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Analysis of Single Server Finite Queue with Deterministic Service and Balking

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Abstract: ***This paper is concerned with the analysis of single server Markovian queue with deterministic service and balking. The behaviour of customers assumed independently who find the server busy may or may not enter in the queue for service depending on the number of jobs present in the system or which on entering in the queue, depart from the queue without being served due to impatient. Cohen [1] obtained the results for M/M/c retrial queue with balking where customers join the system on first come first serve (FCFS) discipline and only the customer at the head of the queue is allowed access to the server. The analytic results for queue size distribution in closed form as well as some performance measure of the system are obtained.***

Keywords: FCFS, Markovian, distribution, simulation

1. Introduction

In evaluating the performance of any mechanical system, reliability is an important issue. The system reliability decreases as the complexity of the system increase unless some precautionary measures are taken into account. In many system such as computer, electric or electronic system, scientific equipments etc., the continuous deterministic service time single server queue with balking have been widely used as a modelling tools in modern era. In communication system, which is intended for high speed transfer of data through optical fibber cable (OFC) or asynchronous transfer mode (ATM) are expected to handle variety of data traffic type are the best example of single server finite queue with balking and reneging. The service interruption or the behaviour of impatient customers which upon arrival in the system may or may not join the queue for service depending on the number of customers waiting for service in the system. Study of single server queue and balking is studied by several researchers in different frame-works.

Finch [2] discussed balking in queueing system GI/M/1 in which he consider the arrival pattern follows general independent while service pattern follows Poisson process with single server. Queueing with balking is analyzed by Haight [3]. Rao [4] deals with queueing model with balking, reneging and limited server availability; in [5] he studied queueing models with balking, reneging and interruptions in services at any finite stage. Satyamurti [6] analysed the queues and balking, a simple method to study the transient behaviour. Dick [7] defined some theorems on single server queues with balking. Haghighi et al [8] discussed on multi-server Markovian queueing system with balking and reneging. Blackburn [9] discussed optimal control of single server queue with balking and reneging. Ke and Wang [10] discussed cost analysis of M/M/R machine repair problem with balking, reneging and server breakdown. Shawky [11] defined the single server machine interface model with balking, reneging and addition server for longer queues, while in [12] he studied the machine interface model M/M/C/K/N with balking, reneging and spares. Barun and Garcia [13] consider analytic solution of finite capacity M/D/1 queues. Artalejo and Lopez-Herraro [14] discussed on the single server retrial queues with balking. Alfa and Isotupa [15] studied an M/PH/K retrial queues with finite number of servers.

2. Model Description

In the present paper we consider the single server queue with deterministic service and balking. The waiting room is assumed to be finite and due to more customers in the queue, it works as accumulator to the customers. We are interested to deriving the queue length distribution at departure using embedded Markov chain with prescribed queueing model. In the case of finite capacity, if the probability distribution is obtained at departure state does not happen but still valid at all point in case of infinite capacity. At the departure state the probability distribution can be used to derive the stationary probability distribution at all points in same time. In this connection we assume the customer arrive at the system one by one and follows Poisson's process with average arrival rate λ; on arrival in the system then corresponding to queue length customer decides either join to queue or balk. The customer does not join the queue if he fined the service is expected to be available after a wait of long time. The expected balking rate is β (constant) and when customer join the finite queue with capacity N then it has to wait a certain length of time T for service completion. If it has not begun then due to impatience, customer leaves the system without getting service. The customers served as first come first serve (FCFS) discipline and once service started, it always proceeds to completion and service time assume to be deterministic with rate μ. Let $X_N(t)$ be the number of customers at any time t, and t_n be time of n^{th} customer leave the system at completed his service. It is noted that $\{X_N(t_n)\}_{n\geq 0}$ stochastic process is Markove chain when [n = 0, 1,…. N-1]. Let $q_i(t_n)$ be the probability at i customers are left in the queue after n^{th} customer depart from the system.

Hence $\alpha_n = \frac{\rho^n e^{-\lambda n}}{n!}$ but

$\rho = \lambda\beta T$, hence $\alpha_n = \frac{(\lambda\beta T)^n}{n!} e^{-\lambda n}$ (1)

Definition 1: Gross & Harris

Let $t \in T$ is a set with parameter t, and X(t) be a random variable, then $\{X(t): t \in T\}$ is called a stochastic process.

