Hopf-Cole Transformation with Effect of the Dissipation Coefficient on Nonlinearity in the Benjamin-Bona-Mahony (BBM) Generalized Pseudo-Parabolic Equation

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Abstract: ***In this paper, we consider the general form of the Benjamin-Bona-Mahony's pseudo-parabolic nonlinear equation (BBM) in which we use the Hopf-Cole transformation [10]. We also set up a conjecture to get rid of several terms throughout our development. Finally, we present a comparative analysis of solutions found either by the aforementioned transform or by the functional variable method formulated by A. Zerarka [11]-[12]. Hence, we show that the result obtained by the approach of Hopf- Cole transformation has some significant features.***

Keywords: pseudo-parabolic nonlinear equation, generalized Benjamin-Bona-Mahony equation (BBM), functional variable method, Hopf-Cole transformation

1. Introduction

The Benjamin-Bona-Mahony equation (BBM) [1] has been studied for the first time as a regularized version of the Korteweg De Vries (KdV) equation for modeling low amplitude waves moving over a long surface under the action of gravity; hence the name of regularized long wave equation (RLW). The term "regularized" refers to the fact that, from the point of view of such properties as existence, uniqueness and stability, the BBM equation offers considerable technical advantages over the KdV equation.

As the authors H. Zhang, GM Wei and Y.T. Gao [2] point out, unlike the KdV equation, the BBM equation is not an equation of evolution in the strict sense of the term, following the appearance of u_{xxt}. The latter makes this equation pseudo-parabolic [9] since the term of the highest order of this equation has mixed derivatives (i.e. derivatives dependent on both time and space). Investigations on exact solutions of traveling waves of nonlinear evolution equations play an important role in the study of non-linear physical phenomena. Solitons are the most important types of solutions among all traveling wave solutions. The existence of multi-soliton, in particular two-soliton type solutions, is crucial in information technologies: it makes possible the simultaneous undisturbed propagation of several pulsations in both directions [13].

In this paper, we attempt to linearize the following generalized BBM equation:

$$u_t + \alpha u_x + \beta u u_x - \delta u_{xxt} = 0 \quad (1)$$

with the coefficients $\alpha, \beta, \delta \in R$, which respectively correspond to the transport, nonlinearity and dispersion coefficients. This equation describes the unidirectional propagation of long waves.

- u : the amplitude or the velocity (or the speed of the flow)
- x : is proportional to the distance in the direction of fluid flow
- t : the time of fluid flow

The classical form of Hopf-Cole transformation is applied in the search for solution of the nonlinear diffusion equation. It should be noted that Eq. (1) incorporates non-linear dispersive and dissipative effects.

Although there are several ineluctably efficient methods by which the multi-solution solutions of Eq. (1) can be derived, the main aim of this paper is to present the remarkable strength of the Hopf-Cole transformation [10]. We perform a comparative analysis of our solution with a solution obtained by the functional variable method formulated by A. Zerarka [11]-[12].

2. Hopf-Cole Transformation

2.1 Description

Below are some essential steps to apply to the cited transformation.

Step 1: Consider the differential equation given by
$P(x, t, u, u_t, u_x, u_{xxt}, \ldots) = 0$
where P is a given function, $u(t, x)$ is the independent variable.

Step 2: Find a smooth solutions to the form $\omega = \phi(u)$

Step 3: Substitute the expression resulting from step 2 into the PDE of step 1.

Step 4: Use the notation $\omega = \phi(u)$ to get a polynomial equation having ω as variable.

Step 5: By equaling the coefficients of most of the powers of $\phi(u)$ to zero, we obtain an algebraic equation of unknowns α, β and δ.

Step 6: Find solutions of the linearized equation to the form

$$\eta(x,t) = 1 + \exp(Ax + Bt + C)$$

Where $\omega = \eta_{xt}$

Step 7: By solving this equation and interchanging, we finally get the solutions of our PDE.

