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Free Euler Lagrange Equation

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Abstract: ***In this work, the free analogue of the Euler- Lagrange equation called free-Euler- Lagrange equation is introduced. The basic examples of this new type of equations are studied.***

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1. Introduction

A variational principle is a scientific principle used with in the calculus of variations, which develops general methods for finding functions which extremizing the value of quantities that depend upon those functions. For example, to answer this question: What is the shape of a chain suspended at both ends? We can use the variational principle that the shape must minimize the gravitational potential energy. In the calculus of variations, the Euler-Lagrange equation, see equation (1), is a second-order partial differential equation which solutions are the functions for which a given functional is stationary. It was developed by Swiss mathematician Leonhard Euler and Italian-French mathematician Joseph-Louis Lagrange in the 1750s. Any body seeks the function minimizing or maximizing, the Euler-Lagrange equation is useful for solving optimization problems in which, given some functional. This is analogous to Fermat's theorem in calculus, stating that at any point where a differentiable function attains a local extremum its derivative is zero. In Hamilton's principle of stationary action, the evolution of a physical system is described by the solutions to the Euler-Lagrange equation for the action of the system. In classical mechanics, it is equivalent to Newton's laws of motion, but it has the advantage that it takes the same form in any system of generalized coordinates, and it is better suited to generalizations. In classical field theory there is an analogous equation to calculate the dynamics of a field. Typically, mathematicians are interested in free-analogue that arise naturally, rather than in arbitrarily contriving free-analogue of known results. The free analogues are most frequently studied in the mathematical fields of combinatorics and special functions. It finds applications in a number of areas, including the study of fractals and multi-fractal measures, and expressions for the entropy of chaotic dynamical systems. The relationship of fractals and dynamical systems results from the fact that many fractal patterns have the symmetries of Fuchsian groups in general (see, for example Indra's pearls and the Apollonian gasket) and the modular group in particular.

Free analogues also appear in the study of quantum groups and in free super algebras. The connection here is similar, in that much of string theory is set in the language of Riemann surfaces, resulting in connections to elliptic curves. This article organized as follow: In Section 2, we present a variational principle of Euler-Lagrange differential equation. In Section 3, we study the free Euler-Lagrange equation. In Section 4, we introduce some examples of free Euler-Lagrange equation.

2. Preliminaries

2.1. Euler-Lagrange Differential Equation

The Euler-Lagrange differential equation is the fundamental equation of calculus of variations. It states that if is defined by an integral of the form

$$J = \int f(t, y, y')dt$$

where

$$y' = \frac{dy}{dt}$$

then J has a stationary value if the Euler-Lagrange differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) = 0$$

is satisfied. If time-derivative notation is replaced instead by space-derivative notation , the equation becomes

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) = 0$$

The Euler-Lagrange differential equation is implemented as Euler equations [f,u[x], x] in the Wolfram Language package variational methods. In many physical problems, the partial derivative of with respect to turns out to be 0, in which case a manipulation of the Euler-Lagrange differential equation reduces to the greatly simplified and partially integrated form known as the Beltrami identity,

$$f - y_x\left(\frac{\partial f}{\partial y_x}\right) = c$$

Problems in the calculus of variations often can be solved by solution of the appropriate Euler Lagrange equation.

2.2. Free Derivative

Here is a nice diversion for anyone who knows what the derivative of a simple function is $f(x)$. The modern theory of differential and integral calculus began in the 20th century with the works of Newton and Leibniz. As it is well known, the derivative of a function $f(x)$ with respect to the variable x is by definition:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

Now, let us consider the following expression:

$$Df(x) = \begin{cases} \dfrac{f(x) - f(0)}{x} & if\ x \neq 0 \\ f'(0) & if\ x = 0 \end{cases}$$

D will be called the free derivative. As an example we compute the free derivative of x^n. If $x \neq 0$, we have

$$D(x^n) = \frac{x^n - 0}{x} = x^{n-1}$$

and if $x = 0$, we have $D(x^n) = 0$. One can easily check that the free-derivative operator is linear:

