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Comparison of Some Methods for the Solution of Linear Fredholm Integral Equations of the Second Kind

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Abstract: ***This paper concern study of some solution technics for the explanation of linear Fredholm integral equations. In this research paper our aim is to compare some new and traditional method that are using for solution of linear Fredholm integral equations. Our essential goal in this paper is to investigate the advantage of each method for solution and solution process of a linear Fredholm integral equation. This research article will focus on Fredholm determinant method, Adomian decomposition method, Modified decomposition method, successive approximation method and direct computational method. Finally, we want to apply this method on a problem and compare the result of the research.***

Keywords: Fredholm integral equations, Fredholm determinant, Fredholm minor, Fredholm resolvent kernel

1. Introduction

Abel was an Italian mathematician; he presents integral equation in 1825 with connection to the Tautochrone problem for first time [31,32,38,39]. This problem concerns with determination of the curve on which a particle sliding and descends to its lowest point, such that the descending time is function of its initial location.

An equation is called the integral equation (IE), where the unknown function $\varphi(x)$ to be determined under the integral symbol. The IE is a very useful tool in all branches of mathematics [28, 31]. Integral equations have many applications in various fields of physics. Many IBVPs associated with ODEs and PDEs can be transmuted into problems to solve some approximate IEs [23.27.28.40]. The growth of technology and scientific knowledge has managed to the creation of numerous laws of physics, which are re-presented in mathematical form, usually expressed as differential equations. Most technical complications can be modeled as differential equations. Therefore, differential equations play central and significant role in solving the most applied complications. Most of the problems that occur in the electrical circuit, the heat transfers and chemical kinetics in the medium can be characterized mathematically as DE [28,29,38,39]. The standard form of an IE with unknown function $\varphi(x)$ is

$$\varphi(x) = \psi(x) + \lambda \alpha \int_{\alpha(x)}^{\beta(x)} \mathrm{K}(x,t)\varphi(t)\,dt. \qquad (1)$$

Here $\mathrm{K}(x,t)$ is known as the kernel of the above IE, also $\alpha(x)$ and $\beta(x)$ are the bounds of integration. It is clear to say that the $\varphi(x)$ happens inside the integral symbol. This is needful to note that $\mathrm{K}(x,t)$ and $\psi(x)$ are given functions and λ is parameter [1,28,31]. Here the essential objective is to illuminate the unknown function $\varphi(x)$ which may satisfy equation (1) applying some solution methods. We shall allocate remarkable diligences in discovering these methods to discover solutions of the $\varphi(x)$.

Integral equations can be solving by analytical and numerical methods, suchlike Adomian decomposition method (ADM), modified decomposition method (MDM), successive approximations method (SAM), direct computational method (DCM) [18-21], [28,29,31,36] and some other methods. Avazzadeh [9], Bakodah [10] and wazwaz [36-39] studied comparison of some certain analytical solution techniques for integral equations.

2. Types of the Integral Equations

An integral equation can be classified as a linear IE and non linear IE as we have seen in the ordinary and partial differen tial equations. It is clear that a differential equation can be represent by an equivalent IE. Therefore, there is a good connection between differential equations and IEs. The most important and major classes of the integral equations are **Volterra** integral equations **(VIE)** and **Fredholm** integral equations (**FIE**). Also we can classify this tow classes as homogeneous or nonhomogeneous; also linear or nonlinear. In some practical problems, we face with singular form of integral equations. Now we have to outline these types of IEs as follows.

Volterra integral equations
The normal form of VIE is

$$u(x)\varphi(x) = \psi(x) + \lambda \int_{a}^{x} \mathrm{K}(x,t)\varphi(t)\,dt \qquad (2)$$

Here the integration limits are functions of x and the unknown function $\varphi(x)$ occurs under the integral symbol. If we have that $u(x) = 1$, then equation (2) becomes

$$\varphi(x) = \psi(x) + \lambda \int_{a}^{x} \mathrm{K}(x,t)\varphi(t)\,dt \qquad (3)$$

and this equation is called VIE of the second kind; whereas if $u(x) = 0$, then equation (2) becomes

$$\psi(x) + \lambda \int_{a}^{x} \mathrm{K}(x,t)\varphi(t)dt = 0 \qquad (4)$$

which is the VIE of the first kind.

