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Stability of Stochastic Delay Logistic Model Incorporating Ornstein-Uhlenbeck Process

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Abstract: ***This paper is concerned about the stability of stochastic delay logistic model with Ornsttein-Uhlenbeck process. First, current paper introduces a general theory to determine the stability of SDDEs for zero solution in term of equation (1) via Lyapunov function. Aftermath, this result verified by stochastic logistic model with delayed feedback and several equations. Furthermore, this research uses the 4-stage stochastic Runge-Kutta (SRK4) method to evaluate the numerical solution and reflects the reality of the results.***

Keywords: Stability, Ornsttein-Uhlenbeck Process, Logistic model, Stochastic Runge-Kutta method

1. Introduction

Stochastic logistic models are the most useful mathematical models in dynamics processes [3], [7]. Most SDDEs do not have analytical solution. Nevertheless, the SDDEs can be solved numerically. There are some previews researches on numerical methods of SDDEs, [4], [5], [15], [16], [19], [31]. Though, three numerical methods are available to approximate the solution of the SDDEs namely, Euler-Maruyama (EM), Milstein scheme and SRK4, with 0.5, 1.0 and 1.5 order of convergence respectively. Baker in [4] proposed the numerical method of EM, Küchler et al., in [19] introduced Milstein scheme and final numerical method of SDDEs is SRK and it was proposed by Norhayati [31]. This method is the strongest and more accurate numerical method among of these numerical methods. We used SRK4 method to simulate the results. SRK method has 1.5 order of convergence which is still the highest convergence and SRK is based on the increments of Brownian motion.

Brownian motion was introduced by Robert Brown in (1827). He described this motion based on random movements of pollen grains in liquid or gas. However, Norbert Wiener (1923) explained full mathematical theory of Brownian motion [29]. It is a simple continuous stochastic process which extensively used to model phenomena in different arenas such as industry, dynamic process and fermentation process [3], [26]. Brownian motion is the of cause instability (turbulence) in every dynamic process. Due to, the Ornsttein-Uhlenbeck Process is a better model of Brownian motion [9, P86]. However, no specific research on stability stochastic delay logistic model with Ornsttein-Uhlenbeck process. Nevertheless, there are some pervious researches work on stability of stochastic logistic models with white noise, [12], [21], [22], [23], [33]. Therefore, this research investigates the stability of stochastic logistic delay model incorporating Ornsttein-Uhlenbeck Process via Lyaponuve function. Owing to, the Ornsttein-Uhlenbeck process has batter performance than Brownian motion [9, pp86].

Aleksandr Mikhailovich Lyapunov in (1892), proposed the sense of stability for nonlinear dynamic system. He introduced an approach to determine the stability of the system without solving the system. The stability theory for stochastic differential equations (SDEs) was introduced by Khasminskii [17] and Mao in [24], [25] explained some basic principles for different types of SDEs. There is no research on stability stochastic delay logistic model by incorporating Ornsttein-Uhlenbeck process via Lyapunov function. Therefore, current research does. In addition, this paper proposed a sufficient condition for general system of SDDEs in zero solution by using Lyapunove function and verified via stochastic delay logistic model and several examples considers in Appendix A.

This paper is organized in five main sections; section (1) is the introduction which provides some essential information and previews research works, section (2) corresponding to the preliminaries and models description that illustrates some fundamental concepts, theories and models, section (3) indicates the main results, where some new theories are proved, section four presents the Numerical Simulation to reflect the reality of our research and section (5) section is conclusion which shows the extract of current research.

2. Preliminaries and Models Description

Throughout this paper; the notations $\lambda_{\max} x(t)$ and $\lambda_{\min} x(t)$ are maximum and minimum eigenvalues of $x(t)$ respectively, $x(t)^T$ represents transpose of $x(t)$, $LV(x(t))$ denotes the differential operator, M indicates the symmetric matrix, $E\left[V(x(t))\right]$ is the expectation of $V(x(t))$, T shows the terminal time, t is time, t_0 illustrates the initial time, r indicates time delay or lag time, $x(t)$ corresponds to the highest growth rate in dynamic process, x_0 is initial data, $\mu_{\max}$ denotes the maximum specific growth rate, $x_{\max}$ illustrates a carrying capacity, $W(t)$ is white noise, $b>0$ shows the coefficient friction, σ indicates the random fluctuation.

