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Commutation Properties of Dilation

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Commutation properties are not only restricted to the traditional operators rather it also hold for dilations. In the following we have established some interesting commutations of dilations. If T ε B (H) and S ε B (H) where H and H are Hilbert spaces and if S be a dilation of T then we have also established the results that T commutes with S on H.

1. Preliminaries

Let Hand H_1 be Hilbert spaces and let T ε B (H) and S ε B (H1. We define T=pr (S) and read as T is projection of S in H, if (i) H is a subspace of H_1 and (ii) (Tx, y) = (Sx, y) for all x, y ε H. If T^8=pr S^n forn=1, 2.... then S is called a dilation of T.

The following result on dilation is due to Singh [2] which is useful in deriving further results.

Theorem (1.1): Let H and H_1 be Hilbert spaces. Let T ε B (H) and S ε B (H1).

Then S is a dilation of i: if and only if Tx=PSx for all x£ H, where P is the orthogonal projection of H_1 onto H.

2. Results

In the following we give a related result is the form of necs. and sufficient condition.

Theorem (2.1): Let H and H_1 be Hilbert spaces where H H_1 Let T ε B (H) and S ε B (H1), then S is a dilation of T if and only if S* is a dilation of T*.

Proof: First we suppose that S is a dilation of T. Then we have by Theorem (1. 1)

Tx=PSx for all x ε H,

Where H is a subspace of H1 and P is the orthogonal projection of H_1 onto H. We have for x, y ε H (Tx, y) = (PSx, y) => (x, T*y) = (Sx, P*y)

=> (x, T*y) = (Sx, Py) = (Sx, y), (as y ε H)
=> (x, T*y) = (x, S*y) = (x, PS*y), (asS*y ε H)
=> T*y=PS*y, for all y ε H.

By theorem (1. 1), it implies that S* is a dilation of T*.

Conversely, we suppose that S* is a dilation of T*, then we have T*X=PS*x

For all x ε H, where P is the orthogonal projection of H1 onto H. Hence H H_1 and we have for all x, y ε H,

(T*x, y) = (PS*x, y) => (T*x, y) = (S*x, P*y)
=> (T*x, y) = (S*x, y), (as P*y=Py=y for all y ε H)

=> (x, Ty) = (x, Sy) = (x, PSy), (as S y ε H)

Ty =PSy, for all y ε H which shows that S is a dilation of T.

We have the following commutation properties of dilation:

Theorem (2. 2): Let T, A ε B (H) and S ε B (H1).

Let S be a dilation of T then

[A, T]=O if and only if [A, S]=O
[A, T*]=O if and only if [A, S*]=0.

Proof: Since S is a dilation of T, we have

Tx =PSx, …… **(2. 1)**

For all x ε H where P is a northogonal projection of H_1 onto H.

Let [A, T]=0 then At=TA. For x ε H, we have

APS (x) =AT (x) =TA (x) =T (h), where A (x) =h ε H
APS (x) =PS (h) [by (l. l) j
=PSA (x)
(APSx, y) = (PSA, y), for all y ε H (PSx, A*y) = (SAx, P*y) = (SAx, y)
(Sx. P*A*y) = (SAx, y)
(Sx, A*y) = (SAx, y), [·:A*:>EHJ,
(ASx, y) = (SAx, y), for all x, y ε H.
=> AS (x) =SA (x) for all x ε H, i.e., AS=SA i.e., [A, S]=0.

Conversely, we suppose that [A, S]=0, i.e., AS=SA. For x ε H, we have TA (x) =T (h) where A (x) =h ε H. It implies that

TA{x) =PS (h), [by 2. 1)
TA (x);:::PSA (x) [·: A (x) =h]
TAx, y) = (PSAx, y), for all x, y ε H
(TAxy) = (PASx, y), [·: AS=SA]
=> (TAx, y) = (ASx, P*y) = (ASx, Py) =ASx, y)
= (Sx, A*y)
= (Sx, h), where A*y=ht:H
= (Sx, P*h) = (PSx, h)
= (PSx, A*y) =APSx, y)
= (ATx, y) [·: PSx=Tx, for all x ε HJ
=> TA (x) =AT (x) for all x ε H, i.e.,

TA=AT, i.e. [A, TJ=0:

Since S is a dilation of T we have from (2. 1)

T=P T*=S*P.

First we suppose that [A, T*]=0, i.e., Af*=T*ATA*=A*T. For all x, y ε H we have

(SA*x, y) = (SA*x, Py) [∵ Py=y as y ε HJ

= (P*SA*x, y) =PSA*x, y) ….. **(2. 2)**

(PSh, y), where A*x=h ε H
(Th, y), (by (2. 1))
(TA*x, y) = (A*Tx, y)
(Tx, Ay) = (PSx, Ay) = (Sx, PAy)
(Sx, Ay) = (A*Sx, y)

=> SA*=A*S=> AS*=S*A, i.e., [A, S*]=O.

Conversely, we suppose that [A, S*]=0. For all x, y ε H we have (T*Ax, y) = (Ax, Ty) = (Ax, I'Sy)

= (P*Ax, Sy) = (P*h, Sy), (where AX=h ε H)
= (Ph, Sy) = (h, Sy), (∵Ph=has h ε H)
= (S*h, y) = (S*Ax, y)
= (AS*x, y) = (S*x, A*y)
(x, SA*y) = (x, Sh), (where A*y=h_1= (Px, Sh_1) = (S*Px, h_1 ε H)
= (S*Px, A*y) = (T*x, A*y), (∵ S*P=T*)
= (AT*x, y) =>T*A =AT*, i.e., [A, T*]=0.

Theorem (2. 3): Let T ε B (H), S ε B (H_1) and let S is a dilation of T, then

i) T*S=S*T,
ii) TS =ST,
iii) (T+T*) S= (S+S*) T

Proof: Since S is a dilation of T, by Theorem (1. 1) we have Tx=PSx for all x ε H where P is an orthogonal projection of H1 onto H. It follows that T=PS=>T*=S*P.

We have

i) S*T=S*PS and T*S=S*PS
=> S*T=T*S.

ii) For x, y ε H, we have
(TSx, y) = (Sx, T*y) = (Sx, S*Py)
= (S*x, Py) = (S^2x, y), (y ε H)
= (Sx, S*y) = (h, S*y), (where Sx=h ε H)
= (Ph, S*y) = (PSx, S*y)
= (Tx, S*y) = (STx, y) =>TS=ST.

iii) For x, y ε H, we have
((T+T*) Sx, y) = (TSx, y) + (T*Sx, y)
= (STx, y) + (S*Tx, y) [by (i) and (ii)]
= ((S+S*) Tx, y)
=> (T+T*) S = (S+S*) T.

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