Fredholm Integral Equations (FIEs)

The normal form of Fredholm integral equations is

$$u(x)\varphi(x) = \psi(x) + \lambda \int_a^b \mathrm{K}(x,t)\varphi(t)dt \qquad (5)$$

Here the bounds of integral are constants, here the functions $\psi(x)$ and $\mathrm{K}(x,t)$ both are real valued, also the function $\varphi(x)$ happens inside the integral symbol [22]. If $u(x) = 1$, then (5) becomes as

$$\varphi(x) = \psi(x) + \lambda \int_a^b \mathrm{K}(x,t)\varphi(t)\,dt \qquad (6)$$

and this equation known as FIE of second kind; whereas if $u(x) = 0$, then (5) becomes

$$\psi(x) + \lambda \int_a^b \mathrm{K}(x,t)\varphi(t)\,dt = 0 \qquad (7)$$

Which is called FIE of the first kind. For more details, see [1,22,28,31].

Remark

It is valuable to point out that IEs arising in physics, engineering, chemistry, and biological complications. Many IBVPs allied with the ordinary and partial differential equations may be covert to integral equations of Volterra and Fredholm types, respectively. If $\varphi(x)$ is presented in functional form $F(\varphi(x))$ such that the power of $\varphi(x)$ is not equal to one then the VIEs and FIEs are classified as nonlinear integral equations [28,38,39,41]. As for examples, the following IEs are nonlinear:

$$\varphi(x) = \psi(x) + \lambda \int_a^x \mathrm{K}(x,t)\ \varphi^3(t)\ dt$$

$$\varphi(x) = \psi(x) + \lambda \int_a^b \mathrm{K}(x,t)\ cos\ (\varphi(t))\ dt$$

$$\varphi(x) = \psi(x) + \lambda \int_a^b \mathrm{K}(x,t)\log(\varphi(t))\ dt$$

Next, if in an IE we have $\psi(x) = 0$, then the resulting VIEs or FIEs is called a homogeneous integral equation, otherwise it is known as nonhomogeneous integral equation. So now we will focus on solution technics of FIEs.

3. Methods for Solution of FIEs

Method of the Fredholm Determinants

The solution of the FIE (6) is provided by the following formula

$$\varphi(x) = \psi(x) + \lambda \int_a^b R(x,\ t;\lambda)f(t)\,dt \qquad (8)$$

Here the kernel $R(x,\ t;\lambda)$ is known as the *Fredholm resolvent kernel* of equation (6) and is given by

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} \qquad (9)$$

Where $D(\lambda) \neq 0$ and the numerator $D(x,t;\lambda)$ and denominator $D(\lambda)$ are power series of λ:

$$D(x,t;\lambda) = \mathrm{K}(x,t) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} B_n(x,t)\lambda^n \qquad (10)$$

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} C_n\ \lambda^n \qquad (11)$$

Whose coefficients are stated as follow

$$B_n(x,t) = \underbrace{\int_a^b \cdots \int_a^b}_{n\ times} \begin{vmatrix} \mathrm{K}(x,t) & \mathrm{K}(x,t_1) & \cdots & \mathrm{K}(x,t_n) \\ \mathrm{K}(t_1,t) & \mathrm{K}(t_1,t_1) & \cdots & \mathrm{K}(t_1,t_n) \\ \mathrm{K}(t_2,t) & \mathrm{K}(t_2,t_1) & \cdots & \mathrm{K}(t_2,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{K}(t_n,t) & \mathrm{K}(t_n,t_1) & \cdots & \mathrm{K}(t_n,t_n) \end{vmatrix} dt_1 \ldots dt_n \qquad (12)$$

and

$$B_0(x,t) = \mathrm{K}(x,t)$$

$$C_n = \underbrace{\int_a^b \cdots \int_a^b}_{n\ times} \begin{vmatrix} \mathrm{K}(t_1,t_1) & \mathrm{K}(t_1,t_2) & \cdots & \mathrm{K}(t_1,t_n) \\ \mathrm{K}(t_2,t_1) & \mathrm{K}(t_2,t_2) & \cdots & \mathrm{K}(t_2,t_n) \\ \mathrm{K}(t_3,t_1) & \mathrm{K}(t_3,t_2) & \cdots & \mathrm{K}(t_3,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{K}(t_n,t_1) & \mathrm{K}(t_n,t_2) & \cdots & \mathrm{K}(t_n,t_n) \end{vmatrix} dt_1 \ldots dt_n \qquad (13)$$

In equation (9) the numerator function $D(x,\ t;\lambda)$ is known as Fredholm minor, and denominator function $D(\lambda)$ is the Fredholm determinant. When $\mathrm{K}(x,t)$ is bounded or the integral

$$\int_a^b \int_a^b \mathrm{K}^2(x,t)dxdt$$

has a finite value, the series (10) and (11) converges for all values of λ and hence are analytic functions with respect to λ.