Definition 2.1 [4]: Let (Ω, F, P) be a complete probability space with a filtration $F_t,\ t \ge t_0$ satisfy the usual conditions $t \ge t_0$ which is right continuous and each $F_t,\ t \ge t_0$ involve all P-null sets in F. For time delay, there is Banach space $C([-r,\ t])$ of all continuous path from $[-r,\ t_0] \to R$ equipped. Let $F(t)$ be an F_0-measurable $C([-r,\ t])$-valued random variable such that $E\|\varphi\|^2 < \infty$. Thus, based on definition (2.1) we write the general from of SDDE in bellow

$$\begin{aligned} dx(t) &= f(t,\ x(t),\ x(t-r))dt \\ &\quad + g(t,\ x(t),\ x(t-r))dW(t) t \in [t_0, T] \qquad (1) \\ x(t) &= \varphi(t),\ t \in [-r,\ t_0] \end{aligned}$$

where r is lag time or fixed delay, $\varphi(t)$ illustrates the initial function defined on $t \in [-r,\ t_0]$, which is independent of $W(t)$, $W(t)$ is d-dimensional Weiner process or Brownian motion. $f: R^m \times R^m \times R^+ \to R^m$ and $g: R^m \times R^m \times R^+ \to R^{m\times n}$ are assumed to be continuous.

Equation (1) has additive noise when $g(t,\ x(t),\ x(t-r))$ depends on $x(t)$. Otherwise, it has multiplicative noises. In equation (1), $f(t,\ x(t),\ x(t-r))$ and $g(t,\ x(t),\ x(t-r))$ admit the Lipschitz condition. Therefore, equation (1) has a unique global solution such us $x(t) = \varphi(t)$, for any initial function $\varphi(t) \in C([-r,\ t_0],\ R^m)$ and $r \ge t_0$. For all $t \ge 0$, equation (1) admits the trivial solution $x(t,0,0)$ if, $f(t,0,0) = 0$ and $g(t,0,0) = 0$ see [6, pp404].

Definition 2.2: The trivial solution of equation (1) is said to be sable in probability;

1. if for all $(\varepsilon,\ r)$, there is a $\delta(\varepsilon,\ r)$, where, $\varepsilon \in (0\ ,1)$ and $r \ge 0$ such that $P\left[\left|x(t;\varphi(t)) - \varphi(t) < r,\ \forall t \ge 0\right|\right] \ge 1-\varepsilon$, otherwise, it is unstable.
2. The trivial solution of equation (1) is said to be asymptotically stochastically stable in large if it is stable in probability. In addition, for all initial function $\varphi(t) \in \mathrm{C}([-r, t_0],\ R)$ and $r \ge t_0 \ge 0$, satisfies the
$$P\left[\lim_{t\to\infty} x(t;\varphi(t)) = \varphi(t)\right] = 1$$

This definition is defined based on stochastic stability [17, pp145].

2.1 Logistic Models

Logistic models are the most useful mathematical models in dynamic processes [2], [3], [7], [13], [18], [27], [31]. The simple logistic model is:

$$dx(t) = \mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt,\ t \in [0, T] \qquad (2)$$

Murray in [30] and May in [27] were proved that equation (2) is stable. It was proposed by Verhulst in (1838). This model is inadequate to describe the dynamic process because this process retains random fluctuation. Hence, Bahar and Mao in [3] proposed the

$$dx(t) = \mu_{\max}\left(1 - \frac{x(t)}{x_{\max}}\right)x(t)dt + \sigma x^2(t)dW(t), \quad t \in [0, T] \qquad (2)$$

stochastic logistic model. Liu and Wang in [21] and Golec et al., in [12] under different conditions were showed that equation (3) is stable. This stochastic logistic model extremely is used in [7]. One of the most affected factor on dynamic process is time delay due to equation (3) is not sufficient equation to compute the time delay. The time delay can be modelled via bellow equation

$$\begin{aligned} dx(t) &= \mu_{\max}\left(1 - \frac{x(t-r)}{x_{\max}}\right)x(t)dt, \quad t \in [t_0, T] \\ x(t) &= \varphi(t), \quad t \in [-r, t_0] \end{aligned} \qquad (3)$$

