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Numerical Forms of Fractional Differential Equations

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Abstract: ***In this paper we discuss fractional Laplace transform method, whose formulas and properties applying on different type of example of fractional order. And try to solve it with this method.***

Keywords: Fractional Laplace transform, fractional Laplace formulas and properties

1. Introduction

In past our published paper we represent the fractional Laplace transform, we derived with the help of fractional complex transform and also with the help of this transformation we derived so many properties and formulas for fractional Laplace transform and inverse fractional Laplace transform. The fractional Laplace transform of a function $f(t)$ for $t \geq 0$ is denoted by the symbol $L_\alpha\{f(t)\}$ or $F(s^\alpha)$, and is defined by the integral

$$L_\alpha\{f(t)\} = F(s^\alpha) = \int_0^\infty e^{-s^\alpha T^\alpha} f(t)\,(dT)^\alpha$$

The integral exists, where s is a parameter real or complex. If $f(t)$ has a fractional Laplace transform then it is said to be transformable.

$\boldsymbol{f(t)}$	$\boldsymbol{L_\alpha\{f(t)\} = F(s)}$
1	$\dfrac{1}{s^\alpha}$
t^n	$\dfrac{\lceil n+1}{s^{(n+1)\alpha}}$
e^{at}	$\dfrac{1}{s^\alpha - a^\alpha}\quad a > 0$
$sinat$	$\dfrac{a^\alpha}{(s^{2\alpha} + a^{2\alpha})}$
$cosat$	$\dfrac{s^\alpha}{(s^{2\alpha} + a^{2\alpha})}$
$sinhat$	$\dfrac{a^\alpha}{s^{2\alpha} - a^{2\alpha}}$
$coshat$	$\dfrac{s^\alpha}{s^{2\alpha} - a^{2\alpha}}$

Properties	$L_\alpha\{f(t)\} = F(s^\alpha) \Rightarrow f(t) = L_\alpha^{-1}\{F_\alpha(s^\alpha)\}$
1st Derivative	$L_\alpha\{f'(t)\} = s^\alpha L_\alpha\{f(t)\} - f(0)$
nth order derivative	$L_\alpha\{f^n(t)\} = s^{n\alpha} L_\alpha\{f(t)\} - \sum_{k=1}^{n} s^{(k-1)\alpha}\, s^\alpha f^{n-k}(0);\ n > 0$
First Shifting Theorem	$L_\alpha\{e^{at} f(t)\} = L_\alpha\{f(t)\}(s^\alpha - a^\alpha) = F(s^\alpha - a^\alpha)$
Second Shifting Theorem	If $L_\alpha\{f(t)\} = F(s^\alpha)$ with $G(t) =$ $f(t-a); t > a$ $0 \quad ;t < a$ Then $L_\alpha\{G(t)\} = e^{-a^\alpha s^\alpha} F(s)$
Convolution Theorem	$L_\alpha\{h(t) = (g * f)(t)\} = \int_0^\infty \left[\int_T^\infty f(T^\alpha - \tau^\alpha) e^{-s^\alpha T^\alpha} g(\tau^\alpha)(dT)^\alpha\right](d\tau)^\alpha$

$f(t) \;==L_\alpha^{-1}\{F(s)\}$	$F(s) \;=\; L_\alpha\{f(t)\}$
$L_\alpha^{-1}\left\{\frac{1}{s^\alpha}\right\}$	1
$L_\alpha^{-1}\left\{\frac{1}{s^{(n+1)\alpha}}\right\}$	$\frac{t^n}{\lceil n+1}$
$L_\alpha^{-1}\left\{\frac{1}{s^\alpha - a^\alpha}\right\};\quad a > 0$	e^{at}
$L_\alpha^{-1}\left\{\frac{a^\alpha}{(s^{2\alpha}+a^{2\alpha})}\right\}$	$sinat$
$L_\alpha^{-1}\left\{\frac{s^\alpha}{(s^{2\alpha}+a^{2\alpha})}\right\}$	$cosat$
$L_\alpha^{-1}\left\{\frac{a^\alpha}{s^{2\alpha}-a^{2\alpha}}\right\}$	$sinhat$

Also we apply it so may application like on lie group, on the oscillator in the presence of an external forces, on Tautochrone curve etc. and successfully solved with the help fractional Laplace transform.