The recursion formulas are

$$B_n(x,t) = C_n\mathrm{K}(x,t) - n\int_a^b \mathrm{K}(x,s)B_{n-1}(s,t)ds \qquad (14)$$

$$C_n = \int_a^b B_{n-1}(s,s)ds\ \ (n = 1,2,\ldots),$$

$$C_0 = 1,\ B_0(x,t) = \mathrm{K}(x,t) \qquad (15)$$

Which are practically using for calculation of B_n and C_n, [20,21,24,25].

The Adomian Decomposition Method

The Adomian decomposition method (ADM) was presented and expended by G Adomian in [2-8], [11,16,17,32], [37-39]. This method can be applying to an extensive class of ODE, PDE, integral equations and integro-differential equations [15,26,30], [33-35]. Here we want to introduce briefly the Adomian method. The ADM consists of decomposition for the function $\varphi(x)$ into a sum of infinite number of components provided by the following series

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x)\,, \qquad (16)$$

or in expended form

$$\varphi(x) = \varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \cdots \qquad (17)$$

Here the components $\varphi_n(x)$, $n \geq 0$ can be determined recurrently. The ADM concerns itself to find out the $\varphi_0,\ \varphi_1,\ \varphi_2,\ \ldots$ separately. The determination of $\varphi_0,\ \varphi_1,\ \varphi_2, \ldots$ can be accomplished in a simple manner by using a recurrence formula, that commonly contains simple integrals. To form the recurrence formula, we have to set the value of $\varphi(x)$ from series (16) into the FIE (17) to get

$$\sum_{n=0}^{\infty} \varphi_n(x) = \psi(x) + \lambda \int_a^b \mathrm{K}(x,t)\left(\sum_{n=0}^{\infty} \varphi_n(t)\right) dt, \qquad (18)$$

or in expanded form

$$\varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \cdots = \psi(x) + \lambda \int_a^b \mathrm{K}(x,t)[\varphi_0(t) + \varphi_1(t) + \cdots]dt\,. \qquad (19)$$

the initial component $\varphi_0(x)$ is identified by all those terms that are without the integral symbol. This shows that

$\varphi_j(x),\ j \geq 0$ for $\varphi(x)$ are totally determined by substituting the recurrence formula

$$\varphi_0(x) = \psi(x),\ \varphi_{n+1}(x) = \lambda \int_a^b K(x,t)\varphi_n(t)dt, n \geq 0, \quad (20)$$

or

$$\begin{aligned}\varphi_0(x) &= \psi(x),\\ \varphi_1(x) &= \lambda \int_a^b K(x,t)\varphi_0(t)dt,\\ \varphi_2(x) &= \lambda \int_a^b K(x,t)\varphi_1(t)dt,\\ \varphi_3(x) &= \lambda \int_a^b K(x,t)\varphi_2(t)dt, \end{aligned} \quad (21)$$

and so on. From formula (21), all of the components $\varphi_0(x),\ \varphi_1(x),\ \varphi_2(x),\ \varphi_3(x), \ldots$ are totally determined. Finally the solution $\varphi(x)$ of FIE (6) is achieved in the series form with use of the series (16).

It is clear that the ADM converts the IE into an elegant determination of calculable components. This is important to note that the achieved series converges very quickly to exact solution if it exists. The idea of convergence for the decomposition series was considered by Cherruault and several other researchers to approve the quick convergence of the consequential series [12-14], [30].

The Modified Decomposition Method

As detailed earlier, the ADM offers the solutions in series form. Here $\varphi_j,\ j \geq 0$ are easily calculated if $\psi(x)$ in the FIE

$$\varphi(x) = \psi(x) + \lambda \int_a^b K(x,t)\varphi(t)dt, \quad (22)$$

is a monomial or binomial. However, if the function $\psi(x)$ consists of a combination of two or more of polynomials, trigonometric functions, hyperbolic functions, and others, the evaluation of $\varphi_j,\ j \geq 0$ needs extra work. As previously stated, the MDM depends essentially on the division of $\psi(x)$ into two parts, so that the MDM cannot be used if $\psi(x)$ is composed of a single term. The MDM will be briefly outlined here, but will be used in this section of paper. The standard employs the recurrence formula

