New Results for Impulsive Nonlocal Fractional Integro-Differential Equations

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Abstract

In this paper, we study the uniqueness of solutions for impulsive fractional integro differential equations with nonlocal conditions. The main results are discussed through the theory of fixed point.

Keywords: Fractional differential equations, Fractional Derivative and integral, fixed point theorem, uniqueness.

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1 Introduction

In this paper, we study the nonlocal impulsive fractional integro differential equations of the form

$$^{c}D^{\alpha}_{0^+}y(t) = a(t)y(t) + f(t, y(t)) + \int_0^t K(t, s, y(s))ds, \quad t \in J = [0, b], \tag{1}$$

$$x(t_k^+) = x(t_k^-) + y_k, \quad k = 1, 2, ...m, \quad y_k \in \mathbb{R} \tag{2}$$

$$y(0) = y_0 - \eta(t), \tag{3}$$

where $^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α, $0 < \alpha \leq 1$, $a \in C([0, b], \mathbb{R})$, $f \in C([0, b] \times \mathbb{R}, \mathbb{R})$, $K \in C([0, b] \times [0, b] \times \mathbb{R}, \mathbb{R})$. The theory of fractional differential equations is a new branch of mathematics by valuable tools in the modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [14, 15, 18, 19]) and reference therein [1, 3, 5, 20].

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Impulsive differential equations have become an important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments and the references therein [7–13]. In [2, 4, 6] M. S. Abdo et. al., studied the fractional integro-differential equation with Caputo fractional derivative and Ψ-Hilfer fractional derivative, continuous dependence for fractional neutral functional differential equations.

S. Suresh and G. Thamizhendhi contributed to studied the existence and uniqueness of solution for non-local impulsive fractional integro-differential equations. The arguments are based upon contraction mapping principle and Krasnoselskii's fixed point theorem (see [16, 17] and the references therein).

In this paper is Organized as follows. In section 2, definitions and elementary results of the fractional calculus are given. In section 3, the existence and uniqueness results for impulsive fractional integro differential equations involving nonlocal conditions are proved.

2 Preliminaries

Let J=[0,b] and $C(J,\mathbb{R})$ be the Banach space endowed with the infinity norm $||g||_\infty = \sup\{|g(t)| : t \in J\}$, for any $g \in C(J,\mathbb{R})$, we also $C^n(J,\mathbb{R}^+)$ be space of all real valued continuous function which are continuously differentiable on J.

Definition 1. *The fractional derivative of order $\vartheta > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by*

$$D^{\vartheta}_{0+}f(t) = \frac{1}{\Gamma(n-\vartheta)}(\frac{d}{dt})^n \int_0^t \frac{f(s)}{(t-s)^{\vartheta-n+1}}ds,$$

where $n = [\vartheta] + 1$, provided the right side is pointwise defined on $(0,\infty)$.

Definition 2. *The fractional integral of order $\vartheta > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by*

$$I^{\vartheta}_{0+}f(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} f(s)ds,$$

provided the right side is pointwise defined on $(0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 3. *The Riemann-Liouville fractional integral of order $\vartheta > 0$ of the function $f \in C(J,\mathbb{R})$ is given by*

$$I^{\vartheta}_{0+}f(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} f(s)ds, \quad t \in J,$$

where Gamma denotes the Gamma function.

Definition 4. *Let $n-1 < \vartheta < n$. The Caputo's fractional operator of the funtion $f \in C^n(J,\mathbb{R})$ defined as*

$${}^cD^{\vartheta}_{0+}f(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{n-\vartheta-1} \frac{d^n}{dt^n} f(s)ds = I^{n-\vartheta-1}_{0+} \frac{d^n}{dt^n} f(t), \quad t \in J.$$

In particular, in $0<\vartheta<1$, *then* ${}^cD^{\vartheta}_{0^+}g(t) = I^{1-\vartheta}_{0^+}\frac{d}{dt}f(t), \quad t\in J.$