2.2 Application

Suppose that u is a smooth solution of Eq. (1). We define

$$\omega = \phi(u)$$

where $\phi : R \to R$ is a smooth function, not yet specified.

We will try to choose ϕ such as ω solve a linear equation.

We should have the following special format

$$\omega(x,t) = \partial_x^m \partial_t^n \{\phi[u(x,t)]\}$$

Where m and n are integers (≤ 2) determined as follows:

$$\omega_t = \phi'(u)u_t \quad (2)$$

$$\omega_x = \phi'(u)u_x, \; \omega_{xt} = \phi''(u)u_t u_x + \phi'(u)u_{xt}$$

$$\omega_{xx} = \phi''(u)u_x^2 + \phi'(u)u_{xx} \text{ and}$$

$$\omega_{xxt} = \phi'''(u)u_t u_x^2 + \phi''(u)2u_x u_{xt} + \phi''(u)u_t u_{xx} + \phi'(u)_t u_{xxt}$$

and therefore, (2) implies

$$\omega_t = \phi'(u)[-\alpha u_x - \beta u u_x + \delta u_{xxt}]$$

Putting all the term together, we have

$$\omega_t = -\alpha\omega_x - \beta\phi'(u)uu_x + [-\delta(\phi'''(u)u_t u_x^2 + \phi''(u)u_t u_{xx}) - 2\delta\phi''(u)u_x u_{xt}] + \delta\omega_{xxt} \quad (3)$$

Explaining the cancelling effects of different mechanisms that act to change waveforms to result in a soliton wave, Authors (2) pointed out the fact that these mechanisms included dispersion, dissipation and nonlinearity either separately or in various combinations.

We now make the conjecture that the cancelling act concentrate on terms mixing up coefficient of dispersion δ acting on such nonlinear terms as $u_t u_x^2, u_t u_{xx}$ and $u_x u_{xt}$ in Eq. (3) In one hand, Conservation laws clearly shows that

$$-\delta(\phi'''(u)u_t u_x^2 + \phi''(u)u_t u_{xx}) - 2\delta\phi''(u)u_x u_{xt} = 0$$

Which is equivalent to

$$-\delta\phi'''(u)u_t u_x^2 - \delta\phi''(u)u_x u_{xt} - \delta\phi''(u)u_x u_{xt} = 0$$

That implies

$$-\delta\phi'''(u)u_t u_x - \delta\phi''(u)u_{xt} = 0 \text{ for } \delta \neq 0$$

Then

$$-\delta\phi''(u)u_t u_x - \delta\phi'(u)u_{xt} = 0$$

In the other hand, to satisfy (3), we choose ϕ so that $\beta\phi'(u) = 0$.

We solve this PDE by establishing $\phi = K$ constant with respect to u.

So we see that if u solve (1), then

$\omega = K$ "Hopf-Cole transformation"

Solve this IVP for the linearized form of the BBM equation:

$$\begin{cases} \omega_t + \alpha\omega_x - \delta\omega_{xxt} = 0 & dans\ R^n \times (0,\infty) \\ \omega = K & sur\ R^n \times \{t = 0\} \end{cases} \quad (4)$$

Considering the following transformation

$\omega = \eta_{xt}$ (4) becomes $\eta_{xtt} + \alpha\eta_{xxt} - \delta\eta_{xxxtt} = 0$

whose trail solution has the form

$$\eta(x,t) = 1 + \exp(Ax + Bt + C)$$

Where A, B and C are arbitrary constants, with A and B satisfying the following relations $[2]$:

$$A \neq 0,\; B \neq 0 \text{ and } B(1 - \delta A^2) + \alpha A = 0$$

Therefore,

$$\eta(x,t) = \begin{cases} 1 + \exp\left[A\left(x - \dfrac{\alpha}{1 - \delta A^2}t + C_1\right)\right] & if\ \alpha \neq 0 \\ & (with\ 1 - \delta A^2 \neq 0) \\ 1 + \exp\left(\pm\delta^{-\frac{1}{2}}x + Bt + C_2\right) & otherwise \end{cases}$$

Simultaneously, Eq. (4) and (3) are automatically satisfied with the substitution of the above $\eta(x,t)$, which is thus a special solution for both equations.