$$\begin{aligned}\varphi_0(x) &= \psi(x),\\ \varphi_{k+1}(x) &= \lambda \int_a^b K(x,t)\varphi_k(t)dt,\ k \geq 0, \end{aligned} \quad (23)$$

here the solution $\varphi(x)$ is presented by

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x) \quad (24)$$

with aid of formula (23), all of $\varphi_n(x),\ n \geq 0$ can be easily achieved. The MDM shows an insignificant variant in the recurrence formula (23) to determine the components of $\varphi(x)$ more easily and rapidly. In several cases, $\psi(x)$ can be defined as the sum of two partial functions $\psi_1(x)$ and $\psi_2(x)$, presented by

$$\psi(x) = \psi_1(x) + \psi_2(x). \quad (25)$$

In equation (25), we present a qualitative change in the construction of the formula (23). The MDM recognizes the initial component $\varphi_0(x)$ from a part of $\psi(x)$, i.e. $\psi_1(x)$ or $\psi_2(x)$. The other part of $\psi(x)$ can be added to $\varphi_1(x)$ existing in the typical recursion formula. The MDM allows the use of the modified recursion formula:

$$\begin{aligned}\varphi_0(x) &= \psi_1(x),\\ \varphi_1(x) &= \psi_2(x) + \lambda \int_a^b K(x,t)\varphi_0(t)dt,\\ \varphi_{k+1}(x) &= \lambda \int_a^b K(x,t)\varphi_k(t)dt,\ k \geq 1. \end{aligned} \quad (26)$$

This evident that the modification of the typical recursion formula (23) and modified recursion formula (26) rests solely on the creation of $\varphi_0(x)$ and $\varphi_1(x)$ only. The remaining components $\varphi_j,\ j \geq 2$ stay unchanged in both recursion formulas. While this variation in the construction of $\varphi_0(x)$ and $\varphi_1(x)$ is very small, however this can show that it accelerates the convergence of the solution and reduces the volume of computations. Moreover, reduction in the number of terms in $\psi_1(x)$ affects not only on $\varphi_1(x)$, but also the remaining components. Here we highlight two key facts. First, by suitable choice of $\psi_1(x)$ and $\psi_2(x)$, the exact solution $\varphi(x)$ can be achieved by using a small number of iterations, and occasionally by evaluating just two components. The achievement of this adjustment rest on only on the suitable selection of $\psi_1(x)$ and $\psi_2(x)$, and this can be done by trials. A canon that can service for the appropriate selection of ψ_1 and ψ_2 cannot yet be found. Secondly, if $\psi(x)$ contain a single term, in this case it's not possible to use MDM.

The Successive Approximations Method (SAM)

The ***Successive Approximations Method,*** also known as the **Picard iteration method**. This method offers a structure that will used to solve IVP or IEs. This method can solve every problem by discovering successive approximations to the solution by beginning with an initial supposition as $\varphi_0(x)$, titled the initial approximation. As we will see, the initial approximation is any real valued function and will be used in the recursion formula to define the next approximations. The utmost usually used values for the initial approximations are zero, one or x. Of course, other real values can be taken as well. Consider the following FIE

$$\varphi(x) = \psi(x) + \lambda \int_a^b K(x,t)\varphi(t)dt, \quad (27)$$

The SAM presents the recursion formula

$$\begin{aligned}\varphi_0(x) &= \textit{any selected real value function},\\ \varphi_{n+1}(x) &= \psi(x) + \lambda \int_a^b K(x,t)\varphi_n(t)dt,\ n \geq 0. \end{aligned} \quad (28)$$

By using the limit, the solution is

$$\varphi(x) = \lim_{n\to\infty} \varphi_{n+1}(x). \quad (29)$$

It is remarkable to note that the ADM allows the usage of a repetition formula as follow

$$\begin{aligned}\varphi_0(x) &= \textit{all terms that are outside the integral sign},\\ \varphi_1(x) &= \lambda \int_a^b K(x,t)\varphi_0(t)dt,\\ \varphi_2(x) &= \lambda \int_a^b K(x,t)\varphi_1(t)dt\\ &\vdots\ \vdots\ \vdots\\ \varphi_{n+1}(x) &= \varphi_n(x) + \lambda \int_a^b K(x,t)\varphi_n(t)dt. \end{aligned} \quad (30)$$

The difference between (28) and (30) can be summarized as

1) The SAM provides successive approximations of the $\varphi(x)$, however the ADM provides successive components of $\varphi(x)$.