$D(f + g) = Df + Dg$

$D(\lambda(f)) = \lambda(D(f)),$

the product rule is slightly modified but it approaches the usual product rule:

$(D(fg))(x) = f(0)(Dg)(x) + (Df)(x)g(x)$

3. Free Euler Lagrange Equation

As analogous of the classical Euler Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad (1)$$

Where

$$y' = \frac{dy}{dt}$$

We introduce the free Euler Lagrange equation as follows:

$$D_y f - D_t\left(D_{y_0'} f\right) = 0 \qquad (2)$$

Where $f = f(t, y, y_0'),$

$$y_0'(t) = D_t y(t) = \begin{cases} \dfrac{y(t) - y(0)}{t} & if\, t \neq 0 \\ y'(0) & if\, t = 0 \end{cases}$$

$$D_y f(t, y(t), y_0'(t)) = \begin{cases} \dfrac{f(t, y(t), y_0'(t)) - f(t, 0, y_0'(t))}{y(t)} & if\, y(t) \neq 0 \\ \dfrac{\partial f}{\partial y}(t, 0, y_0'(t)) & if\, y(t) = 0 \end{cases} \qquad (3)$$

and

$$D_{y_0'} f(t, y(t), y_0'(t)) = \begin{cases} \dfrac{f(t, y(t), y_0'(t)) - f(t, y(t), 0)}{y_0'(t)} & if\, y_0'(t) \neq 0 \\ \dfrac{\partial f}{\partial y_0'}(t, y(t), 0) & if\, y_0'(t) = 0 \end{cases} \qquad (4)$$

From the above discussion we obtain the following theorem.

Theorem 3.1. *If* $t \neq 0$, $y(t) \neq 0$ *and* $y_0'(t) \neq 0$, *the free Euler Lagrange equation* (2) *is equivalent to*

$$f(t, y(t), y_0'(t))\left(\frac{y(0)}{ty_0'(t)}\right) - f(t, 0, y_0'(t)) + f(t, y(t), 0)\left(\frac{y(t)}{ty_0'(t)}\right)$$
$$= \left(f(0, y(0), y_0'(0)) - f(0, y(0), 0)\right)\left(\frac{y(t)}{ty_0'(0)}\right).$$

And if $t = 0$, $y(t) \neq 0$ *and* $y'(t) \neq 0$, *then, equation*(2) *is equivalent to*

$$l'(0) = \frac{f(0, y(0), y_0'(0)) - f(0, 0, y_0'(0))}{y(0)}$$

where $l(t)$ is given by

$$l(t) = \begin{cases} \dfrac{f(t, y(t), y_0'(t)) - f(t, y(t), 0)}{y_0'(t)} & if\, y_0'(t) \neq 0 \\ \dfrac{\partial f}{\partial y_0}(t, y(t), 0) & if\, y_0'(t) = 0 \end{cases} \qquad (5)$$

Proof.• First case: *If* $t \neq 0$, $y(t) \neq 0$ *and* $y_0'(t) \neq 0$. We substitute (3) and (4) in (2), we get

$$\left(\frac{f(t, y(t), y_0'(t)) - f(t, 0, y_0'(t))}{y(t)}\right) - \frac{l(t) - l(0)}{t} = 0.$$

This gives

$$\left(\frac{f(t, y(t), y_0'(t)) - f(t, 0, y_0'(t))}{y(t)}\right) - \frac{\frac{f(t, y(t), y_0'(t)) - f(t, y(t), 0)}{y_0'(t)} - \frac{f(0, y(0), y_0'(0)) - f(0, y(0), 0)}{y_0'(0)}}{t} = 0$$

Then, we get

$$\left(\frac{f(t, y(t), y_0'(t)) - f(t, 0, y_0'(t))}{y(t)}\right)$$
$$= \frac{f(t, y(t), y_0'(t)) - f(t, y(t), 0)}{ty_0'(t)} + \frac{f(0, y(0), y_0'(0)) - f(0, y(0), 0)}{ty_0'(0)}$$

which implies that

$f(t, y(t), y_0'(t)) - f(t, 0, y_0'(t))$

$$f(t, y(t), y_0'(t)) - f(t, 0, y_0'(t))$$
$$= y(t)\frac{f(t, y(t), y_0'(t)) - f(t, y(t), 0)}{ty_0'(t)} +$$
$$y(t)\frac{f(0, y(0), y_0'(0)) - f(0, y(0), 0)}{ty_0'(0)}.$$