Lemma 1. *If* $y\in C([0,b],\mathbb{R})$, *then* y *satisfies the problem (1)-(2) if and only if* y *satisfies the integral equation:*

$$y(t)=\begin{cases} & y_0-\eta(t)+\frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}a(s)y(s)ds+\frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}f(s,y(s))ds\\ + & \frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}\int\limits_0^t K(\tau,s,y(s))d\tau ds, t\in[0,t_1)\\ & y_0-\eta(t)+y_1+\frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}a(s)y(s)ds+\frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}f(s,y(s))ds\\ + & \frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}\int\limits_0^t K(\tau,s,y(s))d\tau ds, t\in[t_1,t_2)\\ & y_0-\eta(t)+y_1+y_2+\frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}a(s)y(s)ds+\frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}f(s,y(s))ds\\ + & \frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}\int\limits_0^t K(\tau,s,y(s))d\tau ds, t\in[t_2,t_3)\\ \vdots & \\ & y_0-\eta(t)+\sum\limits_{i=0}^{m}y_k+\frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}a(s)y(s)ds+\frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}f(s,y(s))ds\\ + & \frac{1}{\Gamma(\vartheta)}\int\limits_0^t(t-s)^{\vartheta-1}\int\limits_0^t K(\tau,s,y(s))d\tau ds, t\in(t_m,b] \end{cases} \tag{4}$$

3 Main results

To prove the existence and uniqueness results we need the following assumptions :

- (A_1) $a: J\to\mathbb{R}$ is continuous function.
- (A_2) $f: J\times\mathbb{R}\to\mathbb{R}$ is continuous such that

  $$|f(t,u)-f(t,v)|\le\psi(|u-v|),\quad t\in J,\quad u,v\in\mathbb{R}.$$

  for $t\in J, u,v\in\mathbb{R}$, where $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing continuous function with $\psi(0)=0$.
- (A_3) $K: J\times J\times\mathbb{R}\to\mathbb{R}$ is continuous on D such that

  $$\int\limits_s^t|K(\tau,s,u(s))-K_1(\tau,s,v(s))|d\tau\le M\psi(|u-v|),$$

  where $\int\limits_0^{\mathbb{R}}\frac{ds}{\psi(0)}=+\infty,\quad 0<x<\mathbb{R},\quad D=\{(t,s):0\le s\le t\le b\}$and M is positive constant.

- (A_4) $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $L > 0$ such that

$$|f(t,u) - f(t,v)| \leq L||u - v||, \quad t \in J, u, v \in \mathbb{R}.$$

- (A_5) $K : J \times J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $L^* > 0$ such that

$$|K(\tau, s, u(s))| - K_1(\tau, s, v(s))| \leq L^*||u - v||, \quad (t,s) \in D, u, v \in \mathbb{R},$$

Theorem 1. *Assume that the hypotheses $(A_1), (A_2)$ and (A_3) hold. If $0 \leq \frac{||a||_\infty b^\vartheta}{\Gamma(\vartheta+1)} < 1$, then the fractional integro-differential equation (1)-(3) has a unique solution in $C(J, \mathbb{R})$.*

Proof:

By Lemma 1, we know that the function y is a solution to (1)-(3) iff y satisfies

$$y(t) = y_0 - \eta(t) + \sum_{i=0}^{m} y_k + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} a(s) y(s) ds + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} f(s, y(s)) ds$$
$$+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \int_0^t K(\tau, s, y(s)) d\tau ds, \quad t \in J.$$

Let $y_1, y_2 \in C(J, \mathbb{R})$ and for any $t \in J$ such that

$$y_1(t) = y_0 - \eta(t) + \sum_{i=0}^{m} y_k + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} a(s) y_1(s) ds + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} f(s, y_1(s)) ds$$
$$+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \int_0^t K(\tau, s, y_1(s)) d\tau ds, \quad t \in J.$$

and

$$y_2(t) = y_0 - \eta(t) + \sum_{i=0}^{m} y_k + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} a(s) y_2(s) ds + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} f(s, y_2(s)) ds$$
$$+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \int_0^t K(\tau, s, y_2(s)) d\tau ds, \quad t \in J.$$

