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Topological Venn Diagrams and Graphs

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Abstract: ***This paper is an attempt to expound Venn diagrams and its applications in Topology and Graph Theory. John Venn introduced Venn diagrams in 1880. Venn diagrams can be called as Eulerian Circles. Euler invented this in the 18th century. Venn diagram consists of curves and circles. Venn diagrams are similar to Euler diagrams. Edwards discusses the dual graph of a Venn diagram is a maximal planar subgraph. Edwards-Venn diagrams are topologically equivalent to diagrams with graphs. 2D and 3D Venn diagrams consist of labeled simple closed curves. 3D Venn diagrams and 3D Euler diagrams are all combinations of surface intersections. In brief this paper will give a vivid picture of Venn diagrams and its close relationship with Topology and Graph Theory.***

Keywords: Venn diagram–Euler diagram – planar Venn diagram – Topologically faithful Venn diagram – 2D and 3D Venn diagram

1. Introduction

A Venn diagram is a diagram that shows all possible logical relations between a finite collection of different sets. These diagrams depict elements as points in the plane, and sets as regions inside closed curves. A Venn diagram consists of multiple overlapping closed curves, usually circles, each representing a set. The points inside a curve labeled S represent elements of the set S, while points outside the boundary represent elements not in the set S. This lends to easily read visualizations, for example, the set of all elements that are members of both sets S and T, $S \cap T$, is represented visually by the area of overlap of the regions S and T. In Venn diagrams the curves are overlapped in every possible way, showing all possible relations between the sets. They are thus a special case of Euler digrams, which do not necessarily show all relations. Venn diagrams were conceived around 1880 by John Venn. They are used to teach elementary set theory, as well as illustrate simple set relationships on probability, logic, statistics, linguistics and computer science.

1.1 Venn Diagram

A Venn diagram in which the area of each shape is proportional to the number of elements it contains is called an area proportional or scaled Venn diagram.

Venn diagrams were named and introduced in 1880 by John Venn.

A simple closed curve in the plane is a non-self-intersecting curve, which, by a continuous transformation of the plane, is identical to a circle. This transformation is achieved when we stretch or shrink all or parts of the plane, without tearing, twisting or pasting it to itself [10].

An n-Venn diagram in the plane is a collection of simple closed curves $C=C_1, C_2,\dots, C_n$, such that each of the 2^n sets $X_1, X_2,\dots, X_n$ is a non empty and connected region, where each X_i is either the bounded interior or the unbounded exterior of C_i. This intersection can be uniquely identified by a subset of 1, 2, …, n, indicating the subset of the indices of the curves whose interiors are included in the intersection. Pairs of curves are assumed to intersect only at a finite number of points, meaning that intersections occur at points and not curve segments.

We say that two Venn diagrams are isomorphic if, by continuous transformation of the plane, one of them can be changed into the other or its mirror image [16].

A simple closed curve is convex if any two interior points can be joined by an interior line segment. A Venn diagram is convex if its curves are all convex. A potentially convex Venn diagram is isomorphic to a convex Venn diagram. Thus, a potentially convex Venn diagram's curves are not necessarily convex.

A planar graph can be drawn in the plane with edges, or curves, intersecting only at vertices. A Venn diagram V is a planar graph whose vertices, called Venn vertices, are the intersections of the curves, and whose edges are the line segments connecting these vertices [21]. A planar graph embedded in the plane is called a plane graph. The actual drawing V of the Venn diagram is a plane graph. The plane graph V is often called the Venn diagram. [9].

Example:

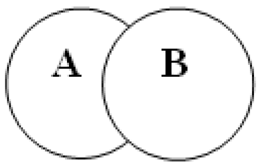

Figure 1: Sets A (creatures with two legs) and B (creatures that can fly)

This example involves two sets A and B, represented as circles. Set A, represents all living creatures that are two-legged. Set B, represents the living creatures that can fly. Each separate type of creature can be imagined as a point somewhere in the diagram. Living creatures that both can fly and have two legs-for example, parrots-are then in both sets, so they correspond to points in the region where circles overlap. It is important to note that this overlapping region would only contain those elements that are members of both set A and are also members of set B.

Humans and penguins are bipedal, and so are then in the set A, but since they cannot fly they appear in the left part of the set A, where it does not overlap with the set B. Mosquitoes have six legs, and fly, so the point for mosquitoes is in the part of the B that does not overlap with A. Creatures that are not two-legged and cannot fly would all be represented by points outside both circles.

The combined region of sets A and B is called the union of A and B, denoted by A∪B. The union in this case contains all living creatures that are either two-legged or that can fly.

The region in both A and B, where the two sets overlap, is called the intersection of A and B, denoted by A∩B. For example, the intersection of the two sets is not empty, because there are points that represent creatures that are in both the set A and B.

