International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

Mathematical Modelling of Fluid Flow in an Open Channel with an Elliptic Cross-Section

Mose Isaac[1], Johana Kibet Sigey[2], Jeconia Abonyo Okelo[3], Emmah Marigi[4]

[1]Pure and Applied Mathematics Department, Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya

[2]School of Science, Department of Mathematics and Physical Science, Dedan Kimathi University of Science and Technology (DeKUT), Nyeri, Kenya

Abstract: ***This study was essentially concerned with fluid flow in an open channel with an elliptic cross-section. The fluid flow considered uniform unsteady Newtonian fluid. The flow parameters such as hydraulic radius, friction slope and channel slope were studied and their effects on fluid velocity and fluid depth. The centered finite difference numerical method of approximation was used to solve the continuity and momentum governing equations using the MATLAB computational method. The results obtained from the investigation were presented in tabulated and graphical form. The findings were that, an increase in the hydraulic radius caused the fluid flow depth to increase. The frictional force on the walls of the channel caused retardation of the smooth flow of water molecules and as a result, the velocity was decreasing systematically.***

Keywords: Elliptic channel, velocity, depth, open channel

1. Introduction

Most fluids possess a unique property that they cannot be compressed under ordinary conditions. Effects of climate change coupled with increase in human population have posed challenges in water harvesting and control especially in urban areas. The challenges include excesses, shortages and quality of water. They are more pronounced in urban centers located in low lying geographic areas where storm waters occur whenever it rains. Sewers and the overland flow network require proper simulation to enhance its proper gathering and disposal or reuse. For instance, storm waters due to heavy rains causes flooding; traffic jams in our roads, blockage of sewers and also breaking of water pipes carrying water for domestic and industrial use. Also, silt contained by the water at high speed causes wear and tear on the channel beds leading to formation of ditches that hampers the smooth flow of water. An elliptic conduit that is horizontal provides a wider surface for the collection of the excess water becoming economical especially where vertical clearance and minimal cover conditions are expected. Transportation of water, oil and gas is done through a network of circular ducts, at times over long distances. The choice of transporting fluids in circular conduits is due to their ability to withstand large pressure differences on their surfaces without any distortion. If the conduit is fully filled by a fluid, then the flow is referred to as closed flow which is driven by pressure difference otherwise it is open channel flow that is driven by the gravitational force. In this case the channel was expected to be able to convey maximum discharge under fixed flow parameters at minimal costs. The horizontal elliptic channel cross-section considered had a reduced depth but, its top width was enlarged as these results from the fact that an ellipse is a circle which has been squashed. The destructive power of water can be reduced if the flow channel is well designed taking into account the flow parameters that determine velocity of fluid flow such as slope, top width as well as flow depth. Great heights also causes pressure to build that can result to rapture of channel walls leading to water surge that causes havoc in the environment. Jomba [3] investigated fluid flow in an open channel with a horseshoe cross-section. From the study he established that for a fixed flow area, the flow velocity increases as depth increases towards the free stream. Also he established that an increase in hydraulic radius and roughness coefficient results to a reduction of velocity due to increased shear stresses. Ojiambo [9] did an investigation that focused on unsteady non-uniform flow on open channels with circular cross-section. The findings were that an increase in the cross-sectional area and depth of flow leads to decrease in the flow velocity. An increase in the lateral inflow rate per unit length of the channel leads to a decrease in the flow velocity. Marangu [7] did an investigation on open channel fluid flow having a trapezoidal cross-section and a segment base and found out that increase in cross-sectional area, channel radius and flow depth leads to a decrease in flow velocity. Also he established that an increase in the bed slope causes an increase in flow velocity. Omari [10] did an investigation on closed channel with circular cross-sectional area. The results obtained showed that an increase in the cross sectional area of sewer flow results to a decrease in the sewer depth. It was observed that a decrease in the friction slope leads to an increase in the sewer flow velocity. Also it was found out that an increase in tunnel angle of inclination results to an increase in sewer velocity. Tsombe [13] investigated flow in open channels having circular cross-sectional area. He found out that when the flow depth increases, it results to reduced fluid velocity. Also reduction in slope leads to a decrease in flow velocity. .Macharia [6] from where he found out that an increase in the angle of the lateral inflow channel does not necessarily mean an increase in the velocity in the main open channel. Also increasing the cross-sectional area of the lateral inflow channel leads to decrease in the flow velocity in the main open channel. Increasing the velocity in the lateral inflow channel leads to an increase in the flow velocity in the main open channel. Adepoju and Nalluri [8] carried out an experiment on the magnitude of resistance to flow in smooth channels of circular cross-section. They established that the measured resistance forces were bigger than those of a pipe of the same radius. They established that the shape effects are well investigated when flow depth is compared to wetted perimeter than relying on a single parameter like the hydraulic radius, R. Sinha and Meena [11] investigated the laminar flow of a viscous incompressible fluid in a straight circular pipe. From the result they observed that the velocity is higher at the entry point compared to the downstream flow. Kwanza [5] investigated the effects of channel slope, lateral discharge and width of the channel on fluid velocity and discharge for both rectangular and trapezoidal channels. He observed that increase on channel slope and lateral discharge led to an increase in fluid velocity and discharge. Thiong'o [12] did an investigation on effects of channel slope, flow depth, top width and roughness coefficient on fluid velocity and they found out that an increase in roughness coefficient, leads to a decrease in fluid velocity. Also fluid velocity increases with depth and becomes maximum slightly below the free surface. This study was conducted to investigate the effects of hydraulic radius on fluid flow depth of an open channel having an elliptical cross section and also the effects of varying channel slope and friction slope on fluid flow velocity. The design of culverts, water pipes and sewerage pipes have to take into account the escalating climatic changes that have led to increased floods, storm waters as well as overland runoff, hence providing the area of application of the result obtained from the study. The objectives of this study were to determine the effect of variation of the hydraulic radius on flow depth, investigate the effect of channel slope and friction slope on fluid velocity in an elliptic conduit.