As we had $-\delta\phi''(u)u_t u_x - \delta\phi'(u)u_{xt} = 0$

Finally, the set of exact solutions of the general form of Eq. (1) are clearly

$$u(x,t) = \phi''\omega_x\omega_t + \phi'\omega_{xt}$$

$$u(x,t) = \begin{cases} \dfrac{3\delta\alpha A^2}{\beta(1 - \delta A^2)}\sec h^2\left\{\dfrac{1}{2}A\left(x - \dfrac{\alpha}{1 - \delta A^2}t\right) + C_1\right\} if\ \alpha \neq 0 \\ (with\ 1 - \delta A^2 \neq 0) \\ \dfrac{3\delta^{-\frac{1}{2}}B}{\beta}\sec h^2\left[\dfrac{1}{2}\left(\pm\delta^{-\frac{1}{2}}x + Bt + C_2\right)\right] otherwise \end{cases} \quad (5)$$

Which are the solitary waves having the $\sec h^2$ profiles. A and B are parameters specifying the amplitude and speed of the wave.

3. Functional Variable Method

3.1 Description

Step 1: Consider the nonlinear differential equation given by

$$P(x,t,u,u_t,u_x,u_{xxt},...)=0$$

where P is a given function, $u(t,x)$ is the independent variable or the functional variable to be determined..

Step 2: Define a new wave variable such that

$$\xi=\theta+\sum_{i=0}^{m}c_i x_i$$

Where θ and c_i are free parameters (with c_i which represents the pulsation of the wave). x_i are independent variables that represents the wave vector . In this case $x_0=x$ and $x_1=t$.

Step 3: Introduce the following transformation for wave solutions of the main equation

$$u(x_0,x_1,...)=U(\xi)$$

Step 4: introduce the following derivation rule

$$\frac{\partial}{\partial x_i}(.)=c_i\frac{d}{d\xi}U(.),\quad \frac{\partial^2}{\partial x_i\partial x_j}(.)=c_i c_j\frac{d^2}{d\xi^2}U(.),...$$

Step 5: transformation from Partial Differential Equation (PDE) to Ordinary Differential Equation (ODE). Using the transformations from step 4, our equation from step 1 converts to the following ODE

$$Q(U,U_\xi,U_{\xi\xi},U_\xi U_{\xi\xi},U_{\xi\xi\xi},...)=0$$

Step 6: transformation in which the unknown function U is a function-dependent variable

$$U_\xi=F(U)$$

Where F is a function of U alone.

Step 7: We give some successive derivatives of U by posing $G=F^2$.

$$U_{\xi\xi}=\frac{1}{2}G'(U),\ U_{\xi\xi\xi}=\frac{1}{2}G''(U)\sqrt{G},$$

$$U_{\xi\xi\xi\xi}=\frac{1}{2}G'''(U)G+G''(U)G'(U),...$$

Step 8: Using these transformations, our equation of step 5 takes the form

$$R(U,G'(U),G''(U),G'(U)G''(U),G'''(U),...)=0$$

The interest of the obtained form is that it admits analytical solutions for a large class of nonlinear equations.

Step 9: the integration of the expression obtained in step 8 provides the expression of F through G.

Step 10: the use of the formula of step 6 allow to construct the possible solutions of the original equation of step 1.

3.2 Application

Consider the Eq. (1). We use the following wave variable $\xi=x-ct$,and we choose $u(x,t)=U(\xi)$.