2) The SAM makes it possible to use a choosy function with real values for the initial approximation φ_0, while the ADM allocates all terms that are not under the integral symbol for $\varphi_0(x)$. Remember that this allocation was adjusted when using the MDM.
3) The SAM provides the solution in exact form, if it available,
$$\varphi(x) = \lim_{n\to\infty} \varphi_{n+1}(x). \quad (31)$$
4) But, the ADM provides the solution in the form of an infinite series of components by
$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x). \quad (32)$$

This solution converges quickly to the exact solution if such a solution is available. The SAM, or the iteration method will be illustrating by evaluating an example at the end of methods.

The Direct Computational Method (DCM)
In this section of the paper, we will apply the DCM to solve the FIEs. This method approaches FIEs in a direct way and provides an exact solution in closed form for the given IE. It is important to note that we can apply this method for the separable and degenerate kernels [28,38,39], of the form
$$\mathrm{K}(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \quad (33)$$
The DCM can be applied as following manner:

- At the beginning of this method we substitute (33) into the FIE of the form
$$\varphi(x) = \psi(x) + \int_a^b \mathrm{K}(x,t)\varphi(t)\,dt. \quad (34)$$
- After from the substitution of kernel (33) in FIE (34) we have the following form for equation (34)
$$\varphi(x) = \psi(x) + g_1(x)\int_a^b h_1(t)\varphi(t)dt + g_2(x)\int_a^b h_2(t)\varphi(t)\,dt + \cdots + g_n(x)\int_a^b h_n(t)\varphi(t)dt. \quad (35)$$
- In view of equation (35) all of the integrals at the RHS depends only on the variable t with constant bounds of integration for variable t. This show that every integral at the RHS of (35) is equal to a constant value. Based on this, equation (35) becomes
$$\varphi(x) = \psi(x) + \lambda\alpha_1 g_1(x) + \lambda\alpha_2 g_2(x) + \cdots + \lambda\alpha_n g_n(x), \quad (36)$$
wherever
$$\alpha_i = \int_a^b h_i(t)\varphi(t)dt,\ 1 \le i \le n. \quad (37)$$
- If we substitute the equation (36) into equation (37), we obtain a system of n algebraic equations that may solved for determination of α_i, $1 \le i \le n$. Setting the achieved numerical values of α_i into equation (36) the solution $\varphi(x)$ of the FIE (34) is facilely obtained[38].

To accede our aim of the comparison between above discussed methods, we illustrate this comparison by discussion on the solution of following FIE, applying the above methods.

Example: We want to study the solution of the following integral equation by above methods.
$$\varphi(x) = \frac{5}{6}x + \frac{1}{2}\int_0^1 xt\varphi(t)\,dt \quad (i)$$

Method of the Fredholm Determinant: As we discussed before this is necessary to find B_n, C_n by using the equations (12) and (13).
Here $\psi(x) = \frac{5}{6}x$ and $\lambda = \frac{1}{2}$
$$B_0(x,t) = xt,\ B_1(x,t) = \int_0^1 \begin{vmatrix} xt & xt_1 \\ t_1 t & t_1 t_1 \end{vmatrix} dt_1 = 0$$
$$B_2(x,t) = \int_0^1\int_0^1 \begin{vmatrix} xt & xt_1 & xt_2 \\ t_1 t & t_1 t_1 & t_1 t_2 \\ t_2 t & t_2 t_1 & t_2 t_2 \end{vmatrix} dt_1 dt_2 = 0,\ B_n = 0$$
for $n = 1,2,3,\ldots$
$$C_0 = 1,\ C_1 = \int_0^1 t_1^2\,dt_1 = \frac{1}{3}t_1^3\Big|_0^1 = \frac{1}{3},$$
$$C_2 = \int_0^1 \begin{vmatrix} t_1 t_1 & t_1 t_2 \\ t_2 t_1 & t_2 t_2 \end{vmatrix} dt_1 dt_2 = 0,\ C_n = 0 \text{ for all } n \ge 2$$
Now in view of equations (10) and (11) we have
$$D(x,t;\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n(x,t)\lambda^n = \mathrm{K}(x,t) = xt \quad (ii)$$
$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} C_n\,\lambda^n = 1 + (-1)\frac{1}{3}\frac{1}{2} = \frac{5}{6} \quad (iii)$$
From equation (9)
$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} = \frac{xt}{\frac{5}{6}} = \frac{6}{5}xt \quad (iv)$$
Finally, the solution of equation (i) is
$$\varphi(x) = \frac{5}{6}x + \frac{1}{2}\int_0^1 \frac{6}{5}xt\,\frac{5}{6}t\,dt = \frac{5}{6}x + \frac{1}{2}x\int_0^1 t^2\,dt = \frac{5}{6}x + \frac{1}{2}x\left(\frac{1}{3}t^3\right)_0^1 = \frac{5}{6}x + \frac{1}{6}x = x \quad (v)$$
The Adomian Decomposition Method: In ADM, we have to substitute the initial component φ_0 instead of $\psi(x)$, therefore we have
$$\varphi_0(x) = \frac{5}{6}x$$
Using this value of initial component to find next component $\varphi_1(x)$ by
$$\varphi_1(x) = \frac{1}{2}x\int_0^1 \frac{5}{6}t^2 dt$$
$$\Rightarrow\ \varphi_1(x) = \frac{5}{6^2}x$$
Continue the process on this manner we can get $\varphi_2(x) = \frac{5}{6^3}x$, and so on
Remember that from decomposition method we know that