2. Mathematical Analysis

Governing equations

The equation of continuity and momentum are used to solve the partial differential equations in this study. The equation of continuity is derived from the Navier-stoke's equation which is given as,

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \qquad (1)$$

The resultant equation could be very difficult to solve. Hence for practical purposes, the spatial variations in lateral and transverse directions are neglected, Tuitoek and Hicks [14].

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0 \qquad (2)$$

By the law of mass conservation, equation (2) becomes;

$$\frac{\partial \rho A dx}{\partial t}=\rho(AV+qdx)-\rho(AV+\frac{\partial AVdx}{\partial x}) \qquad (3)$$

The flow is unsteady and has no lateral inflow. Also flow area is a function of depth where, T=2a=$\frac{\partial A}{\partial y}$,

Given that from from Chezy formula, $V=C\sqrt{RS}$ equation (3) can be simplified to give the equation of continuity governing our flow problem as;

$$\frac{\partial y}{\partial t}+\frac{\pi b}{4}\frac{\partial V}{\partial x}+C\sqrt{RS}\frac{\partial y}{\partial x}=0 \qquad (4)$$

Cecen [1]

The equation of momentum is derived from Newton's second law of motion and is given as;

$$\rho\frac{\partial}{\partial x}\beta Q\mathrm{v}dx+\rho\frac{\partial Q}{\partial t}dx=(\rho gA\mathbf{S}_0-\rho gA\frac{\partial y}{\partial x}-\rho gA\mathbf{S}_f)dx \qquad (5)$$

Since the flow is unsteady and the momentum coefficient, $\beta=1.0$ for straight prismatic channels, the equation reduces to;

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}+g\frac{\partial y}{\partial x}=g\left(S_o-S_f\right) \qquad (6)$$