Many problems of physical interest are described by ordinary or partial differential equations with appropriate initial or boundary conditions. These problems are usually formulated as initial-boundary value problems that seem to be mathematically more rigorous and physically realistic in applied and engineering sciences. The fractional Laplace transform method is particularly useful for finding solutions of these problems. This method is very effective for the solution of the response of a linear system governed by an ordinary differential equation to the initial date and/or to an external input function.

This paper is deal with the solutions of ordinary and partial differential equations that arise in mathematical, physical, and engineering sciences. The examples given in this paper are only representative of a variety of problems which can be solved by the use of the fractional Laplace transform.

We consider first order partial differential equation

$$f(u, X, Y, Z, \ldots, u_X, u_Y, u_Z, \ldots.) = 0$$

$$\frac{du}{dx} = \frac{du}{dX}\frac{dX}{dx} = \frac{du}{dX}\frac{d}{dx}\left(\frac{x^\alpha}{\Gamma\alpha+1}\right) = \frac{d^\alpha u}{dX^\alpha}$$

So from this we take first order linear differential equation,

$$\frac{d^\alpha x}{dt^\alpha} + px = f(t); \qquad t > 0$$

With initial condition x=a and t=0.

Here p and a are constant, f (t) is external input function so we can apply fractional Laplace transform.

$$L_\alpha\left\{\frac{d^\alpha x}{dt^\alpha}\right\} + pL_\alpha\{x\} = L_\alpha\{f(t)\}$$

$$s^\alpha \bar{x}(s) - x(0) + p\bar{x}(s) = \bar{f}(s)$$

$$s^\alpha \bar{x}(s) - a + p\bar{x}(s) = \bar{f}(s)$$

$$(s^{\alpha}+p)\bar{x}(s)=\bar{f}(s)+a$$

$$\bar{x}(s)=\frac{a}{s^{\alpha}+p}+\frac{\bar{f}(s)}{s^{\alpha}+p}$$

With the help of inverse fractional Laplace transform

$$x(t)=ae^{-pt}+\int_0^t f(t-\tau)(d\tau)^{\alpha}$$

Therefore we can say that the solution splits into two terms. The first term corresponds to the response of the initial condition and the second term is entirely due to the external input function f (t).

Example

The second order linear ordinary differential equation has the general form,

$$\frac{dx}{dt}+2p\frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}}+qx=f(t), t>0$$

With initial conditions are,

$x(t)=a$ And $\frac{dx}{dt}=b$ at t=0

Apply fractional Laplace transform we obtain,

$$L_{\alpha}\left\{\frac{dx}{dt}\right\}+2pL_{\alpha}\left\{\frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}}\right\}+qL_{\alpha}\{x\}=L_{\alpha}\{f(t)\}$$

$$s\bar{x}(s)-s^{\frac{1}{2}}x(0)-x'(0)+2p\left\{s^{\frac{1}{2}}x(0)-x(0)\right\}+q\bar{x}(s)=\bar{f}(s)$$

$$\bar{x}(s)=\frac{\left(s^{\frac{1}{2}}+ap\right)+(b+ap)+\bar{f}(s)}{s\bar{x}(s)+2ps^{\frac{1}{2}}a+q}$$

Example: 2

Now we consider higher fractional order ordinary differential equation of order $n\alpha$ with constant coefficients as,

$f(D^{\alpha})\{x(t)\}=D^{n\alpha}x+a_1D^{n-1}x+a_1D^{n-1}x+\cdots+a_nx=\emptyset(t)\ :t>0\ldots(1)$

With the initial conditions

$x(t)=x_{0,}\ D^{\alpha}x(t)=x_1,\ldots,D^{(n-1)\alpha}x(t)=x_{n-1}\ \ ;at\ t=0\ldots(2)$