A stochastic process $\{X(n): n \in N\}$ is said to be Markov chain if $n \in N$ for all states (i_0, i_1,……………..i_n) s.t. $P\{X_n = i_n/X_0 = i_0, X_1 = i_1 \ldots\ldots X_{n-1} = i_{n-1}\}$ = $P\{X_n = i_n/X_{n-1} = i_{n-1}\}$, i.e. we may make prediction about its future state without consulting past states when given the present state of the system.

Definition 2: Takacs

Using conditional probability $P\{X_n = j/X_{n-1} = i\}, \forall\, i, j \in S$, then the transition probability is denoted by $P_{ij}(n)$ s. t. $0 \le P_{ij} \le 1$, $1 \le i$ and $j \le N$, and $\sum_{j=1}^{N} P_{ij} = 1$, $1 \le i \le N$. Hence the probability transition matrix of embedded Markov chain $\prod = P_{ij}$ as

$$\prod = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & .. & .. & \alpha_{N-2} & & & 1-\sum_{i=0}^{N-2}\alpha_i \\ \alpha_0 & \alpha_1 & \alpha_2 & .. & .. & \alpha_{N-2} & & & 1-\sum_{i=0}^{N-3}\alpha_i \\ .. & .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & .. & \alpha_0 & & & 1-\alpha_0 \end{bmatrix}$$

When $\prod$ is ergodic, {Cohen [65]} then there exist the stationary distribution Q = [q_0, q_1, ….q_{N-1}] which is eigenvector of the matrix $\prod$, s.t. $\prod Q = Q$. The stationary distribution Q varies the system of the linear equation as given below

$$\alpha_0 q_0 = q_0$$

$$\alpha_0 q_0 + \alpha_1 q_1 = q_1$$

$$\alpha_2 q_0 + \alpha_1 q_1 + \alpha_3 q_2 = q_2$$

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$$\alpha_{N-2} q_0 + \alpha_{N-2} q_1 + \ldots\ldots\ldots\ldots \alpha_0 q_{N-1} = q_{N-2}$$

Assuming a_0, a_1, ….. a_{N-1} , s.t. $q_n = a_n q_0$ $\forall\, n \ge 0$; from the set of N-1 linear equations involving N unknowns q_0, q_1, ….q_{N-1}. Then a_0, a_1, ….. a_n can be obtained using

$$a_n = e^{\lambda\beta T}\left(a_{n-1} - \sum_{i=1}^{N} \alpha_i a_{n-i} - \alpha_{n-1}.a_0\right) \quad (2)$$

This can be solved using moment generating function

$$A(Z) = \sum_{k=0}^{\infty} a_k . z^k \quad (3)$$

The explicit formula for generating function A(Z) in the form of lemma is given below:

LEMMA 1: Let $(a_n)_{n\ge 0}$ be an infinite sequence with term a_0, a_1, ….. a_{N-1} and A(Z) be Z-transform of the sequence then the generating function A(Z) takes the form (1-Z).B(Z), where B(Z) = $[1-ze^{\beta\lambda T(1-z)}]^{-1}$.

Proof: Using recursion equation (2) & (3), we get

A(Z)= $a_0 e^{-\lambda\beta T} \sum_{n=0}^{\infty} \frac{(\lambda\beta T z)^n}{n!} + \frac{1}{z}\sum_{i=1}^{\infty} a_i z_i . \sum_{n=0}^{\infty} \alpha_n z^n$

$= a_0 e^{\lambda\beta T(z-1)} + \frac{e^{\lambda\beta T(z-1)}}{z}(A(z) - a_0)$

$= \frac{1-z}{1-ze^{\lambda\beta T(1-z)}}$,

where we assumes that the coefficient of Z-transform of B(Z) by $(b_n)_{n\ge 0}$.