Consequently, by (A1),(A2) and (A3), then for $t \in J$, we have,

$$\begin{aligned}
|y_1(t) - y_2(t)| \leq & \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} |a(s)||y_1(s) - y_2(s)| ds \\
& + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} |f(s, y_1(s)) - f(s, y_2(s))| ds \\
& + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \int_s^t |K(\tau, s, y_2(s))||K(\tau, s, y_2(s))| d\tau ds \\
\leq & \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \sup_{s\in J} |a(s)||y_1(s) - y_2(s)| ds \\
& + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \psi(|y_1(s) - y_2(s)| ds \\
& + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} M\psi(|y_1(s) - y_2(s)|) ds \\
\leq & \frac{||a||_\infty b^\vartheta}{\Gamma(\vartheta+1)} |y_1(t) - y_2(t)| + \frac{1+M}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \psi(|y_1(s) - y_2(s)|) ds \\
\leq & \frac{1+M}{1 - \frac{||a||_\infty b^\vartheta}{\Gamma(\vartheta+1)}} \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \psi(|y_1(s) - y_2(s)| ds \\
< & \epsilon + \frac{M}{1 - \frac{||a||_\infty b^\vartheta}{\Gamma(\vartheta+1)}} \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \psi(|y_1(s) - y_2(s)| ds
\end{aligned}$$

where $\epsilon = \frac{1+M}{1 - \frac{||a||_\infty b^\vartheta}{\Gamma(\vartheta+1)}} > 0$.

$$\begin{aligned}
|y_1(t) - y_2(t)| \leq & \psi^{-1} \left[\psi(\epsilon) + \frac{M}{1 - \frac{||a||_\infty b^\vartheta}{\Gamma(\vartheta+1)}} \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} ds \right] \\
\leq & \psi^{-1} \left[\psi(\epsilon) + \frac{M}{1 - \frac{||a||_\infty b^\vartheta}{\Gamma(\vartheta+1)}} \frac{b^a}{\Gamma(\vartheta+1)} \right],
\end{aligned}$$

where $\psi(.)$ is a primitive of $\frac{1}{\psi(.)}$ and ψ^{-1} is called the inverse of $\psi(.)$

It follows that $\psi^{-1}\left[\psi(\epsilon)+\frac{M}{1-\frac{||a||_\infty b^\vartheta}{\Gamma(\vartheta+1)}}\frac{b^a}{\Gamma(\vartheta+1)}\right]\to 0.$

Consequently, $y_1(t)=y_2(t)$, for $t\in[0,b]$. So, $y(t)\in C(J,\mathbb{R})$ is unique solution to fractional integro differential equation (1)-(3).

Theorem 2. *Assume that the hypotheses (A_1), $(A_4),(A_5)$ are satisfied. Let λ and η be two positive real numbers such that $0<\lambda<1$ and*

$$\left(\frac{||a||_\infty+L}{\Gamma(\vartheta+1)}+\frac{L^*b}{(\vartheta+1)\Gamma(\vartheta)}\right)b^\vartheta=\lambda,$$

$$|y_0|+\left(\frac{p}{\Gamma(\vartheta+1)}+\frac{p^*b}{(\vartheta+1)\Gamma(\vartheta)}\right)b^\vartheta=(1-\lambda)N.$$

Then the fractional integro-differential equation (1)-(3) has a unique solution continuous on [0,b], where $p=\{\max|f(t,0)|:t\in J\}$ and $p^*=\{\max|K(\tau,s,0)|(\tau,s)\in D\}$

Proof:

Let the operator $\mathcal{M}:C(J,\mathbb{R})\to C(J,\mathbb{R})$ be defined by

$$(\mathcal{M}y)(t)=y_0-\eta(t)+\sum_{i=0}^m y_k+\frac{1}{\Gamma(\vartheta)}\int_0^t(t-s)^{\vartheta-1}a(s)y(s)ds+\frac{1}{\Gamma(\vartheta)}\int_0^t(t-s)^{\vartheta-1}f(s,y(s))ds$$
$$+\frac{1}{\Gamma(\vartheta)}\int_0^t(t-s)^{\vartheta-1}\int_0^t K(\tau,s,y(s))d\tau ds.$$

and defined $B_N=\{y\in C(J,\mathbb{R}):||y||_\infty\le N\}$ for some $N>0$. Now, we need to prove that the operator $\mathcal{M}$ has a fixed point on $B_N\subset C(J,\mathbb{R})$. This fixed point is the unique solution of (1.1)-(1.2). In order that, the proof in two steps:

Step 1: We show that $\mathcal{M}B_N\subset B_N$.