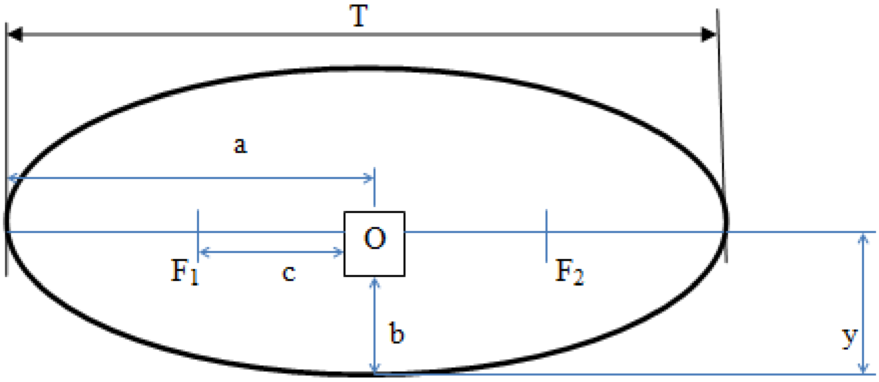


Figure 1: Cross - section of an elliptic conduit

Equations (4) and (6) are subjected to discretization to obtain the CDS, Viessman [15] to give;

$$70\sqrt{Ry_{i+1,j}}-70\sqrt{Ry_{i-1,j}}-1400\,y_{i,j}=55bV_{i-1,j}-55bV_{i+1,j}-1400\,y_{i,j+1} \qquad (7)$$

and $$-50V_{i,j}+V_{i+1,j}-V_{i-1,j}=-50V_{i,j+1}+10y_{i-1,j}-10y_{i+1,j}+5(S_0-S_f) \qquad (8)$$

respectively which are then solved using the initial and boundary conditions;

V(0, t)=5 and y(0,t)=1.0 for all t > 0
V(x, t) = 5 and y(x, t) = 1.0 for all t > 0
V (0, t) = 2, y (0, t) = 2 for all t > 0
V(x, t) = 0 and y(x, t) = 0 for all t > 0

3. Results and Discussion

Effects of hydraulic radius on flow depth

Table 1: Elliptic pipe length for varying hydraulic radius

Pipe length , x **Radius (m)**	**0**	**1**	**2**	**3**	**4**	**5**
R = 4	0.656967	1.024957	0.999341	0.999262	0.990162	0.900983
R = 9	0.706477	1.029055	0.999326	0.999304	0.990167	0.900983
R = 16	0.755987	1.033153	0.999511	0.999316	0.990174	0.900983
R = 25	0.814691	1.040340	0.999987	0.999366	0.990177	0.900983

The results for varying elliptic pipe hydraulic radius and its effects on flow depth is presented in the figure 2 below

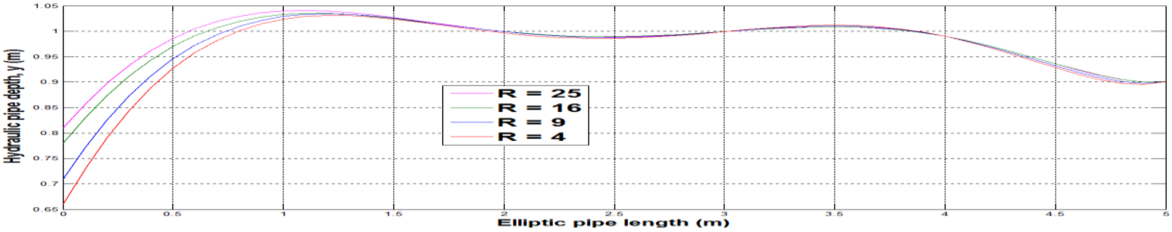


Figure 2: Fluid flow depth against elliptic pipe length

From figure 2, we observed that increasing the hydraulic radius resulted to an increase in the flow depth between the elliptic pipe lengths 0m to 1m. The flow depth then assumed a sinusoidal wave. Hydraulic radius increases with increase in flow depth since depth is a function of area and the initial velocity was found to be relatively high and as a result corrosion effects were higher.