⇨ A(Z) = (1-z). $[1-ze^{\beta\lambda T(1-z)}]^{-1}$. (4)

Preposition 1:

The coefficient a_n defined by recursion equation (4), may be obtained as $n \ge 0$ and $a_0 = 1$, s.t. $a_n = b_n - b_{n-1}$, where $b_0 = 1$ and

$$b_n = \sum_{i=0}^{n} (-1)^i . \frac{(n-i)^i}{i!} . e^{(n-i)\lambda\beta T} . (\lambda\beta T)^i.$$

Proof: As given $a_n = b_n - b_{n-1}$, then

$\sum_{n=0}^{\infty} a_n . z^n = \sum_{n=0}^{\infty} b_n . z^n$ -z.$\sum_{n=1}^{\infty} b_{n-1} . z^{n-1}$

$= b_0 + \sum_{n=1}^{\infty} b_n . z^n - \sum_{n=1}^{\infty} b_{n-1} . z^n$

$= b_0 + \sum_{n=1}^{\infty} (b_n - b_{n-1}) . z^n$ (5)

We consider z-transform F(Z) such as

F(Z) = $\sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^i . \frac{(n-i)}{i!} (n-i)^i . e^{(n-i)\lambda\beta T} (\lambda\beta T)^i z^n$

Hence, F(Z) = $\sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^i}{i!} . m^i . e^{m\lambda\beta T} (\lambda\beta T)^i . z^i z^n$

By equation (5), a_0 =1 and $a_n = b_n - b_{n-1}$ for $n \ge 1$, with the property

$\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} . m^i . (\lambda\beta T)^i . z^i = e^{-m\beta\lambda Tz}$

⇨ F(Z) = $\sum_{m=0}^{\infty} e^{-m\lambda\beta T(1-z)} z^m$

In Z- transform if $|e^{-m\lambda\beta T(1-z)} . z| < 1$, then F(Z) converges at $|z| < 1$, hence

⇨ F(Z) = $[1 - ze^{\lambda\beta T(1-z)}]^{-1}$, here we may define F(Z) = B(Z) at b_0 =1 and

$$b_n = \sum_{i=0}^{n} (-1)^i . \frac{(n-i)^i}{i!} . e^{(n-i)\lambda\beta T} . (\lambda\beta T)^i,$$

for all $n \ge 1$.

Let $P_n(t)$ be the probability that there are n customers in system at any time t, arrive according to Poisson's process wit average arrival rate λ; then

$b_n = \sum_{i=0}^{n} P_i[(i-n)T]$, for all n > 1.

Preposition 2: The linear system of (N-1) probabilities q_0, q_1, ….q_{N-1} of the number of customers may be obtained by

$$q_0 = \frac{1}{b_{N-1}} \text{ and } q_n = \frac{b_n - b_{n-1}}{b_{N-1}},$$

for all n = 1, 2, …… N-1.

Proof: When we consider the linear system of (N-1) probability distribution, then

$\sum_{n=0}^{N-1} q_n = 1$, hence we may find

$$q_0 = \sum_{n=0}^{N-1} a_n = \frac{1}{b_{N-1}} \text{ and}$$

$$q_n = a_n q_0, \text{ i.e. } q_n = \frac{b_n - b_{n-1}}{b_{N-1}},$$

for all n = 0, 1, 2….., N-1.

3. Main Results

At the steady-state the probability distribution Q of number of customers left after departure be differ from the queue length probability distribution in case of finite capacity queueing system. Cohen [65] was discuss the M/M/c queue with balking and reneging, holds the result

$$q_j = \frac{P_j(N)}{1-P_n(N)} \quad (6)$$

On the basis of finite capacity, the probability distribution of the number of customers in the system may be obtained at steady state

Theorem 1: If stochastic matrix is ergodic and stationary distribution Q = { q_0, q_1, ….q_{N-1}} exist, then the probability distribution of number of customers in the system at steady-state is $P_n(N) = 1 - \frac{b_{N-1}}{1+\beta\lambda T b_{N-1}}$.

Proof: Using equation no. (1), we know

$$q_j = \frac{P_j(N)}{1-P_n(N)}, \text{ where } j = 1,2,3…., N-1$$

and conversation Law of probability

$$\frac{1}{T}[1-P_0(N)] = \lambda\,[1-P_N(N)].$$

Hence $[1-P_N(N)] = \frac{1}{\lambda\beta T}[1-(1-P_N(N))q_0]$,

$$= \frac{1}{\lambda\beta T}\left[1-(1-P_N(N))\frac{1}{b_{N-1}}\right],$$

$$= \frac{1}{\lambda\beta T} - (1-P_N(N))\frac{1}{\lambda\beta T b_{N-1}}.$$

Or, $$\left[1+\frac{1}{\lambda\beta T b_{N-1}}\right][1-P_N(N)] = \frac{1}{\lambda\beta T}$$

