rosenhead[6] provides details of how this method is used to solve the Blasius equation. Here we give a brief description of the method and discuss its suitability for finding numerical solutions to equation (1.9).

It can be verified that equation (1.9) is scale-invariant to the transformation defined by

$$\bar{s} = a^{2-n} s$$
$$\bar{f}(\bar{s}) = a^{1-2n} f,$$

where a is an arbitrary 'constant of homology'. When this transformation is applied to equation (1.9) the following associated ordinary differential equation is obtained

$$\bar{f}''' + \frac{1}{n}\bar{f}\left(\bar{f}''\right) = 0, \qquad (1.10)$$

Where the prime indicates differentiation with respect to $\bar{s}$. The boundary conditions given at $s = 0$ become $\bar{f}(0) = \bar{f}'(0) = 0$, while the asymptotic boundary condition becomes

$$\lim_{s\to\infty} f'(s) = a^{n+1} \lim_{\bar{s}\to\infty} \bar{f}'(\bar{s})$$

Since $f'(\infty) = 1$, the far-field boundary condition for equation (1.10) takes the form

$$\lim_{\bar{s}\to\infty} \bar{f}'(\bar{s}) = \frac{1}{a^{n+1}}$$

Hence, the associated differential equation (1.10) the same initial conditions as equation (1.9), but the asymptotic boundary condition requires the solution to converge to a different and unknown value, viz. $\frac{1}{a^{n+1}}$

It would seem that little benefit has been gained from the use of this transformation, as it is still necessary to specify either $f''(0)$ or $\bar{f}''(0)$ to solve equation (1.9) or equation (1.10) respectively. However, we note that $f''(0) = a^3\bar{f}''(0)$, where with a specified $\bar{f}''(0)$ and a known value of 'a' we can calculate the required $\bar{f}''(0)$. Since $\bar{f}''(0)$ is completely arbitrary we set it equal to unity. The value of a is determined from the far-field solution of equation (1.10). Hence, the additional initial condition for $f''(0)$ needed to solve equation (1.9) is given by

$$f''(0) = \left\{\lim_{\bar{s}\to\infty} \bar{f}'(\bar{s})\right\}^{-\frac{3}{n+1}} \qquad (1.11)$$

The numerical method for finding the solution to equation (1.9) consists of two stages. First, solve the initial value problem posed by equation (1.10) for $\bar{f}(\bar{s})$ subject to the initial conditions $\bar{f}(0) = \bar{f}'(0) = 0$ and $\bar{f}''(0) = 1$. The integration is carried out to a suitably large $\bar{s}_\infty$ at which the value of $\bar{f}'$ is considered to have satisfied the asymptotic boundary condition. This is indicated by a plateau on the plot of $\bar{f}'$, as well by meeting an appropriate stopping condition. The value of $\bar{f}''(0)$ is then calculated with sufficient accuracy according to equation (1.11). Secondly, solve equation (1.9) for f(s) starting with initial values $f(0) = f'(0) = 0$, and the newly found value of $f''(0)$. This two stage numerical process eliminates the iterative search for $f''(0)$ that is common in standard shooting methods. It is also described as being quite stable and as having no significant build-up of error.

This numerical method was implemented as a Matlab script employing the Runge-Kutta solver ode45, essentially to rapidly prototype the method and check its effectiveness. It was also implemented as a Fortran program based on a standard fourth order Runge-Kutta single-step integrator. The method was tried for various values of n and far-field locations $\bar{s}_\infty$ to check the behaviour of the value of $\bar{f}'$ in the far-field. The values of $\bar{f}'(\bar{s}_\infty)$ obtained are shown in Table 1.1.

The results in Table 1.1 indicate that for values of the fluid index slightly less than unity the solution to equation (1.10) in the far-field, $\bar{f}'(\bar{s}_\infty)$, has converged to a constant value. For fluid index values below 0.7 there is still some variability in the values of $\bar{f}'(\bar{s}_\infty)$. However, as $\bar{s}_\infty$. takes progressively larger values it is seen that more digits in the value of $\bar{f}'(\bar{s}_\infty)$ remain unchanged. For example, for $n = 0.4$ we have $\bar{f}'(\bar{s}_\infty{=}50)$- $\bar{f}'(\bar{s}_\infty{=}25)$=0.000179, whereas $\bar{f}'(\bar{s}_\infty{=}150)$- $\bar{f}'(\bar{s}_\infty{=}125)$=0.0000017. Hence, for small values of n we can ensure that $\bar{f}'(\bar{s}_\infty)$ has converged to an appropriate accuracy by choosing a sufficiently large value of $\bar{s}_\infty$. By calculating $\bar{f}'(\bar{s}_\infty)$ to a high level of accuracy, we are then able to ensure that the initial condition $f''(0)$ for the solution of equation 1.9 is also known to a high accuracy.