Effects of friction of elliptic pipe on fluid velocity

Table 2: Elliptic pipe length for varying elliptic pipe friction slope

Pipe length **Slope m**	**0**	**1**	**2**	**3**	**4**	**5**
S_f =0.01	0.557396	-0.015105	-0.000229	-0.000823	-0.000798	-0.000768
S_f =0.02	0.458178	-0.015552	-0.000627	-0.001223	-0.001197	-0.001152
S_f =0.03	0.357804	-0.014913	-0.000028	-0.000623	-0.000598	-0.000576
S_f =0.04	0.257346	-0.016351	-0.001427	-0.002023	-0.001996	-0.001920

The results for varying elliptic pipe friction slope on fluid velocity is presented in the figure 3 below

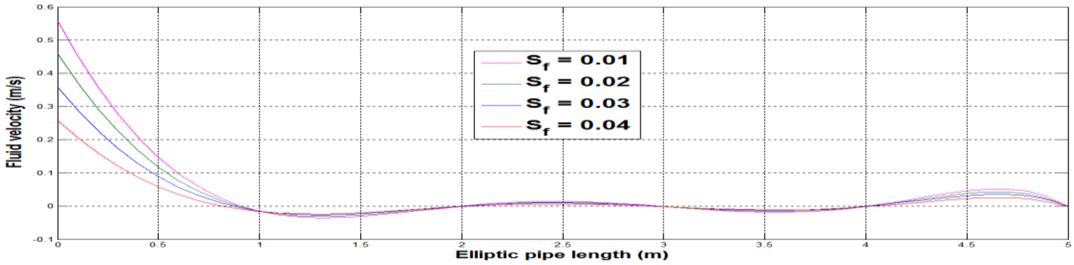


Figure 3: Fluid velocity against elliptic pipe length at varying friction slope

It was seen from figure 3 that an increase in channel friction resulted to a decrease in fluid velocity. This is because of the shear forces that exert resistance to the smooth flow of water. However, the accumulation of eroded particles causes a rise incenter of gravity resulting to instability in the fluid particles. Similarly fluid particles always bombard each other and therefore their individual velocities are never uniform along the flow. Hence the velocity changes in a wave form as shown in the graph. In order to maintain a specific fluid velocity, then the friction forces along the channel should be varied.

4. Discussion

The deposition of the accumulated eroded particles interferes with the smooth flow of the fluid downstream.

These particles reduce both the velocity of water and the flow depth. Therefore, as the length of the elliptic pipe increases, the passage becomes narrower.This shows that an increase in the depth of flow causes an increase in the flow rate of the fluid particles towards the free stream. When a fluid is flowing through a conduit, friction force between the conduit and the fluid causes a pressure loss. Fluids have viscosities that make different layers to flow at different velocities whereby the fluid directly in contact with the surface is held to it by adhesive forces. This force retards the flow which in turn retards the flow of the adjacent layer. Variation in velocity along the channel results from pressure difference, where the pressure and velocity are inversely related.