$$[1-P_N(N)] = \frac{b_{N-1}}{1+\lambda\beta T b_{N-1}}$$

$$P_N(N) = 1 - \frac{b_{N-1}}{1+\lambda\beta T b_{N-1}} \quad (7)$$

Similarly using preposition 2, for P_j (N) and equation (6) $P_0(N)$ and P_j may be obtain as $P_0(N) = \frac{1}{1+\beta\lambda T b_{N-1}}$,

$$P_j(N) = \frac{b_j - b_{j-1}}{1+\beta\lambda T b_{N-1}}.$$

Corollary 1: If $P_N(N)$ be the probability distribution of number of customers in the system at steady-state, then the expected number of customers in the system are

$$E_N(N) = N\left[1+\frac{\sum_{n=0}^{N-1} b_n}{1+\beta\lambda T b_{N-1}} - b_{N-1}\right].$$

Proof: Law of probability, Gross and Harris [118],

$$E_N(N) = \sum_{n=0}^{N-1} nP_n(N)$$

$$= N + \frac{N\sum_{n=1}^{N-1} b_n}{1+\beta\lambda T b_{N-1}} - \frac{Nb_{N-1}}{1+\beta\lambda T b_{N-1}}$$

$$= N\left[1+\frac{\sum_{n=1}^{N-1} b_n - b_{N-1}}{1+\beta\lambda T b_{N-1}}\right] \quad (8)$$

In case of infinite capacity, the expected number of customers who does not quit the system and wait for service at departure epoch happen also to be valid at all points in time and which is not true in case of finite capacity.

Theorem 2: If customers do not quit the queue without getting service, then the expected waiting time is

$$E_N(w) = \left[N-1-\frac{\sum_{n=1}^{N-1} b_n - b_{N-1}}{\lambda\beta T b_{N-1}}\right].T$$

Proof: If customers wait for service in the queue, then from Little's formula

$$E_N(N) = \lambda(1-P_N(N))T_N$$

$$T_N = \frac{E_N(N)}{\lambda(1-P_N(N))},$$

hence using equations (7) and (8), we have

$$= \frac{1}{\lambda}\frac{N\left[1+\dfrac{\sum_{n=1}^{N-1} b_n - b_{N-1}}{1+\beta\lambda T b_{N-1}}\right]}{\left[\dfrac{b_{N-1}}{1+\beta\lambda T b_{N-1}}\right]}$$

$$= \frac{N(1+\beta\lambda T b_{N-1}) + N\sum_{n=1}^{N-1} b_n - N b_{N-1}}{\lambda b_{N-1}}$$

Hence, $E_N(w) = T_N - T$

$$= \frac{N(1+\beta\lambda T b_{N-1}) + \sum_{n=1}^{N-1} b_n - b_{N-1}}{\lambda b_{N-1}} - T \tag{9}$$

$$= \left[N-1-\frac{\sum_{n=1}^{N-1} b_n - b_{N-1}}{\lambda\beta T b_{N-1}}\right].T$$

Some Performance Measures

In the given queueing model, some performance measure may be obtained as

(i) Expected Number of customers in the system is

$$E_N(n) = \sum_{n=0}^{N-1} nP_N(N) \tag{10}$$

(ii) Expected number of customers in the queue is

$$E_N(m) = \sum_{m=0}^{N-1} mP_N(N) \quad \text{for } m = (n-1) \text{ for single server queue} \tag{11}$$

(iii) Variance of queue length

$$V(N) = \sum_{n=0}^{N-1} n^2 P_N(N) - [E_N(n)]^2 \tag{12}$$

4. Conclusion

In this paper we discussed the analysis of single server finite queue with deterministic service and balking in which The behaviour of customers assumed independently who find the server busy may or may not enter in the queue for service depending on the number of jobs present in the system or which on entering in the queue, depart from the queue without being served due to impatient. we obtain The formula for the probability distribution of the number of customers in the system, expected number of customers in the system, expected number of customers in the queue and variance of queue length for finite capacity single server queue. Method to calculate the performance measure for the given queueing model has been provided which may be applicable modern communication and machining system.

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