$\varphi(x) = \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \cdots,$
So that
$$\varphi(x) = \frac{5}{6}x\left(1 + \frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \cdots\right)$$
This provides the exact solution as
$\varphi(x) = x$

The Successive Approximations Method: Here in SAM we have to choose the initial approximation as $\varphi_0(x) = 0$. If we follow the process discussed in SAM, the remaining approximations of $\varphi(x)$ can be find as
$$\varphi_1(x) = \frac{5}{6}x$$
$$\varphi_2(x) = \left(\frac{5}{6} + \frac{5}{6^2}\right)x$$
$\varphi_2(x) = \left(\frac{5}{6} + \frac{5}{6^2} + \frac{5}{6^3}\right)x$, and so on
Finally, the nth approximation can be obtained as
$$\varphi_2(x) = \left(\frac{5}{6} + \frac{5}{6^2} + \frac{5}{6^3} + \frac{5}{6^4} + \cdots + \frac{5}{6^n}\right)x,\ n \geq 1$$
$$\Rightarrow \varphi(x) = \lim_{n\to\infty} \varphi_n(x)$$
$$= \lim_{n\to\infty} \frac{5}{6}x\left(1 + \frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4} + \cdots\right)$$
$$= x \qquad (vii)$$

Direct Computational Method: In view to discussion on DCM, we substitute
$$\alpha = \int_0^1 t\varphi(t)\,dt \qquad (viii)$$
If we set the above value of α in to equation (i), we have
$$\varphi(x) = \left(\frac{5}{6} + \frac{1}{2}\alpha\right)x \qquad (ix)$$
For determination of the value of α, we substitute the above value of $\varphi(x)$ in equation $(viii)$ then we have
$$\alpha = \int_0^1 \left(\frac{5}{6} + \frac{1}{2}\alpha\right)t^2 dt$$
$$= \frac{1}{3}\ (x)$$
Substituting the value of α from (x) in to equation (ix) yields
$$\varphi(x) = x \qquad (xi)$$
which is the exact solution of equation (i).

From above computation we see that the solution in each method is same to the solution of other one but the process and solution manner are different. At the end of paper, we want to summarize and conclude the above methods as follow

4. Conclusion

Now that we have completed the mathematical analysis of the technics that evaluate the FIEs, now we are ready to compare the studied methods. When selecting a preferable method among the studied technics for linear FIEs, we cannot to mention a particular method. On the other hand, we can say that if the kernel of FIE is the algebraic function of degree n or is the resolvent kernel it is profitable to use Fredholm determinant method. We also found that if the kernel $K(x,t)$ of the IE is separable and is a monomial or binomial, the DCM might be the finest selection since it gives the exact solution for IE with the slightest volume of computations. For other forms of kernels, and if in addition the inhomogeneous part $\psi(x)$ is a polynomial of three or more terms we found that the ADM, is verified to be more active, unfailing and yields a rapid convergent series for the solution. The series achieved with the decomposition method can provide the solution in an exact form or we can provide very precise approximation using a shortened series for actual problems. It is valuable to remember that the decomposition method expands $\varphi(x)$ around a function, rather than a point as in Taylor theorem. To match the methods of ADM and MDM with the SAM, it is obvious that the decomposition methods is simple and very stress-free since in decomposition methods we integrate some terms to get the successive components, while in the other method we integrate several terms to calculate the successive approximations after choosing the initial approximation. Both technics provides the solution as a series. It should be noted that, in this paper, we introduced just five methods to handle FIEs, but there are some traditional methods that are using to handle FIEs.

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