Where r is time delay, $\varphi(t),\ t \in [-r,\ t_0]$ shows initial function. There is an extensive literature concerned with the stability and dynamic of equation (4) [10], [13], [18], [20], [32], [34]. Dynamic processes are controlled by noise and delayed feedback. Therefore, model (3) and (4) are not adequate to model time delay and randomness jointly. Hence, Ayoubi in [3] introduced the

$$\begin{aligned} dx(t) &= \mu_{\max}\left(1 - \frac{x(t-r)}{x_{\max}}\right)x(t)dt + \sigma x(t)dW(t),\ t \in [t_0, T] \\ x(t) &= \varphi(t), \quad t \in [-r, t_0] \end{aligned} \qquad (4)$$

to model time delay and random fluctuations jointly in fermentation process.

Remarks 2.1: Liu and Wang in [21] Liu et al., [22] were proved that the stochastic logistic with distributed delay and stochastic logistic with infinite delay are stable. Both [21], [22] are consider the white noise. Nevertheless, this research investigates the stability of stochastic delay logistic model with Ornsttein-Uhlenbeck process via Lyapunove function. The Ornstein-Uhlenbeck process is:

$$\begin{aligned} y''(t) &= -by'(t) + dW(t) \\ y(0) &= y_0,\ y'(0) = y_1 \end{aligned} \qquad (6)$$

where $y(t)$ indicates the Brownian motion at time t, $W(t)$ is white noise, $b > 0$ shows the coefficient friction and σ illustrates the diffusion coefficient.

By substituting

$$y'(t) = W(t) \qquad (7)$$

into the equation (6) hence, the Ornstein-Uhlenbeck process becomes:

$$dW(t) = -bW(t)dt + \sigma dW(t) \qquad (8)$$

The expectation and variance of Ornstein-Uhlenbeck process is:

$$E\left(y(t)\right)=E\left(y_0\right)+\left(\frac{1-e^{-bt}}{b}\right)E\left(y_1\right)$$

$$V\left(y(t)\right)=V\left(y_0\right)\frac{\sigma^2}{b^2}t+\frac{\sigma^2}{b^3}\left(-3+4e^{-bt}-e^{-2bt}\right)$$

and the normal distribution follows $N\left(0,\ \frac{\sigma^2}{b^2}t+\frac{\sigma^2}{b^3}\left(-3+4e^{-bt}-e^{-2bt}\right)\right)$

Whereas, the normal distribution of white noise is a normal Gaussian. Furthermore, the new model is equation (5) with Ornstein-Uhlenbeck process.

by substituting the equation (8) into equation (5) yields:

$$dx(t)=\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)x(t)dt-\sigma x(t)bW(t)dt+\sigma^2x(t)dW(t),$$

$$t\in\left[0,T\right],\quad x(t)=\varphi(t),\quad t\in[-r,t_0] \qquad (9)$$

Equation (9) is stochastic delay logistic model with Ornstein-Uhlenbeck process.

3. Main Results

The general theory is obtained based on Lyapunov theory to determine the stability of differential equations in term of equation (1). Suppose the Lyapunov quadratic function is given

$$V\left(x(t)\right)=x\left(t\right)^T Mx\left(t\right) \qquad (10)$$

$$M=\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{m1} & \dots & a_{m\times n}\end{pmatrix}_{n=m}$$

is symmetric positive definite matrix and $V\left(x(t)\right)$ is:

where

$$\lambda_{\min}M\left|x^2(t)\right|\le\left|x(t)^T Mx(t)\right|\le\lambda_{\max}M\left|x^2(t)\right|$$

Theorem 3.1: The differential operator $dV\left(x(t)\right)$, associated with equation (1) is:

$$\begin{aligned}dV\left(x(t)\right)&=x(t)^T Mf\left(t,x(t),x(t-r)\right)\\&+f\left(t,x(t),x(t-r)\right)^T Mx(t)\\&+Mg\left(t,x(t),x(t-r)\right)^T\times\\&g\left(t,x(t),x(t-r)\right)=LV\left(x(t)\right)dt\end{aligned} \qquad (11)$$

Based $V\left(x(t)\right)$ we sense that the equation (1) is stochastically asymptotically stable in large, stochastically asymptotically stable or unstable in trivial solution (equilibrium point).