Table 1.1: Far-field convergence values for solutions to equation (1.10) for a selection of fluid index values n and 'infinity' $\bar{s}_\infty$

n	$\bar{s}_\infty = 25$	$\bar{s}_\infty = 50$	$\bar{s}_\infty = 75$	$\bar{s}_\infty = 100$	$\bar{s}_\infty = 125$	$\bar{s}_\infty = 150$
0.1	1.0460786	1.0478182	1.0483143	1.0485413	1.0486691	1.0487502
0.2	1.2581674	1.2593406	1.2596233	1.2597402	1.2598014	1.2598381
0.3	1.3803195	1.3808658	1.3809717	1.3810102	1.3810286	1.3810388
0.4	1.4609016	1.4610806	1.4611066	1.4611145	1.4611179	1.4611196
0.5	1.5178824	1.5179191	1.5179227	1.5179236	1.5179240	1.5179242
0.6	1.5600880	1.5600917	1.5600919	1.5600920	1.5600920	1.5600920
0.7	1.5924359	1.5924360	1.5924360	1.5924360	1.5924360	1.5924360
0.8	1.6179111	1.6179111	1.6179111	1.6179111	1.6179111	1.6179111
0.9	1.6384072	1.6384072	1.6384072	1.6384072	1.6384072	1.6384072
1.0	1.6552030	1.6552030	1.6552030	1.6552030	1.6552030	1.6552030
1.1	1.6691869	1.6691869	1.6691869	1.6692706	1.6692722	1.6692737
1.2	1.6809933	1.6809935	1.6809936	1.6810992	1.6811097	1.6811202
1.3	1.6910838	1.6910839	1.6910840	1.6911653	1.6911753	1.6911852

Alternatively, the solution of equation (1.10) can be found at a moderate value of $\bar{s}_\infty$ and the initial condition $f''(0)$ used in the second stage of the solution procedure can be refined by a Newton-Raphson iteration (akin to the usual shooting method). The slow convergence of $\bar{f}'$ in the far-field for small values of 'n' is attributable to the asymptotic form of the solution in the far-field.

Shear-thinning Fluids

Power-law fluids having the fluid index in the range $0 < n < 1$ are often referred to as shear-thinning or pseudoplastic. The numerical method described above was used to find solutions to equation (1.9) for this class of fluids. The fluid index values considered was $n = 1.0, 0.8, \ldots, 0.2$ The results given in Table 1.1 were used to properly select values for $\bar{s}_\infty$. The value $n = 1$ corresponds to a Newtonian fluid for which the solution is obtained from the classical Blasius equation. This case served as a confidence check that the numerical technique being used was performing correctly.

The self-similar solutions to equation (1.9) for different values of n represent the stream-wise velocity in the boundary-layer flow of a shear-thinning fluid. Velocity profiles for the values of 'n' considered . When compared with the Blasius solution, we see that for values of 'n' down to approximately 0.6 the velocity profiles do not show much variability in appearance. In the next section we will show that the solution to equation (1.9) possesses algebraic decay in the far-field and this can be observed for the velocity profiles plotted. This effect becomes more noticeable for lower values of n where the velocity profiles exhibit a more gentle 'shoulder'. These velocity profiles confirm that matching of the boundary-layer velocity with the far-field uniform free-stream takes place at greater distances from the surface as the fluid index n decreases, corresponding to thickening of the boundary-layer.

The two-stage numerical method for solving the two-point boundary value problem (1.9) along with boundary conditions (1.8) was modified to use a Newton-Raphson iteration to refine the value of $f''(0)$**.** For shear-thinning fluids this numerical method was found to be generally quite robust and efficient. However, in using this numerical method some care must be taken to ensure that the solutions obtained exhibit the correct form of asymptotic decay in the far-field. Furthermore, by examining the nature of the asymptotic form of the numerical solutions in the far field we will gain a better understanding of the behaviour of the self-similar solutions of the boundary-layer flow of power-law fluids. We will next examine the asymptotic form of the velocity profile for shear-thinning fluids in the far-field.