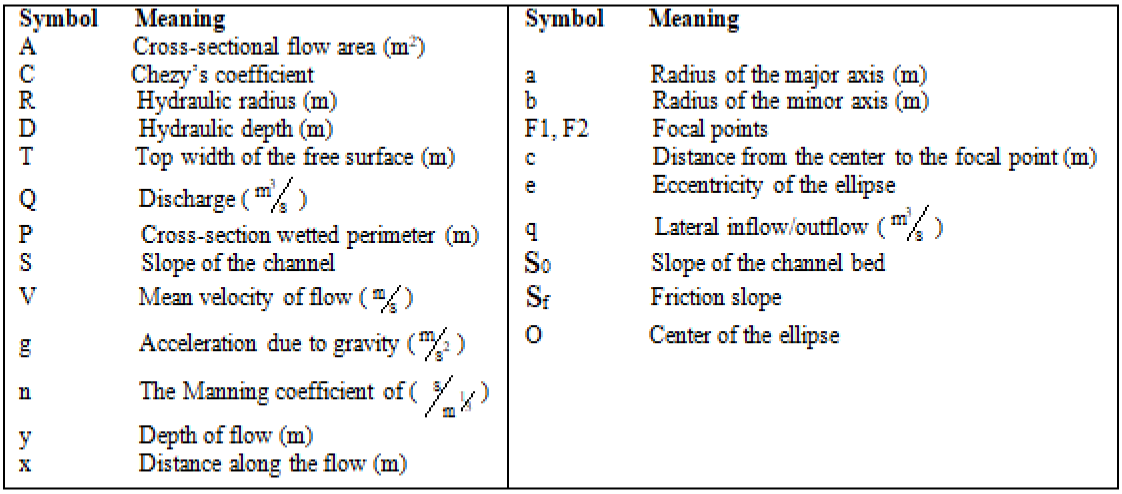
5. Conclusion

The study was conducted over the effects of hydraulic radius on fluid flow depth of an open channel and also the effects of varying channel slope and friction slope on fluid flow velocity. The following conclusions were drawn from the results obtained. When the hydraulic radius increases, there is an increase in the fluid flow depth since depth is a function of area. The fluid flow depth reduces along the channel due to the accumulation of eroded particles which consequently reduces the fluid velocity. Variation of friction slope, also affects flow velocity. When friction is raised, flow velocity is reduced. Friction arises from the shear forces on the walls and channel bed which offers resistance to the smooth flow of water. The results obtained from this study were found to be in line with what other researchers have investigated and found. For instance, Jomba [3] who investigated fluid flow in an open channel with a horseshoe cross-section. His results showed that an increase in depth leads to an increase in flow velocity towards the free surface. Similarly, Omari [10] who did an investigation on closed channel with circular cross-sectional area. His result was that as the angle of inclination increases, flow velocity increases. The results also showed that when the friction slope is higher, the flow velocity becomes less along the elliptic pipe length. Further investigations can be done on the study with consideration of unsteady non uniform fluid flow, elliptic conduit placed vertically as well as subjecting the project to experimental tests.

6. Acknowledgement

I wish to extend my appreciation to the Jomo Kenyatta University of Agriculture and Technology family as a whole and to my supervisors; Prof. Johana Kibet Sigey (JKUAT), Dr. Emmah Marigi (DeKUT) and Prof. Jeconia Okelo Abonyo (JKUAT) for their continued guidance in preparing the project. My sincere gratitude also goes to my course mates and friends for their technical, moral encouragement and continuous support during the entire process.

Nomenclature



Symbol	Meaning	Symbol	Meaning
A	Cross-sectional flow area (m^2)		
C	Chezy's coefficient	a	Radius of the major axis (m)
R	Hydraulic radius (m)	b	Radius of the minor axis (m)
D	Hydraulic depth (m)	F1, F2	Focal points
T	Top width of the free surface (m)	c	Distance from the center to the focal point (m)
Q	Discharge (m^3/s)	e	Eccentricity of the ellipse
P	Cross-section wetted perimeter (m)	q	Lateral inflow/outflow (m^3/s)
S	Slope of the channel	S_0	Slope of the channel bed
V	Mean velocity of flow (m/s)	S_f	Friction slope
g	Acceleration due to gravity (m/s^2)	O	Center of the ellipse
n	The Manning coefficient of ($s/m^{1/3}$)		
y	Depth of flow (m)		
x	Distance along the flow (m)		

References

[1] Çeçen K. (1982)‘‘Hydraulic’’. *İstanbul Teknik Üniversitesi (I.T.U).*

[2] Chow V.T (1959) ‘‘Open channel hydraulics’’. *McGraw hill book Co*, New York, 2-40.

[3] Jomba J, Theuri D.M, MWENDA.E, CHOMBA.C (2015) “Modeling fluid flow in open channel with horseshoe cross-section”. *International Journal of Engineering and Applied Sciences***5.**

[4] Khan A.A. (2000)“Modeling flow over an initially dry bed”. *Journal of Hydraulic Research*.,**5**,38.

[5] Kwanza J.K, Kinyanjui M.N, NKoroi J.M (2007) “Modeling fluid flow in rectangular and trapezoidal open channels”. *Advances and applications in fluid mechanics***2**,149-158.

[6] Macharia K, Theuri D and Kinyanjui M.N. (2014) “Modeling Fluid Flow in an Open Rectangular Channel with Lateral Inflow Channel”. *International Journal of Sciences: Basic and Applied Research* (IJSBAR) **17**, (1), 186-193.

[7] Marangu P.K, Mwenda E, Theuri D.M (2016) “Modeling Open Channel Fluid Flow with Trapezoidal Cross Section and a Segment Base”. *J Appl Computat Math***5**, 292.

[8] Nalluri C. and Adepoju B.A. (1985)“Shape effects on resistance to flow in smooth channels of circular cross section”. *Journal of Hydraulic Research*, **23**, 37-46.