Proof: We use the basic concept of derivative with Lyapunov quadratic function hence;

$$dV\left(x(t)\right)=V\left(x(t)+dx(t)-V\left(x(t)\right)\right)$$

By using equations (1) and (6) in (12), we have

$$\begin{aligned}dV\left(x(t)\right)&=\left(x(t)+\left(dx(t)\right)^T\right)M\left(x(t)dx(t)\right)-x(t)^T Mx(t)\\&=\underbrace{\left(x(t)^T+\left(f\left(t,x(t),x(t-r)\right)\right)^T\right)}_{a}\\&+\underbrace{dt\left(g\left(t,x(t),x(t-r)\right)\right)^T dW(t)}_{a}\times Mx(t)\\&+\underbrace{f\left(t,x(t),x(t-r)\right)dt}_{b}\\&+\underbrace{g\left(t,x(t),x(t-r)\right)dW(t)}_{b}-Mx(t)^T x(t)\end{aligned}$$

To multiply the brackets a and b yields

$$\begin{aligned}dV&=Mx(t)^T x(t)+Mx(t)^T f\left(t,x(t),x(t-r)\right)dt\\&+Mx(t)^T g\left(t,x(t),x(t-r)\right)dW(t)\\&-Mx(t)^T x(t)+Mx(t)f\left(t,x(t),x(t-r)\right)^T dt\\&+Mf\left(t,x(t),x(t-r)\right)^T dtf\left(t,x(t),x(t-r)\right)dt\\&+Mf\left(t,x(t),x(t-r)\right)^T dtg\left(t,x(t),x(t-r)\right)dW(t)\\&+Mx(t)g\left(t,x(t),x(t-r)\right)^T dW(t)\\&+Mg\left(t,x(t),x(t-r)\right)^T dW(t)f\left(t,x(t),x(t-r)\right)dt\\&+Mg\left(t,x(t),x(t-r)\right)^T dW(t)f\left(t,x(t),x(t-r)\right)dW(t)\end{aligned}$$

Based on these facts $dt\times dt=0,\ dt\times dW(t)=0$ and $dW(t)\times dW(t)=dt$, [11, pp87] and somewhat lengthy calculation we get,

$$\begin{aligned}dV&=Mx(t)^T f\left(t,x(t),x(t-r)\right)dt\\&+Mx(t)^T g\left(t,x(t),x(t-r)\right)dW(t)\\&+Mx(t)f\left(t,x(t),x(t-r)\right)^T dt\\&+Mx(t)g\left(t,x(t),x(t-r)\right)^T dW(t)\\&+Mg\left(t,x(t),x(t-r)\right)^T g\left(t,x(t),x(t-r)\right)dt\end{aligned} \qquad (13)$$

By applying the expectation to the system (13), we have

$$\begin{aligned}dE\left[V\left(x(t)\right)\right]&=E\Big[Mx(t)^T f\left(t,x(t),x(t-r)\right)dt\\&+Mx(t)^T g\left(t,x(t),x(t-r)\right)dW(t)\\&+Mx(t)f\left(t,x(t),x(t-r)\right)^T dt\\&+Mx(t)g\left(t,x(t),x(t-r)\right)^T dW(t)\\&+Mg\left(t,x(t),x(t-r)\right)^T g\left(t,x(t),x(t-r)\right)dt\Big].\end{aligned}$$

The expectation of Brownian motion is zero [9, pp78] and assumed M=1. Hence, we obtain the equation (11)

$$\begin{aligned}dE\left[V\left(x(t)\right)\right]&=E\Big[Mx(t)^T f\left(t,x(t),x(t-r)\right)dt\\&+Mx(t)f\left(t,x(t),x(t-r)\right)^T dt+Mg\left(t,x(t),x(t-r)\right)^T\\&\times g\left(t,x(t),x(t-r)\right)dt\Big]=LV\left(x(t)\right)\end{aligned}$$