Venn himself did not use the term “Venn diagram” and referred to his invention as “Eulerian Circles” [17], [21].

Venn diagrams are very similar to Euler diagrams, which were invented by Leonhard Euler in the 18th century.

In the 20th century, Venn diagrams were further developed. D.W. Henderson showed in 1963 that the existence of an n-Venn diagram with n-fold rotational symmetry implied that n was a prime number. He also showed that such symmetric Venn diagrams exist when n is five or seven. In 2002 Peter Hamburger found symmetric Venn diagrams for n=11 and in 2003, Griggs, Killian and Savage showed that symmetric Venn diagrams exist for all other primes. Thus rotationally symmetric Venn diagrams exist if and only if n is a prime number [15].

Venn diagrams and Euler diagrams were incorporated as part of instruction in set theory as part of the new math movement in the 1960s.

The famous three-circle Venn diagram, which is known to most people, had already been used by Euler. Venn himself calls this diagram “Euler’s famous circles.” So why do we speak of Venn diagrams and not Euler diagrams? I believe there are two reasons. It was John Venn who first gave a rigorous definition of the notion (though he did not always follow it consistently); and he was the first to prove that the desired diagrams exist for any number of sets.

1.2 Planar Venn Diagram

A modern definition is this. A planar Venn diagram is a set of n closed non-self-intersecting continuous planar curves, intersecting each other in isolated points, and such that the connected components of the complement (which are bounded by unions of arcs of these curves) are 2^n in number. Then these regions can be assigned distinct binary codes, in the following manner. Label the curves 1, 2, …, n. If a region is inside the curve i, then write 1 in the ith place in its binary code, otherwise write 0. As the n-digit binary codes are exactly 2^n in number, the definition of Venn diagram means that they allow all the codes to be assigned to regions.

Branko Grunbaum wrote the following [8]: Venn diagrams were introduced by J. Venn in 1880 [19] and popularized in his book [20]. Venn did consider the question of existence Venn diagrams for an arbitrary number n of classes, and provided in [19] an inductive construction of such diagrams. However, in his better known book [20], Venn did not mention the construction of diagrams with many classes; this was often mistakenly interpreted as meaning that Venn could not find such diagrams, and over the past century many papers were published in which the existence of Venn diagrams for n classes is proved.

Edwards knows that Grunbaum showed in 1975 [8] the possibly of constructing Venn diagrams with any number of convex curves, and he recognizes that this was a remarkable advance, but he gratuitously disparages others figures in comparison to his own, and he misrepresents the history.

Grunbaum’s results is stronger than Edwards quotes: not only may all the n curves be chosen so that they are convex, but also so that the $2^n – 1$ interior intersection regions, and also their union, are convex.

A Venn diagram is called reducible if there is some one of its curves whose deletion results in a Venn diagram with one less curve. It is called simple if at every intersection at most two curves meet. It is known that there are irreducible Venn diagrams, and Edwards refers to this counterintuitive property with 5 curves can even be simple irreducible but Edwards says falsely that if a Venn diagram can be built up by adjoining n curves one by one, that determines its topological (graph theoretic) structure uniquely. Some of the reducible structures can be realized by curves all of which are convex, and some cannot; even among those which can, there are many graphically different ones. This richness is one of the attractions of the subject to the geometer.

Edwards discusses the dual graph, stating, “The dual graph of a Venn diagram is a maximal planar subgraph of a Boolean cube”. He says he realized this in 1990 but his paper on the subject was rejected. He cites a 1996 paper [2] says “the proof, though trivially short, assumes a knowledge of graph theory and is therefore omitted here”. Someone who would be daunted by a trivial proof in graph theory and yet can cope with maximal planar subgraphs of the hypercube!.

One of the most disturbing mistakes in the book is when Edwards presents an induction argument to prove a statement, namely, every Venn diagram can be colored with two colors such that no regions with common are boundary have the same color.

After publication of the fundamental there was a pause before the study of Venn diagrams was revived by Grunbaum [8] and Peter Winkler [22]. Their deep understanding and challenging conjectures have motivated more recent work. Let me mention two advances here. In [2] the authors show that it is possible to extend any planar

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Venn diagram to a planar Venn diagram with one more curve. In [6] the authors show that for every prime number p there is a planar Venn diagram with p curves and p-rotational symmetry. In both problems, it remains unknown whether the Venn diagrams can be chosen simple. The latter of these problems is surveyed by Barry Cipra Readers may consult an online, regularly updated survey. An accurate essay by M.E. Baron [1] gives the history of representations of logic diagrams upto the time of Venn.