By the hypotheses, then for any $y\in B$ and for $t\in J$, we have

$$|(\mathcal{M}y)(t)|\le|y_0|-|\eta(t)|+\sum_{i=0}^m|y_k|+\frac{1}{\Gamma(\vartheta)}\int_0^t(t-s)^{\vartheta-1}|a(s)||y(s)|ds$$
$$+\frac{1}{\Gamma(\vartheta)}\int_0^t(t-s)^{\vartheta-1}|f(s,y(s))|ds$$
$$+\frac{1}{\Gamma(\vartheta)}\int_0^t(t-s)^{\vartheta-1}\int_0^t|K(\tau,s,y(s))|d\tau ds$$

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$$\leq |y_0| - |\eta(t)| + \sum_{i=0}^{m} |y_k| + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} ||a||_\infty ||y||_\infty ds$$
$$+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} (|f(s,y(s)) - f(s,0)| + |f(s,0)|) ds$$
$$+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \int_0^t (|K(\tau,s,y(s)) - K(\tau,s,0)| + |K(\tau,s,0)| d\tau ds$$
$$\leq |y_0| - |\eta(t)| + \sum_{i=0}^{m} |y_k| + \frac{b^\vartheta ||a||_\infty N}{\Gamma(\vartheta+1)} + \frac{b^\vartheta}{\Gamma(\vartheta+1)}(LN+p) + \frac{b^{\vartheta+1}}{(\vartheta+1)\Gamma(\vartheta)}(L^*N+p^*)$$
$$= |y_0| - |\eta(t)| + \sum_{i=0}^{m} |y_k| + \left(\frac{p}{\Gamma(\vartheta+1)} + \frac{p^*b}{(\vartheta+1)\Gamma(\vartheta)}\right) b^\vartheta + \left(\frac{||a||_\infty + L}{\Gamma(\vartheta+1)} \frac{L^*b}{(\vartheta+1)\Gamma(\vartheta)}\right) b^\vartheta N$$
$$= (1-\lambda)N + \lambda N = N.$$

It follows that $||\mathcal{M}y||_\infty \leq N$,this implies that $\mathcal{M}y \in B_N$ which leads to $\mathcal{M}B_N \subset B_N$.
Step 2: we shall show that $\mathcal{M} : B_N \to B_N$ is a contractions, then for any $y, y^* \in B_N$ and for $t \in J$, we can write

$$|(\mathcal{M}y)(t) - (\mathcal{M}y^*)(t)| \leq \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} |a(s)||y(s) - y^*(s)| ds$$
$$+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} |f(s,y(s)) - f(s,y^*(s))| ds$$
$$+ \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} \int_0^t |K(\tau,s,y(s)) - K(\tau,s,y^*(s))| d\tau ds$$
$$\leq \frac{b^\vartheta ||a||_\infty}{\Gamma(\vartheta+1)} ||y-y^*|| + \frac{b^\vartheta L}{\Gamma(\vartheta+1)} ||y-y^*|| + \frac{b^{\vartheta+1} L^*}{(\vartheta+1)\Gamma(\vartheta)} ||y-y^*||$$
$$\leq \left(\frac{||a||_\infty + L}{\Gamma(\vartheta+1)} \frac{L^*b}{(\vartheta+1)\Gamma(\vartheta)}\right) b^\vartheta ||y-y^*||$$
$$= \lambda ||y-y^*||$$

Since $\lambda < 1$, we get

$$||\mathcal{M}y - \mathcal{M}y^*||_\infty \leq ||y-y^*||.$$

This implies that $\mathcal{M}$ is contraction mapping. As consequence of Banach contraction principle, there exists a fixed point $y \in C(J, \mathbb{R})$ and $\mathcal{M}y = y$ which is the unique solution of (1)-(3) on J.

4 Conclusion

We study the existence and uniqueness of solutions of the nonlocal impulsive fractional integro differential equations. The main results are proved by using the fixed point theorems. Further, the problem (1)-(3) to study the existence of solutions for Caputo-Hadamard fractional integro differential equations involving fractional impulsive conditions.

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