By taking into account Mao [26, pp 108], let $-V\left(x(t)\right)$ to be positive then $V\left(x(t)\right)$ is negative-definite and non-negative continuous function, $V\left(x(t)\right)$ is said to be decrescent. Thus,

$$-LV\left(x(t)\right)\ge KV\left(x(t)\right) \qquad (13)$$

$$\frac{\partial}{\partial t}\left[V(x(t))\right]\leq -KE\left[V(x(t))\right]\Rightarrow E\left[V(x(t))\right]\leq E\left[\exp(-Kt)\right]$$

$$\lim_{t\to\infty}E\left[V(x(t))\right]=\lim_{t\to\infty}E\left[x(t)x(t)^T\right]=\Xi$$

As a result, based on Ξ we say that the equation (1) is stable, asymptotically stable or asymptotically stable in large and prove is complete.

Remark 3.1: This theory is general theory to manifest the stability of all SDDEs in term of equation (1). This result verified via equation (9) and considers several examples in Appendix A.

Before we go through on stability of equation (9). It is really needed to prove that equation (9) has a positive global solution.

Lemma 3.1: For any initial function $x(t)=\varphi(t),\ t\in[-r,t_0]$ and $\varphi(t)\in C([-r,t_0],R^+)$, and $t\geq -r$, equation (9) has a global positive solution $x(t)$ on $t\in[-r,t_0]$.

Proof: Here we do not need to show the explosion time $\tau_e\to\infty$, owing to it has explicit solution. The coefficient functions of equation (9) admits the locally Lipschitz continuous. By using Itô formula [1, pp95], [25] [26, pp31] we have:

$$d(\ln|x(t)|)=\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)dt-\sigma bW(t)dt+\sigma^2dW(t)$$
$$+\underbrace{\frac{-1}{2x(t)^2}\left[\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)x(t)dt-\sigma bx(t)W(t)dt\right.}_{b}$$
$$\underbrace{\left.+\sigma^2x(t)dW(t)\right]^2}_{b}$$

$$d(\ln|x(t)|)=\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)dt-\sigma bW(t)dt+\sigma^2dW(t)$$
$$+\frac{-1}{2x(t)^2}\left[\mu^2_{\max}x^2(t)(dt)^2-2\frac{\mu_{\max}}{x_{\max}}x(t)x(t-r)(dt)^2\right.$$
$$+\left(\frac{\mu_{\max}}{x_{\max}}x(t)x(t-r)dt\right)^2-2\left(\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)x(t)dt\right)$$
$$\times(\sigma bW(t)dt)+(\sigma bW(t)dt)^2+2\left(\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)dt\right.$$
$$\left.-\sigma bW(t)dt\right)\left(\sigma^2x(t)dW(t)\right)+\left.\left(\sigma^2x(t)dW(t)\right)^2\right]$$

By using these facts Based on these facts $dt\times dt=0,\ dt\times dW(t)=0$ and $dW(t)\times dW(t)=dt$, [11, pp87] and somewhat lengthy calculation we have:

$$d(\ln|x(t)|)=\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)dt-\sigma bW(t)dt-\frac{\sigma^4}{2}dt+\sigma^2dW(t) \quad (14)$$

Equation (14) has two parts delay and stochastic. For delay part, we use step method [13, pp5], [14], [32, pp17], [34, pp46]. Therefore,

$$\ln|x(t)|-\ln|x_0|=\mu_{\max}t-\frac{\mu_{\max}}{x_{\max}}(1+\sum_{k=1}^{n}(-1)^k\frac{[t-(k-1)r]^k}{k!}$$
$$(n-1)r\leq t<nr)-\sigma bW(t)t-\frac{\sigma^4t}{2}+\sigma^2W(t)$$
$$\ln|x(t)|=x_0\exp\left(\mu_{\max}t-\frac{\mu_{\max}}{x_{\max}}(1+\sum_{k=1}^{n}(-1)^k\frac{[t-(k-1)r]^k}{k!},\right.$$
$$\left.(n-1)r\leq t<nr)-\sigma bW(t)t-\frac{\sigma^4t}{2}+\sigma^2W(t)\right) \quad (16)$$

Equation (16) is the global solution of equation (9) and ultimately, prove is completed.

It is time to prove equation (9) is stochastically stable.

Theorem 3.2: In Lemma 3.1 we proved equation (9) has a global positive solution for any initial function.

Now, it is the time to investigate the stability of equation (9) in trivial solution.