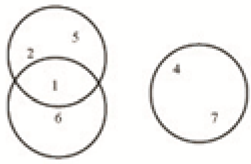
A Venn diagram is constructed with a collection of simple closed curves drawn in a plane. According to Lewis [12] the “principle of these diagrams is that classes or sets be represented by regions in such relation to one another that all the possible logical relations of these classes can be indicated in the same diagram. That is, the diagram initially leaves room for any possible relation of the classes, and the actual or given relation, can then be specified by indicating that some particular region is null or is not-null”.

Venn diagrams normally comprise overlapping circles. The interior of the circle symbolically represents the elements of the set, while the exterior represents elements that are not members of the set. For instance, in a two-set Venn diagram, one circle may represent the group of all wooden objects, while another circle may represent the set of all tables. The overlapping region or intersection would then represent the set of all wooden tables. Shapes other than circles can be employed as shown below by Venn’s own higher set diagrams. Venn diagrams do not generally contain information on the relative or absolute sizes of sets; i.e. they are schematic diagrams.

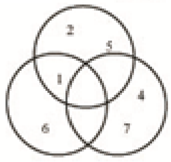
Venn diagrams are similar to Euler diagrams. However, a Venn diagram for n component sets must contain all 2^n hypothetically possible zones that correspond to some combination of inclusion or exclusion in each of the component sets. Euler diagrams contain only the actually possible zones in a given context. In Venn diagrams, a shaded zone may represent an empty zone, whereas in an Euler diagram the corresponding zone is missing from the diagram. For example, if one set represents dairy products and another cheeses, the Venn diagram contains a zone for cheeses that are not dairy products. Assuming that in the context cheese means some type of dairy product, the Euler diagram has the cheese zone entirely contained within the dairy-product zone-there is no zone for non dairy cheese. This means that as the number of contours increases, Euler diagrams are typically less visually complex than the equivalent Venn diagram, particularly if the number of non-empty intersections is small.

The difference between Euler and Venn diagrams can be seen in the following example.

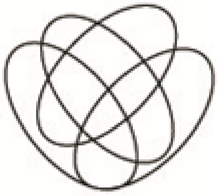
The Venn and the Euler diagram of those sets are:



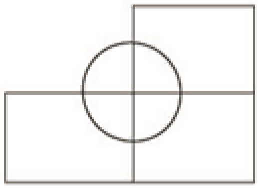
Euler diagram



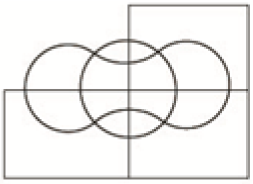
Venn Diagram



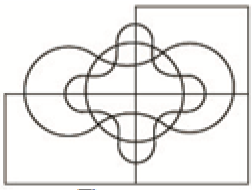
Venn’s four-set diagram using ellipses



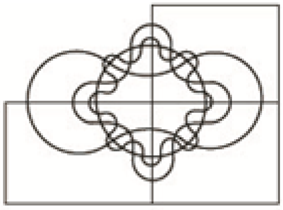
Three sets



Four sets



Five sets



Six sets

Anthony William Fairbank Edwards constructed a series of Venn diagrams for higher numbers of sets by segmenting the surface of a sphere, which became known as Edwards-Venn diagrams. For example, three sets can be easily represented by taking there hemispheres of the sphere at right angles (x=0, y=0 and z=0). A fourth set can be added to the representation by taking a curve similar to the seem on a tennis ball, which winds up and down around the equator, and so on. The resulting sets can then be projected back to a plane to give cogwheel diagrams with increasing numbers of teeth. These diagrams were devised while designing a stained-glass window in memory of Venn [4].

Edwards-Venn diagrams are topologically equivalent to diagrams devised by Branko Grunbaum, which were based around intersecting polygons with increasing numbers of sides. They are also two-dimensional representations of hypercubes.

1.3 Venn diagrams for more than three statements

In three-statement Venn diagram, the three circles and the region outside them partition the “universe” into eight regions. Each region can be characterized by whether its points are inside or outside A, B or C: Each region corresponds to a unique threefold and, running from not A and not B and not C to A and B and C. The usual name for these expressions is conjunctions or, in analogy with multiplication, monomials. Furthermore, in the three-statement Venn diagram, the relative topology of these regions mirrors the relative closeness of the corresponding monomials, in the following precise sense: If two monomials differ by switching one statement to its negation, the two corresponding regions share a common edge. We will call such a Venn diagram topologically faithful. Topologically faithful diagrams exist for one, two or three statements, but not if the number of statements is four or more.

1.4 Topologically faithful Venn diagrams in higher dimensions

A 4-statement topologically faithful Venn diagram cannot be drawn in the plane, but an analogous structure does exist in three dimensions.

This construction can be repeated to give a topologically faithful four-dimensional "Venn diagram" for five statements, etc.