Asymptotic form for Shear-thinning fluids

We proceed by noting that the asymptotic boundary condition (1.8b) allows us to write the solution $f(s)$ in the far-field as:

$$f(s) = s + a + \varphi(s) \quad \text{as } s \to \infty,$$

where a is a constant and $\varphi(s) \ll 1$. To determine the large s structure we define $\varsigma = s + a$ so that the form for $f(s)$ becomes:

$$f(\varsigma) = \varsigma + \varphi(\varsigma), \tag{1.12}$$

with $\varphi(\varsigma) \ll 1 \; as \; \varsigma \to \infty$. Substituting this expression into equation (1.9), and neglecting products of φ with its derivatives, yields, to leading order,

$$n\varphi''' + \varsigma(\varphi'')^{2-n} \sim 0,$$

where the primes denote differentiation with respect to ς. Integrating this equation gives the large ς behaviour of φ as:

$$\varphi(\varsigma) \sim \alpha_1 \varsigma^{\frac{2n}{n-1}},$$

where $\alpha_1 = \left[\frac{1-n}{2n}\right]^{\frac{n}{n-1}} \left(\frac{1-n}{n+1}\right)$. Substituting this expression into equation (1.12) and differentiating with respect to ς gives in the limit $\varsigma \to \infty$ (or equivalently $s \to \infty$)

$$f'(\varsigma) \sim 1 + \alpha_2 \varsigma^{\frac{n+1}{n-1}} + \cdots, \tag{1.13}$$

where the ellipsis denote lower-order terms and

$$\alpha_2 = -\left[\frac{1-n}{2n}\right]^{\frac{n}{n-1}} \left(\frac{2n}{n+1}\right)$$

Hence, equation (1.13) is a first-order approximation to the solution of equation (1.9) in the far-field and its form predicts that the solution in the far-field will display algebraic decay, provided $n < 1$. We note that for $n = 1$ the exponent in (1.13) possesses a singularity that indicates faster than algebraic decay in the far-field. However, the velocity in the far-field of the boundary layer of a Newtonian fluid is known to display exponential decay as it approaches the free-stream velocity; see Rosenhead [6] for details.

For shear-thinning fluids, with $0 < n < 1$, equation (1.13) predicts algebraic decay toward the free-stream velocity value from below. Letting $n = 1-\delta$, so that δ represents the degree of shear-thinning in the fluid with $0 < \delta < 1$, we can express the exponent in equation (1.13) as $1-\frac{2}{\delta}$. For slightly shear-thinning fluids with small values of δ, this exponent will be negative and of large magnitude.

The velocity profile for such fluids will have very rapid algebraic decay in the far-field. For values of δ closer to unity, corresponding to a more shear-thinning fluid, this exponent is smaller in magnitude and is negative. Hence the velocity profile of power-law fluids with a higher degree of shear-thinning will exhibit slower algebraic decay in the far-field.

The algebraic decay of the velocity field for shear-thinning fluids poses some concerns with regard to matching such a solution to an inviscid outer (potential) flow. It is implicitly assumed that such matching is possible when the large Reynolds number limit is applied to the Cauchy equations. However, there is a parallel for this algebraic decay that occurs for Newtonian fluids. General similarity solutions of the Falkner-Skan equation possessing algebraic decay are known to exist. Brown and Stewartson considered the conditions under which such solutions can match onto an outer potential flow and whether they are of relevance in the context of such solutions being an asymptotic description of a real boundary layer. They demonstrated that solutions showing algebraic decay are not appropriate if such solutions are to be matched onto an outer potential flow and, consequently, should be disregarded.

The boundary-layer flow being considered here is somewhat complicated by the additional non-linearity in the apparent viscosity. We will show that this term plays a crucial role in correctly describing the correct matching of the inner boundary layer with an outer potential flow. Now we turn to the matter of matching the boundary-layer solutions of equation 1.9 to an outer flow for fluid index values $0 < n < 1$.

The solutions presented above were derived from the boundary-layer equations on the assumption that they match smoothly onto an outer inviscid (i.e. potential) flow. To ensure that this is the case we first note that combining equations (1.5) and (1.13), when $m = 0$, gives

$$u = 1 + \hat{\alpha}_2 x^{-\frac{1}{n+1}} y^{\frac{n+1}{n-1}} + \cdots, \tag{1.14a}$$

$$v = \gamma x^{-\frac{n}{n+1}} + \alpha_2 \left(\frac{1}{n+1}\right)^{\frac{n}{n-1}} x^{-\frac{n}{n-1}} y^{\frac{2n}{n-1}} + \cdots, \text{ as } y \to \infty, \tag{1.14b}$$

where $\hat{\alpha}_2$ and γ are constants. Next, we recall that under the boundary-layer approximation the apparent viscosity μ_{app} can be expressed as