H1: For $x_{\max}>\sigma>\mu_{\max}>x_0\neq 0$ and $t\geq 0,\ b=1,\ r=1$ equation (9) is stochastically stable then $\lim_{t\to\infty}V(x(t))\leq 0$

Proof: By using equation (9) into system (11) we have

$$dE\left[V(x(t))\right]=\left[Mx(t)^T\left(\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)x(t)\right)dt\right.$$
$$-M\left(\sigma x(t)bW(t)-(\sigma bx(t)W(t))^T\right)dt$$
$$+Mx(t)\left(\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)x(t)\right)^T dt$$
$$\left.-M\sigma x^2(t)(\sigma x(t))^T dt\right]=LV(x(t))dt$$

where M is symmetric positive matrix, if $M=1$.

$$dE\left[V(x(t))\right]=\left[x(t)^T\left(\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)x(t)\right)dt\right.$$
$$-\left(\sigma x(t)bW(t)-(\sigma bx(t)W(t))^T\right)dt$$
$$+x(t)\left(\mu_{\max}\left(1-\frac{x(t-r)}{x_{\max}}\right)x(t)\right)^T dt$$
$$\left.-\sigma x^2(t)(\sigma x(t))^T dt\right]=LV(x(t))dt$$

Based on equation (11) and inequality (14) we will have

$$-LV(x(t))\geq KV(x(t))$$
$$\frac{\partial}{\partial t}\left[V(x(t))\right]\leq -KE\left[V(x(t))\right]$$
$$E\left[V(x(t))\right]\leq E\left[\exp(-Kt)\right]$$
$$\lim_{t\to\infty}E\left[V(x(t))\right]=\lim_{t\to\infty}E\left[\exp(-2t)\right]\leq 0$$

The function $V(x(t))$ is definite, therefore the trivial solution is stable. Eventually prove is completed. □

Remark 3.2: Liu et al., in [21], [22] were dealt the stability of stochastic delay logistic model with white noise. Nevertheless, current research studies the stability of stochastic delay logistic model with Ornstein-Uhlenbeck process due to it has better performance than Brownian motion.

For numerical solution, we used the SRK4 method which presented in below section.

4. Numerical Simulation

In this section we considered a strong and accurate numerical method SRK4 to elaborate the results [2, pp43]. Norhayati in [31] developed the SRK for numerical simulation of SDDEs. The general formula of SRK for SDDEs is:

$$Y_i^{(n-km)} = Y_0^{(n-km)} + \sum_{j=1}^{s} Z_{ij}^{(0)} f\left(Y_j^{(n-km)}, Y_j^{(n-(k+1)m)}\right) + \sum_{j=1}^{s} Z_{ij}^{(1)} g\left(Y_j^{(n-km)}\right)$$

$$Y_i^{(n-(k-1)m)} = Y_0^{(n-(k-1)m)} + \sum_{j=1}^{s} Z_{ij}^{(0)} f\left(Y_j^{(n-(k-1)m)}, Y_j^{(n-km)}\right) + \sum_{j=1}^{s} Z_{ij}^{(1)} g\left(Y_j^{(n-(k-1)m)}\right)$$

$$y_{n+1} = y_n + \sum_{i=1}^{s} z_i^{(0)} f\left(Y_i^{(n-(k-1)m)}, Y_i^{(n-m)}\right) + \sum_{i=1}^{s} z_i^{(1)} g\left(Y_i^{(n-(k-1)m)}\right) \quad . \qquad (17)$$

Where, $k = 1,\ldots,n$ and

$$Z_{ij}^{(0)} = \Delta a_{ij}, \qquad i, j = 1,\ldots,s \qquad (18)$$

$$Z_{ij}^{(1)} = \sum_{l=1}^{q} b_{ij}^{(l)} \theta_l, \quad i, j = 1,\ldots,s \qquad (19)$$

$$z_i^{(0)} = \Delta \alpha_i, \qquad i = 1,\ldots,s \qquad (20)$$

$$z_i^{(1)} = \sum_{l=1}^{q} \gamma_i^{(l)} \theta_l, \qquad i = 1,\ldots s \qquad (21)$$

Figure1 (a) and (b) show the stability of equation (8).