The different derivatives are carried out via the chain rule, Eq.(1) is transformed as

$$cU_\xi+\alpha U_\xi+\frac{\beta}{2}(U^2)_\xi-\delta c U_{\xi\xi\xi}=0 \quad (6)$$

A first integration of Eq. (6) provides

$$cU+\alpha U+\frac{\beta}{2}(U^2)-\delta c U_{\xi\xi}=0 \quad (7)$$

We use transformation $U_\xi=\sqrt{G(U)}$, Eq. (7) transforms as

$$G'(U)=\frac{2}{\delta c}\left[(c+\alpha)U-\frac{\beta}{2}U^2\right] \quad (8)$$

The integration of (8) gives

$$G=\frac{2}{\delta c}\left[\frac{(c+\alpha)}{2}U^2-\frac{\beta}{6}U^3\right] \quad (9)$$

Thus, we get the functional $F(U)$,

$$F(U)=\sqrt{\frac{(c+\alpha)}{2\delta c}}U\sqrt{1-\frac{2\beta}{3(c+\alpha)}U} \quad (10)$$

and the solution U is obtained by integration

$$\sqrt{\frac{(c+\alpha)}{2\delta c}}\xi=\int\frac{dU}{\sqrt{1-\frac{2\beta}{3(c+\alpha)}U}} \quad (11)$$

We obtain the following relation after integration of (11)

$$\sqrt{\frac{(c+\alpha)}{2\delta c}}\xi=-2Arc\tanh\left(\sqrt{1-\frac{2\beta}{3(c+\alpha)}U}\right) \quad (12)$$

it comes then

$$\sqrt{1-\frac{2\beta}{3(c+\alpha)}U}=-\tanh\left(\frac{1}{2}\sqrt{\frac{(c+\alpha)}{2\delta c}}\xi\right) \quad (13)$$

Relationship (13) provides the following expression

$$\frac{2\beta}{3(c+\alpha)}U=\sec h^2\left(\frac{1}{2}\sqrt{\frac{(c+\alpha)}{2\delta c}}\xi\right) \quad (14)$$

We deduce U which is written:

If $\frac{(c+\alpha)}{2\delta c}>0$,

$$U(\xi)=\frac{3(c+\alpha)}{2\beta}\sec h^2\left(\frac{1}{2}\sqrt{\frac{(c+\alpha)}{2\delta c}}\xi\right) \quad (15)$$

If $\frac{(c+\alpha)}{2\delta c}<0$,

$$U(\xi)=\frac{3(c+\alpha)}{2\beta}\sec^2\left(\frac{1}{2}\sqrt{\frac{(c+\alpha)}{2\delta c}}\xi\right) \quad (16)$$

The solutions in terms of x and t are :

Case 1 : If $\frac{(c+\alpha)}{2\delta c} > 0$,

$$U(x,t) = \frac{3(c+\alpha)}{2\beta} \operatorname{sech}^2\left(\frac{1}{2} \sqrt{\frac{(c+\alpha)}{2\delta c}} (x-ct) \right) \quad (17)$$

Case 2 : If $\frac{(c+\alpha)}{2\delta c} < 0$,

$$U(x,t) = \frac{3(c+\alpha)}{2\beta} \sec^2\left(\frac{1}{2} \sqrt{\frac{(c+\alpha)}{2\delta c}} (x-ct) \right) \quad (18)$$

It is clear that in case1, the set of exact solution of the general form of BBM are solitary waves having the profile sech^2. Hence, we have soliton-type solutions. On the other hand, case2 gives us solutions of the type compactons with the profile $\sec^2$. Which clearly are not solitons.

4. Conclusion

As a conclusion, we used the Hopf-Cole transformation [10] to linearize Benjamin-Bona-Mahony's pseudo-parabolic nonlinear generalized equation, and we obtained a suitable form for determining a solution that was difficult to get in the past. In addition, it is appropriate to note the ease of the method, which offers a solution with the soliton profile. We have also presented a comparative analysis of the results obtained using the functional variable method [11]-[12], and we note that solutions are visibly very close. Even if, we have used two different methods, results are fascinating.

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