$$\mu_{app} = Re^{\frac{n-1}{n+1}} \tilde{\mu}$$

$$= Re^{\frac{n-1}{n+1}} \left[\left(\frac{\partial u}{\partial y}\right)^2 + 2\epsilon^2 \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right) + \epsilon^4 \left(\frac{\partial v}{\partial x}\right)^2\right]^{\frac{n-1}{2}} \tag{1.15}$$

where $\epsilon = Re^{-\frac{1}{n+1}}$ is the boundary-layer thickness. Thus, to leading order in powers of ϵ, the apparent viscosity depends solely upon the horizontal shear within the boundary-layer flow. Referring to equation (1.15) for the apparent viscosity, we find that for large y (i.e. in the outer region of the boundary layer) the terms that were previously excluded from our boundary layer analysis now become important. A simple dominant balance of the terms in equation (1.15) indicates that our boundary-layer expansion breaks down when $y = O\left(Re^{\frac{1-n}{n+1}}\right)$. We also note that, on the boundary-layer scale, the outer potential flow occurs when $y = O\left(Re^{\frac{1}{n+1}}\right) \gg O\left(Re^{\frac{1-n}{n+1}}\right)$ (since $0 < n < 1$). Therefore, we define a new stretched co-ordinate by

$$Y = Re^{\frac{n-1}{n+1}} y = Re^{\frac{n}{n+1}} \hat{y},$$

where $\hat{y}$ is the non-dimensional form of the vertical co-ordinate. The co-ordinate Y is large on the boundary-layer scale y but still small on the physical length-scale.

The asymptotic form for the velocity components given by equation (1.14) now suggests that in the new outer, or viscous, region we write

$$U = 1 + Re^{-1} U_1(x, Y) + \cdots,$$

$$V = Re^{-\frac{1}{n+1}} \left(V_0(x) + Re^{-\frac{2n}{n+1}} V_1(x, Y) + \ldots\right).$$

Substituting these expansions into the Cauchy equations gives, at leading order,

$$\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial Y} = 0, \tag{1.16a}$$

$$\frac{\partial U_1}{\partial x} = n \left|\frac{\partial U_1}{\partial Y} + \frac{\partial V_0}{\partial x}\right|^{n-1} \frac{\partial^2 U_1}{\partial Y^2}, \tag{1.16b}$$

where $V_0(x)$ is determined by matching with the inner boundary-layer solution. From equation (1.14) this gives

$$V_0(x) = \gamma x^{-\frac{n}{n+1}}.$$

Equation (1.16b) needs to be solved subject to the matching of U_1 with the inner boundary-layer solution, which from equation (1.14b) is

$$U_1(x, Y) = \hat{\alpha}_2 x^{-\frac{1}{n+1}} Y^{\frac{n+1}{n-1}} \text{ as } Y \to 0, \tag{1.17}$$

and exponential decay as $Y \to \infty$ which ensures a smooth transition between the new outer viscous layer and the uniform flow in the free stream. Having determined the form of U_1, equation (1.14a) can now be integrated to give V_1. However, the exact form of V_1 is not needed in the subsequent analysis and so is not pursued here. The boundary condition (1.17) on U_1 at $Y = 0$, together with the form for V_0, precludes any similarity type solutions of the non-linear diffusion equation (1.14b). Nonetheless, it can readily be shown that this equation admits solutions that satisfy the prescribed (matching) boundary conditions. We proceed by noting that the singular nature of equation (1.17) gives $U_{1Y} \gg V_{0x}$ in the limit $Y \to 0$. Hence, letting $U_1 = x^{-\frac{1}{n+1}} \tilde{U}(Y)$ and ignoring the V_{0x} term in equation (1.16

b) gives, to leading order,

$$-\frac{1}{n+1}\tilde{U} = n\left|\frac{d\tilde{U}}{dY}\right|^{n+1}\frac{d^2\tilde{U}}{dY^2}$$

The solution of this equations gives the correct asymptotic form $\tilde{U}\,(Y) \sim Y^{\frac{n+1}{n-1}}$ as $Y \to 0$.