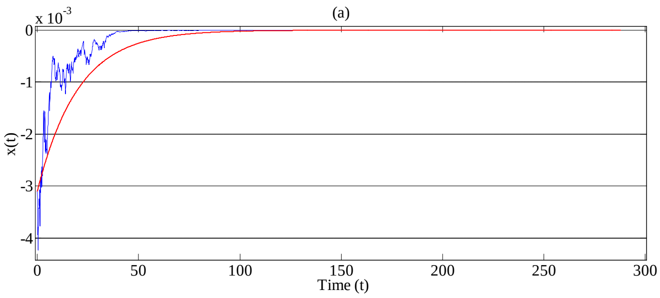

Figure1 (a): Shows the stability of equation (11), for deterministic part we used $\sigma = 0$, $r = 1$, $b = 0$ $x_0 = -0{,}0031$ and $\mu_{\max} = 0.05$, and for stochastic part we used same values only the difference are $b = 1$ and $\sigma = 0.3$.

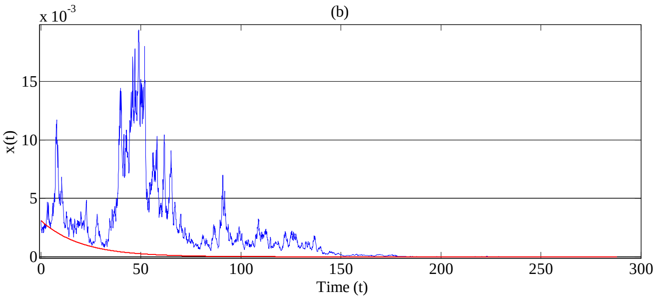

Figure1 (b): Illustrates the stability of equation (11), for deterministic part we used $\sigma = 0$, $r = 1$, $b = 0$ $x_0 = 0.0031$ and $\mu_{\max} = 0.05$, and for stochastic part we used same values only the difference are $b = 1$ and $\sigma = 0.3$

Remark 4.1: Figure1 (a) and (b) are stable in zero solution. If $\sigma \to \pm\infty$, model (11) cannot preserve the sufficient condition of stability means the model will not be stable in trivial solution.

5. Conclusion

This research introduced a general theory to determine the stability of SDDEs for zero solution in term of equation (1) by using Lyapunov function (see Theorem 3.1) and verified by stochastic delay logistic model with Ornstein-Uhlenbeck process (see Theorem 3.2) and several examples (see Appendix A). Moreover, we proved that the equation (9) has positive global solution for any initial function (see Lemma 3.1). Finally, this research used the SRK4 method to reflect the reality of our theory.

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Appendix A

Example 1: For
H1: $t_0 \geq 0$, $0 < a < b$ and $r = 1$, then $\lim_{t\to\infty} V\left(x(t)\right) \leq 0$.

The equation (21) is stable:

$$dx(t) = -bx(t-r)dt + a\exp(-t)dW(t) \qquad (22)$$

By using equation (22) into (11) and assuming ***M***=1, we get

$$dE\left[V\left(x(t)\right)\right] = x(t)^T\left(-x(t-r)\right) + \left(-x(t-r)\right)^T x(t) + \left(\exp(-t)\right)^T \exp(-t) = LV\left(x(t)\right)dt, \qquad (23)$$

after simplification yields

$$\lim_{t\to\infty} LV\left(x(t)\right) = -2\lim_{t\to\infty}\int_0^t x(s)\left(-bx(s-r)\right)ds + \mathrm{a}\lim_{t\to\infty}\int_0^t \exp(-2s)ds$$

For each $x(t) \geq 0$ the $\lim_{t\to\infty} V\left(x(t)\right) \leq 0$, therefore according to the [24], [26] the trivial solution is stable.

Example 2: Investigate the stability of stock price

$$dx(t) = bx(t-r)dt + \sigma dW(t) \qquad (24)$$

we use the equation (24) into (11) to determine the stability of stock price in market and assume *M*=1. So,

$$dE\left[V(x(t))\right] = x(t)^T bx(t-r) + (x(t-r))^T x(t) + \sigma^2 dt = LV(x(t))dt, \qquad (25)$$

By taking limit and integral from both sides of equation (25) we have:

$$LV(x(t)) = 2b \lim_{t\to\infty} \int_0^t x(s)(x(s-r))ds + \sigma^2 \lim_{t\to\infty} \int_0^t ds$$

For any initial function and b and σ the $\lim_{t\to\infty} V(x(t)) > 0$, hence stock price or the trivial solution is unstable.