In the planar 3-statement Venn diagram, let us label a point in each of the eight regions by the corresponding monomial, and draw a line segment between two of these points if the regions share a side. The graph thus obtained is the dual graph of the partition into regions; by its construction no two edges intersect. The graph itself can be redrawn as the vertices and edges of a cube, with displacement in the x-direction corresponding to A↔not A adjacency of the corresponding regions, displacement in the y-direction corresponding to B↔not B adjacency, and displacement in the z-direction corresponding C↔not C adjacency.

Here is where the contradiction can be identified: the graph K cannot be drawn in the plane without two of its edges intersecting. The usual proof of this fact uses the Jordan Curve Theorem (every simple closed curve divides the plane into two regions, one "inside" the curve, and one "outside") and Euler's Theorem (if V,E and F are the numbers of vertices, edges and faces of a planar graph, then V-E+F=2). Here is a more rudimentary argument.

The first three vertices of K share three edges which form a triangle. By the Jordan Curve Theorem (we only need it for curvilinear polygons; much simpler to prove than the general statement), the triangle divides the plane into an inside and an outside. Suppose the fourth vertex goes inside. Then its edges to the first three vertices divide that triangle into three regions. Now there is nowhere to put the fifth vertex. If it is outside, it cannot be connected to the fourth, but if it is inside it must lie in one of the three triangles. That triangle will use vertex four and two of the original vertices, but then the fifth vertex will not be connectible to the remaining original vertex.

1.5 3D Venn and Euler Diagrams

In 2D, Venn and Euler diagrams consist of labelled simple closed curves. As in 2D, these 3D Euler diagrams visually represent the set-theoretic notions of intersection, containment and disjointness. There is only one topologically distinct embedding of wellformed Venn-3 in 2D, there are four such embeddings in 3D when the surfaces are topologically equivalent to spheres. Furthermore, we hypothesize that all data sets can be visualized with 3D Euler diagrams whereas this is not the case for 2D Euler diagrams, unless non-simple curves and/or duplicated labels are permitted.

Euler diagrams represent intersection, containment and disjointness of sets. Currently, these diagrams are drawn in the plane and consist of labelled simple closed curves. These 2D Euler diagrams have been widely studied over the last few years.

3D Euler diagrams consist of labelled orientable closed surfaces drawn in $\mathcal{R}^3$. An example of a 2D and a 3D Euler diagram representing the same information can be seen in Fig. 1. Using the freely available Autodesk Design Review software, one can rotate and explore the 3D diagrams.

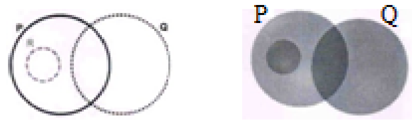


Figure 1: A 2D Euler diagram with an equivalent 3D Euler diagam

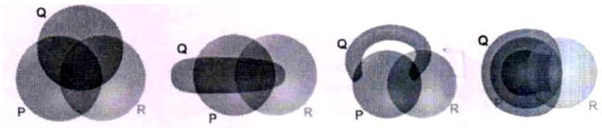


Figure 2: Four topologically distinct wellformed Venn-3s

We define 3D Venn diagrams as 3D Euler diagrams where all combinations of surface intersections are present. An interesting comparison between 2D and 3D is in the common Venn-3 case, i.e. the Venn diagram representing exactly three sets. It is known that there is only one topologically distinct embedding of well formed Venn-3 in 2D [13]. In 3D, there are infinitely many topologically distinct embeddings of wellformed Venn-3 when the surfaces are closed and orientable. When the surfaces are topologically equivalent to the sphere, there are at least four topologically distinct embeddings of wellformed 3D Venn-3, shown in Fig.2.

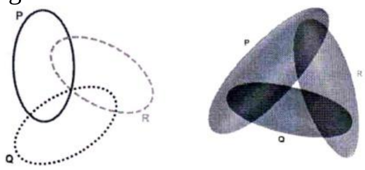


Figure 3: A non-well formed 2D diagram and an equivalent well formed 3D diagram

Well formedness properties are a key aspect of drawing of Euler diagrams. In 2D, they relate to how the curves intersect and to the properties of the regions present. In 3D, we generalize them to how the surfaces intersect and the properties of the solids to which the surfaces give rise. The 2D Euler diagram on the left of Fig. 3 is not well formed because it has a triple point of intersection between the curves. By contrast, the same data can be represented in a well formed manner in 3D, as shown in the right hand side of Fig.3.

3D Euler diagrams are formed from closed surfaces embedded in $\mathbb{R}^3$ rather than closed curves embedded in $\mathbb{R}^2$.

The concept of 3D Euler diagrams, formally defining them as orientable closed surfaces which implies the surfaces are simple. We have compared them 2D Euler diagrams and discovered that 3D Euler diagrams have some benefits over 2D Euler diagrams in terms of drawability when well formedness is considered.

We expect that 3D Euler diagrams will form a useful component in the field.

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