Next we examine the form of U_1 in the limit $Y \to \infty$. If we assume that $U_1 \to 0$ then, upon retaining the dominant terms in equation (1.16 b), we find that U_1, which by assumption is much smaller than unity, is governed by

$$\frac{\partial U_1}{\partial x} = n\left|\frac{dV_0}{dx}\right|^{n-1}\frac{\partial^2 U_1}{\partial Y^2}$$

Making the substitution $\xi = \frac{Y}{\sigma}$, where

$$\sigma^2 = 2n\int\left|\frac{dV_0}{dx}\right|^{n-1}dx$$

gives

$$\frac{d^2U_1}{d\xi^2} = -\xi\frac{dU_1}{d\xi}.$$

The solution of this equation has the asymptotic form

$$U_1 \sim \frac{1}{\xi}(1+\cdots)\,e^{-\frac{\xi^2}{2}},$$

as. $\xi \to \infty$. Hence, we see that equation (1.16b) has solutions that provide a smooth match between the outer viscous layer and the free-stream potential flow.

The analysis performed above demonstrates that equation (1.16b) possesses a solution that satisfies the appropriate matching conditions at $Y = 0$ and as $Y \to \infty$. Though we have not provided numerical solutions of equation (1.16b), we note that since it is a parabolic partial differential equation in x it should, in principle, be possible to develop a numerical scheme to march an initial velocity profile forward in x. However, to provide an appropriate initial velocity profile for such a procedure, we note that $V_0(x)$ becomes unbounded as $x \to 0$ and this would require us to perform a small- x asymptotic analysis of the full Cauchy equations. As $x = 0$ corresponds to the leading edge of the flat plate, the small- x analysis would need to take account of leading-edge effects. Such a study is outside the scope of the current work and will not be pursued further.

Shear-thickening Fluids

Power-law fluids having the fluid index in the range $1 < n < 2$ are referred to as shear-thickening or dilatant. The set of fluid index values n considered for dilatant fluids was $n = 1.0, 1.1, ..., 1.4$. The numerical method described above is not readily applicable to this class of fluids. However, the results from Table 1.1 were used to determine an initial guess for $f''(0)$ and then a standard shooting method was used to find solutions to equation (1.9). Velocity profiles for the values of n considered. The features displayed by these velocity profiles are in agreement with profiles reported by Acrivos et al. and Lee and Ames. From the velocity profiles, it can be seen that as the fluid index n increases, the velocity profile matches onto the free-stream velocity at progressively smaller values of s. Such thinning of the boundary-layer is common among shear-thickening fluids.

During the numerical solution of equation (1.9) for values of $n > 1$ it was found that manual intervention was often required to monitor the convergence criteria, whereas for shear-thinning fluids the numerical scheme converged onto the solution quite readily.

Asymptotic form for Shear-thickening fluids

Using a numerical scheme based on the standard shooting method coupled with Newton-Raphson iteration to find the solution of equation (1.9) for shear-thickening fluids provides results of limited usefulness. However, a more promising approach that provides a better understanding of the nature of the boundary layer of a shear-thickening fluid is to regard the original problem as a free-boundary problem, where the outer limit of the 'boundary layer' now becomes an unknown of the system. Thus we pose the problem for non-zero 'm' as

$$nf''' = \left[\frac{m(n+1)}{m(2n-1)+1}\right]\left[(f')^2 - 1\right]|f''|^{1-n} - f|f''|^{2-n}, \quad (1.18a)$$

to be solved subject to the usual no-slip conditions

$$f = f' = 0 \quad \text{on} \quad s = 0, \qquad (1.18b)$$

and the new boundary conditions

$$f' = 1,\ f'' = 0 \quad \text{on} \quad s = s_c \quad (1.18c)$$

We observe that equation (1.18a) is essentially identical to equation (1.4) along with the appropriate replacement for the pressure gradient parameter. Also the absolute value of f has been introduced in the terms involving exponents which take negative values when $n > 1$.

The boundary condition (1.18c) ensures that the stream wise velocity does not overshoot its far-field value of unity at the 'outer' edge of the boundary layer. Though this system appears to be over-specified, it can be seen that when s_c is treated as an unknown we then obtain an eigen value problem for s_c in the form of a two-point boundary-value problem which can be solved using standard methods.

To make the computation of solutions more convenient, it is useful to take advantage of the autonomous nature of the system (1.18) and to make a shift of coordinates so as to define the origin to be at the critical point $z = s_c - s$. Applying this shift of coordinates gives the transformed equation

$$n\frac{d^3f}{dz^3} = \left[\frac{m(n+1)}{m(2n-1)+1}\right]\left[1 - \left(\frac{df}{dz}\right)^2\right]\left|\frac{d^2f}{dz^2}\right|^{1-n} + \left|\frac{d^2f}{dz^2}\right|^{2-n} f,$$