Example 3: For

H2: $t_0 \geq 0$, $b > 0 > a$ and *r*=1, then $\lim_{t\to\infty} LV(x(t)) \leq 0$ the Langevin model is stable.

$$dx(t) = a\left[x(t) + bx(t-r)\right]dt + \sigma dW(t) \qquad (26)$$

We assume that *M*=1 and use equation (26) into (11), yields:

$$dE\left[V(x(t))\right] = x(t)^T a\left[x(t) + bx(t-r)\right]dt + x(t)a\left[x(t) + bx(t-r)\right]^T dt + \sigma^2 dt$$

$$LV(x(t))dt = 2x(t)a\left[x(t) + bx(t-r) + 2\sigma^2\right]dt \qquad (27)$$

Taking limit and integral from both sides of equation (27);

$$\lim_{t\to\infty} LV(x(t)) = \lim_{t\to\infty} \int_0^t \left[2ax(s)(s(s) + bx(s-r))\right]ds + \sigma^2 \lim_{t\to\infty} \int_0^t ds$$

Under **H2** the $\lim_{t\to\infty} LV(x(t)) \leq 0$, therefore according to the [24, 26], the trivial solution is stable.

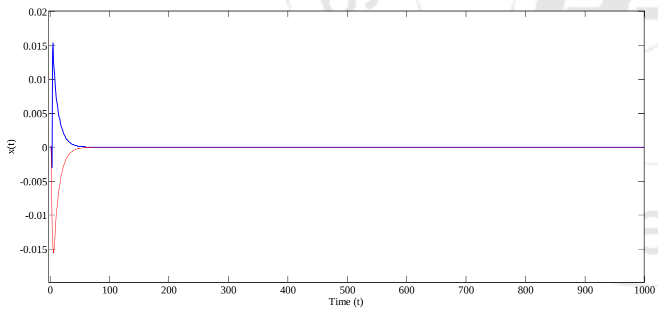

Figure 2: Shows the stability of equation (22) for $t_0 = 0.0$, $x_0 = 0.0001$, $a = 0.22$, $b = 1.99$, and $r = 1$.

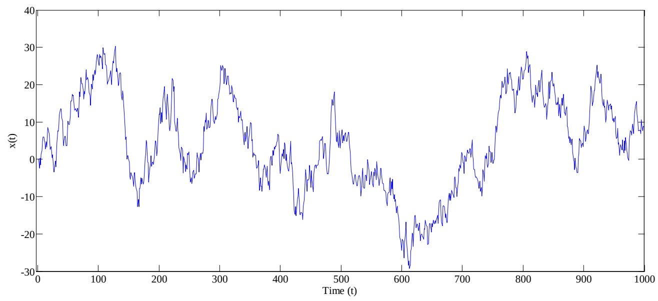

Figure 3: Indicates the instability of equation (24) for $x_0 = 0.0001$, $t_0 = 0.0$, $b = 0.99$, $r = 1$ and $\sigma = 2.22$.

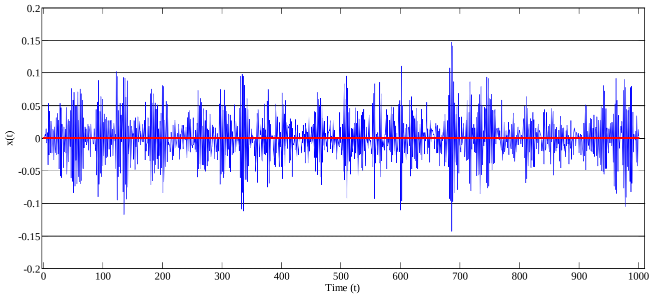

Figure 4: Indicates the stability of Langevin model for $x_0 = 0.0001$, $t_0 = 0.0$, $a = -0.35$, $b = 1.525$, $\sigma = 0.02$ and $r = 1$. The red line shows the Langevin model without noise and the solid blue line is constituted noise.

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