[9] Ojiambo V.N, Kinyanjui M. N, Theuri D.M. Kiogora P.R. Giterere K (2014) "Modeling Fluid Flow in Open Channel with Circular Cross-section", *International Journal of Engineering Science and Innovative Technology*, **3** (5).

[10] Omari P.I**,** Sigey J.K, Okelo J.A and Kiogora R.P (2018) "Modeling circular closed channels forSewer lines".*International Journal of Engineering Science and Innovative Technology* (IJESIT),**7**(2).

[11] Sinha P.C and Meena Aggarwal (1982) "Entry flow in a straight circular pipe", *Journal of Australian Mathematics* (Series BG**),** 59-66.

[12] Thiong'o J.W, D.P, Kinyanjui M.N, and Kwanza J.K, (2011) "Modeling fluid flow in open rectangular and triangular channels". *Jagst College of pure sciences* (304).

[13] Tsombe D.P, Kinyanjui M.N, Kwanza J.K, Giterere K (2011) "Modeling fluid flow in open channel with circular cross-section". *Jagst College of pure sciences***13**. 80-91.

[14] Tuitoek D.K. and Hicks F.E. (2001)"Modeling of unsteady flow in compound channels".*African Journal Civil Engineering*. **4**, 45-53.

[15] Viessman W, Jr. Knapp J.W. Lewis G. L., and Harbaugh T.E. (1972). "Introduction to hydrology", *Second edition, Harper and Row, New York*, 1-60.

[16] Weisstein E.W (2018) 'Ellipse'.*From mathworld-A wolframweb resource. http://mathworld.wolfram.com/Ellipse.html*

Author Profile

Mose Isaac: Mr. Mose holds a B. Ed (Science) degree in Mathematics and Chemistry from Egerton University of 2005. He is working on his final project for the requirement of MSc in Applied Mathematics at Jomo Kenyatta University of Science and Technology (JKUAT), Kisii CBD Campus, Kenya. He is currently the Administrative assistant in charge of timetabling at Kisii University, Kenya. His area of interest is applied mathematics.

Johana K. Sigey: Prof. Sigey holds a Bachelor of Science degree in mathematics and computer science First Class honors from Jomo Kenyatta University of Agriculture and Technology, Kenya, Master of Science degree in Applied Mathematics from Kenyatta University and a PhD in applied mathematics from Jomo Kenyatta University of Agriculture and Technology, Kenya. Affiliation: Jomo Kenyatta University of Agriculture and Technology, (JKUAT), Kenya. He is currently the Director, JKuat, and Kisii CBD. He has been the substantive chairman - Department of Pure and Applied mathematics –Jkuat (January 2007 to July- 2012). He has published 40 papers on heat transfer, MHD and Traffic models in respected journals. Teaching experience: 2000 to date- postgraduate programme: (JKUAT); Supervised student in Doctor of philosophy: thesis (6 completed, 10 ongoing); Supervised student in Masters of Science in Applied Mathematics: (45 completed, 10 ongoing).

Jeconia Okelo Abonyo: Prof Okelo holds a PhD in Applied Mathematics from Jomo Kenyatta University of Agriculture and Technology as well as a Master of Science degree in Applied Mathematics and first class honors in Bachelor of Education, Science; specialized in Mathematics with option in Physics, both from Kenyatta University. He has dependable background in Applied Mathematics in particular fluid dynamics, analyzing the interaction between velocity field, electric field and magnetic field. He has published over Forty four papers in international Journals. He is a Professor in the Department of Pure and Applied Mathematics and Assistant Supervisor at Jomo Kenyatta University of Agriculture and Technology

Emmah Marigi. Dr.Marigi holds PhD in Applied Mathematics (JKUAT-2013), Msc in Applied Mathematics (JKUAT-2007) and B.Ed (Science) from KU-1987. Her Research Interest is Modelling Mathematics and area of expertise is Applied Mathematics. Her responsibilities include: Senior Lecturer (DeKUT-2013 to date), Dean of Science (DeKUT-2012 to date), Head of Mathematics (2012), Examination Coordinator (2010-2012). She has done five Publications in the Applied Mathematics. She has done postgraduate Supervision (one PhD-on going and Msc one completed - in 2014 and five on-going).