along with the transformed boundary conditions

$$\frac{df}{dz} = -1, \quad \frac{d^2f}{dz^2} = 0 \quad \text{on} \quad z = 0,$$

$$f = \frac{df}{dz} = 0 \quad \text{on} \quad z = s_c.$$

In order to numerically integrate the transformed equation we employ the small- z asymptotic form for f to start the calculation at some suitably chosen $\Delta z \ll 1$. This is given by

$$f = \tilde{\beta} - z + \tilde{\gamma} z^{\alpha} + \cdots \qquad (1.19)$$

where $\tilde{\beta}$ is an unknown that is to be determined, with α and $\tilde{\gamma}$ being 'constants' that are dependent on the fluid and flow parameters. Making the appropriate substitutions into the transformed equation and simplifying results in

$$n\alpha(\alpha - 1)(\alpha - 2)\tilde{\gamma} z^{\alpha-3} = \tilde{\beta}\big(\alpha(\alpha - 1)\big)^{2-n}|\tilde{\gamma}|^{2-n} z^{n(2-\alpha)+2\alpha-4}$$

$$+\left\{2\left[\frac{m(n+1)}{m(2n-1)+1}\right]\alpha\tilde{\gamma} - \alpha(\alpha - 1)|\tilde{\gamma}|\right\}(\alpha(\alpha - 1))^{1-n}|\tilde{\gamma}|^{1-n} z^{n(2-\alpha)+2\alpha-3}$$

$$-\left\{\left[\frac{m(n+1)}{m(2n-1)+1}\right]\alpha^2\tilde{\gamma} - \alpha(\alpha - 1)|\tilde{\gamma}|\right\}(\alpha(\alpha - 1))^{1-n}|\tilde{\gamma}|^{1-n}\tilde{\gamma} z^{n(2-\alpha)+3\alpha-4}$$

Performing a balance of the leading terms, we find that $\alpha = \frac{2n-1}{n-1}$, while $\tilde{\gamma}$ is given by the solution of

$$\frac{|\tilde{\gamma}|^{2-n}}{\tilde{\gamma}} = \frac{n}{\tilde{\beta}}(\alpha(\alpha-1))^{n-1}(\alpha-2).$$

This asymptotic form allows us to apply boundary conditions at some suitably small Δz and then integrate out to the location at which the boundary condition $f = f_z = 0$ is satisfied. This was accomplished using a fourth-order Runge-Kutta quadrature routine coupled with Newton-Raphson iteration on the unknowns $\tilde{\beta}$ (or equivalently $\tilde{\gamma}$) and s_c.

It is worth noting at this point that equation (1.9) can be solved analytically when $n = 2$, with the pressure gradient parameter $\beta = 0$, for then the non-linear ordinary differential equation reduces to the linear third-order ordinary differential equation

$$f''' + \frac{1}{n} f = 0. \qquad (1.20)$$

Equation (1.20) has the general solution

$$f(s) = C_1 e^{-2as} + C_2 e^{as}\sin(a\sqrt{3}s) + C_3 e^{as}\cos(a\sqrt{3}s)$$

where $a = (1/2)^{4/3}$ and C_i $(i = 1,2,3)$ are constants of integration. Imposing the boundary condition $f(0) = 0$ gives $C_3 = -C_1$, while $f'(0) = 0$ gives $C_2 = \sqrt{3}C_1$. Hence, the general solution simplifies to the following form

$$f(s) = C_1 e^{-2as} + C_1 e^{as}\left[\sqrt{3}\sin(a\sqrt{3}\,s) - \cos(a\sqrt{3}\,s)\right] \quad (1.21)$$

The constant of integration C_1 and the critical position s_c are then determined from the 'far-field' boundary conditions, namely that $f' = 1$ and $f'' = 0$ on $s = s_c$. It is easiest to apply the second of these to first determine the position s_c, which is readily shown to satisfy the transcendental equation

$$\cos(a\sqrt{3}s_c) = -\frac{1}{2}\exp(-3asc) \qquad (1.22)$$

With s_c determined the value for the constant of integration C_1 appearing in equation (1.21) can finally be determined from the remaining boundary condition, namely $f' = 1$ on $s = s_c$. It is worth noting that the presence of the exponential terms in the general solution (1.21) indicates that it is not possible to satisfy the usual asymptotic boundary condition $f'(\infty) \to 1$. Hence, we conclude from this that a similarity-type solution with the required asymptotic behaviour in the far-field does not exist for $n = 2$.