Here $D^{\alpha}=\frac{d^{\alpha}x}{dt^{\alpha}}\quad ;0<\alpha<1.$

When we apply fractional Laplace transform on the equation (1), we obtain,

$$\begin{aligned}(s^{n\alpha}\bar{x}+s^{(n-1)\alpha}x_0+\cdots+s^{\alpha}x_{n-2}-x_{n-1})&\\+a_1(s^{(n-1)\alpha}\bar{x}-s^{(n-2)\alpha}x_0-s^{(n-3)\alpha}x_1\ldots&\\-x_{n-2})&\\+a_2(s^{(n-2)\alpha}\bar{x}-s^{(n-3)\alpha}x_0-s^{(n-4)\alpha}x_1\ldots&\\-x_{n-3})+\cdots+a_{n-1}(s^{\alpha}\bar{x}-x_0)+a_n\bar{x}&\\=\bar{\emptyset}(s)&\end{aligned}$$

$$\begin{aligned}(s^{n\alpha}+s^{(n-1)\alpha}+\cdots+a_n)\bar{x}(s)&\\=\bar{\emptyset}(s)&\\+(s^{(n-1)\alpha}+a_1s^{(n-2)\alpha}\ldots+a_{n-1})x_0+\cdots&\\+(s+a_1)x_{n-2}+x_{n-1}&\\=\bar{\emptyset}(s)+\bar{\psi}(s).&\end{aligned}$$

Here $\bar{\psi}(s)$ is made up of all terms on the right hand side of equation (1) except $\bar{\emptyset}(s)$, and is a polynomial in a degree $(n-1)$.

$$\bar{f}(s)\bar{x}(s)=\bar{\phi}(s)+\bar{\psi}(s)$$

Where

$$\bar{f}(s)=s^{n\alpha}+a_1s^{(n-1)\alpha}+\cdots+a_n$$

Thus, the fractional Laplace transform solution $\bar{x}(s)$ is

$$\bar{x}(s)=\frac{\bar{\phi}(s)+\bar{\psi}(s)}{\bar{f}(s)}$$

After use inverse fractional Laplace transform

$$x(t)=L_{\alpha}^{-1}\left\{\frac{\bar{\phi}(s)}{\bar{f}(s)}\right\}+L_{\alpha}^{-1}\left\{\frac{\bar{\psi}(s)}{\bar{f}(s)}\right\}.$$

Example:

$$\frac{d^{\frac{3}{2}}x}{dt^{\frac{3}{2}}}+\frac{dx}{dt}-6\frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}}=0$$

$$s^{\frac{3}{2}}\bar{f}(s)-s-5+s\bar{f}(s)-s^{\frac{1}{2}}-6\left(s^{\frac{1}{2}}\bar{f}(s)\right)$$

$$\left(s^{\frac{3}{2}}-6s^{\frac{1}{2}}+s\right)\bar{f}(s)-s^{\frac{1}{2}}-s+1=0$$

$$\bar{f}(s)=\frac{s^{\frac{1}{2}}+s-1}{s^{\frac{3}{2}}-6s^{\frac{1}{2}}+s}$$

$$=\frac{s^{\frac{1}{2}}+s-1}{s^{\frac{1}{2}}\left(s+s^{\frac{1}{2}}-6\right)}$$

$$\bar{f}(s)=\frac{s^{\frac{1}{2}}+s-1}{s^{\frac{1}{2}}\left[\left(s^{\frac{1}{2}}+3\right)\left(s^{\frac{1}{2}}-2\right)\right]}$$

$$\bar{f}(s)=\frac{s^{\frac{1}{2}}+s-1}{s^{\frac{1}{2}}\left[\left(s^{\frac{1}{2}}+9^{\frac{1}{2}}\right)\left(s^{\frac{1}{2}}-4^{\frac{1}{2}}\right)\right]}$$

$$\bar{f}(s)=-4\frac{1}{s^{\frac{1}{2}}}+\frac{13}{2!}\frac{1}{s^{\frac{1}{2}}+9}+\frac{5}{52}\frac{1}{s^{\frac{1}{2}}-4}$$

$$f(x)=-4+\frac{13}{2!}e^{-9t}+\frac{5}{52}e^{4t}$$

2. Conclusion

Using Fractional Laplace transform method we can derive numerical form of fractional order differential equation. Also it is successfully derived the higher order system of fractional differential equation. This technique is also useful for derived applications of Fractional differential equation.

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