We note that the form of the exponential term appearing on the right hand side of equation (1.22) allows us to obtain estimates for s_c by writing (1.22), to a first approximation, as

$$\cos(a\sqrt{3}\,s_c) \approx 0$$

This gives $s_c \approx (1+2k)\frac{\pi}{2\sqrt{3}a}$ $(k = 0,1,2,\dots)$. These approximate values were used as starting values for the numerical solution of the full system (1.20). The equations were solved using both a forward and backward shooting method. Typically, when the forward-shooting method failed to converge to a solution, the backward-shooting method was found to be successful. Only the first three modes are presented; however, the results for $n = 2$ confirm that there is an infinite number of modal solutions of the system (1.20). This is supported by the observation that the transcendental equation (1.22) possesses an infinite number of solutions for s_c. We note that for the first modal solution it appears that s_c is finite for $n = 1$. This is simply an artefact of our numerical scheme, which iterates on the values $f'(s_c) - 1$ and $f''(s_c)$ until these quantities are less than a predefined tolerance, which in obtaining these results was set to 10^{-12}. At this tolerance level the numerical scheme cannot distinguish between a converged solution and the true solution, which for $n = 1$ is known to have exponential decay to the free-stream value of unity.

Mode 1 appears to represent a 'boundary layer' with forward flow throughout the flow domain. However, this solution is non-physical as it lacks the asymptotic behaviour that is characteristic of boundary-layer flows. The higher modes exhibit regions with negative velocity, where $f' < 0$ for some range of s. The solutions for higher mode numbers become increasingly oscillatory with alternating regions of positive and negative velocity. However, there is no physical mechanism whereby the laminar flow over a flat plate can have a region with negative velocity. Consequently, those eigenfunctions which exhibit regions in which $f' < 0$ are not physically realisable and can be ignored.

We now turn our attention to the question of matching the inner solution described above with the outer flow solution.

The phenomenon of a finite-thickness boundary layer is also encountered in hypersonic boundary layers by Bush; Lee and Cheng[7]; Mikhailov *et al*[8]. In such flows the abrupt termination of the boundary layer arises due to the vanishing nature of the temperature, and consequently the fluid viscosity (which is a function of temperature), in the outer regions of a hypersonic boundary layer. For the case of a fluid whose viscosity-temperature relation is described by Sutherland's law (with a non-linear dependence of viscosity upon temperature), Bush demonstrated that this singularity is smoothed out in a thin viscous transition layer which allows uniform matching with an outer inviscid shock layer. Lee and Cheng[7] extended this analysis to the case where the viscosity-temperature relation is given by Chapman's law (with a linear dependence of viscosity upon temperature). Although there are some subtle differences between the two cases, both result in the need for a viscous transition layer at the outer extent of the finite-width boundary layer. The parallels between the structure of the hypersonic boundary-layer and that of the shear-thickening boundary layer are obvious. In the latter case the underlying cause of the existence of the finite-width boundary layer is the vanishing of the leading-order viscosity as $s \to \infty$. The regularisation of the resulting singularity is accomplished through the re-introduction of lower-order terms in the viscosity function. A similar adjustment layer was also observed by Denier and Hewitt[9] in their study of the flow of a power-law fluid above a rotating disk.

In order to determine the structure within this viscous adjustment layer, we first note that, as mentioned above, the underlying cause of the finite-width of the boundary layer is due to the vanishing of the leading-order viscosity as $s \to s_c$. From our original scalings, the terms that were ignored in our leading-order approximation for $\hat{\mu}$ in the boundary layer are of the form

$$Re^{-\frac{2}{n+1}}\left(\frac{\partial v}{\partial y}\right)^2 \quad \text{and} \quad Re^{-\frac{2}{n+1}}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

With the expansion given above for u as $s \to s_c$ we obtain

$$\left(\frac{\partial v}{\partial y}\right)^2 \sim x^{-\frac{2}{n+1}}\,(s - s_c)^{\frac{2}{n-1}} + \cdots,$$

$$Re^{-\frac{2}{n+1}}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \sim x^{-2}\,(s - s_c)^{\frac{1}{n-1}} + \cdots$$

Thus our somewhat naive truncation of the viscosity function breaks down when

$$s - s_c = O\left(Re^{-\frac{2(n-1)}{n+1}}\right)$$

As a result of this observation, we define

$$y = y_c(x) - Re^{-\frac{2(n-1)}{n+1}}\xi,$$

and write

$$u = 1 + Re^{-\frac{2n}{n+1}}\hat{U}(x,\xi) + \cdots,\ v = \hat{V}(x) + \cdots,$$

where the ellipsis denote lower-order terms that do not enter into the subsequent analysis. The leading-order term for v is determined through a trivial match with the 'inner' solution. This gives

$$\hat{V}(x) = A_1 x^{A_2+m+1}\hat{\beta}\,[s_c - (m + A_2)],$$

where the constants $A_1 = \left(\frac{1}{n+1}\right)^{\frac{1}{n+1}}$, $A_1 = \frac{1-m(2-n)}{n+1}$ and $\hat{\beta}$ was defined in previously. From the stream wise momentum equation we obtain, after some simplifications,

$$-\hat{V}(x)\frac{\partial \hat{U}}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\hat{\mu}\frac{\partial \hat{U}}{\partial \xi}\right) \tag{1.23}$$

where the viscosity function $\hat{\mu}$, is given by

$$\hat{\mu} = \left[\left(\frac{\partial \hat{U}}{\partial \xi}\right)^2 - 2\frac{\partial \hat{U}}{\partial \xi}\frac{d\hat{V}}{dx} + \left(\frac{d\hat{V}}{dx}\right)^2\right]^{\frac{n-1}{2}}$$

The boundary conditions appropriate to equation (1.23) are

$$\hat{U} \to 0 \text{ as } \xi \to -\infty,\ \hat{U} \sim A_2^{\frac{n}{n-1}} \text{ as } \xi \to -\infty.$$

These ensure correct asymptotic decay of the streamwise velocity in the far-field ($\xi \to -\infty$) and matching to the algebraic terms in the 'inner' region ($\xi \to \infty$). It proves useful to rescale equation (1.23) by writing $\hat{U} = \alpha_0 F$, $\xi = \alpha_1 \varsigma$, where $\alpha_0 = \frac{(-\hat{V}_x)^{n-1}}{\hat{V}}$ and $\alpha_1 = \frac{(-\hat{V}_x)^n}{\hat{V}}$.

The equation for F (ς) is then

$$\frac{\partial}{\partial \varsigma}\left[|F' - 1|^{n-1}F'\right] = -F'$$

The solutions which satisfy the asymptotic matching conditions are found in the lower-left quadrant, for which F'' is strictly negative.

4. Conclusion

In this paper, we have derived a similarity-type transformation that converts the partial differential equations governing the boundary-layer flow of a power-law fluid into an equivalent ordinary differential equation. The solution of the two-point boundary value problem was obtained by solving an equivalent initial value problem using a numerical scheme consisting of a standard shooting method and coupled with Newton iteration to find the unknown $f''(0)$. This numerical scheme was found to be satisfactory for shear-thinning fluids; however, for shear-thickening fluids the numerical scheme was less effective and the solutions needed to be interpreted with some care.

An asymptotic analysis of the behaviour of the solution in the far-field was also performed. It was shown that, under the original boundary-layer scaling, the solution for shear-thinning fluids exhibited algebraic decay. However, for shear-thickening fluids it was found that the derived asymptotic form did not predict decay in the far-field, hence, suggesting that the shear-thickening boundary layer is of 'finite-width'. It was demonstrated for shear-thinning fluids that by introducing a transition layer between the boundary-layer and the free-stream, via an appropriate rescaling, that a composite solution which matches with the boundary-layer solution as well as exhibiting exponential decay as $s \to \infty$ exists. For shear-thickening fluids we identified an infinite family of solutions. As was the case for shear-thinning flows, shear- thickening fluids require the presence of a viscous transition layer in which the singularity that arises as a result of the abrupt termination of the boundary layer is smoothed out, hence allowing matching with the outer potential flow.

We also considered the relationship between the wall shear $f''(0)$ with the parameter m that influences the pressure gradient in the free-stream flow. The numerical results indicate that for positive values of m both shear-thinning and shear-thickening fluids possess a unique solution. When m takes negative values then both classes of fluid possess non-unique (or multiple) solutions. It was argued that-the non-unique solutions are members of a single family of solutions for shear-thickening flows.

The similarity-type solutions described in this paper cannot be directly compared with the numerical results due to differences in the underlying methods and flow scenarios as captured by the relevant boundary conditions. In particular, we focussed here on flows with zero mass transfer through the surface. Nonetheless, some qualitative agreement between the results obtained using similarity techniques with numerical results can be observed.

The results presented here in this paper clearly demonstrate the significant issues that arise when a simple constitutive relation based on the power-law rheology is used tp model the boundary-layer flow of either shear-thinning or shear-thickening fluids. That the problem for shear-thinning fluids can be made mathematically consistent is perhaps gratifying, however, this does not hide the fact that the underlying model is fundamentally flawed. Interestingly, the mathematical 'fix' described in this paper has now been used in a wide variety of modeling problems.

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