Therefore, we obtain

$$f(t, y(t), y_0'(t))\left(1 - \frac{y(t)}{ty_0'(t)}\right) - f(t, 0, y_0'(t)) +$$
$$f(t, y(t), 0)\frac{y(t)}{ty_0'(t)}$$
$$= \left(f(0, y(0), y_0'(0)) - f(0, y(0), 0)\right)\frac{y(t)}{ty_0'(0)}$$

Hence, we get

$$f(t, y(t), y_0'(t))\left(\frac{y(0)}{ty_0'(t)}\right) - f(t, 0, y_0'(t)) + f(t, y(t), 0)\frac{y(t)}{ty_0'(t)}$$
$$= \left(f(0, y(0), y_0'(0)) - f(0, y(0), 0)\right)\frac{y(t)}{ty_0'(0)}.$$

• Second case: If $t = 0$, $y(t) \neq 0$ *and* $y'(t) \neq 0$,then equation (2) becomes

$$\left(\frac{f(0, y(0), y_0'(0)) - f(0, 0, y_0'(0))}{y(0)}\right) - l'(0) = 0$$

Which completes the proof.

4. Examples

Recall that, the classical standard example, for *f* given by

$f(t, y, y') = \sqrt{1 + y'^2}$

We get,$y = At + B$.That is, the function must have constant first derivative and thus it is graph is a straight line. Now, we will study the free analogue of this standard example.

Theorem 4.1. Let f given by

$$f(t,y,y') = \sqrt{1+y_0'^2}$$

satisfying the free Euler Lagrange equation (2). Then, for $t \neq 0$ *and if* $y_0'(t) \neq 0$ *we get* $y(t) = At + B$, *for constants A and B*$\cdot$

Proof. Using equation (4), we obtain

$$D_{y_0'}f = \frac{\sqrt{1+y_0'^2}-1}{y_0'}\cdot$$

Since, we have $D_y f = 0$. Then, by Eq. (2), we get

$$D_t(D_{y_0'}f) = 0\cdot$$

Therefore, we obtain

$$D_{y_0'}f = c.$$

Then, we get

$$\frac{\sqrt{1+y_0'^2}-1}{y_0'} = c.$$

This gives

$$\sqrt{1+y_0'^2} = 1 + cy_0'\cdot$$

Then, we deduce that

$$1 + y_0'^2 = 1 + 2cy_0' + c^2 y_0'^2$$

which implies that

$$y_0'^2(1-c^2) - 2cy_0' = 0\cdot$$

Which gives

$$y_0' = 0 \quad or \quad y_0' = \frac{2c}{1-c^2}\cdot$$

Let $A = \frac{2c}{1-c^2}\cdot$ Then, we get

$$\frac{y(t)-y(0)}{t} = A$$

which gives

$$y(t) = At + B$$

where $A = \frac{2c}{1-c^2}$ and $B = y(0)\cdot$ Which completes the proof.

Using Euler-Lagrange equation (1) and by taking the following function

$$f = ty'(t) + y^2(t)$$

we obtain the following solution

$$y(x) = \frac{-1}{4}x^2 + c_1 x + c_2$$

As free-deformation of this example we get the following theorem

Theorem 4.2. Let f given by

$$f(t,y,y_0') = ty_0' + y_0'^2$$

satisfying the free-Euler Lagrange equation (2). Then,

$t \neq 0$ *and* $y'(t) \neq 0$, *we get*

$$y(t) = -t^2 + c_1 t + c_2$$

Where

$$c_2 = y(0).$$

Proof. Using equation (4), we obtain

$$D_{y_0'}f(t,y,y_0') = \frac{f(t,y,y_0') - f(t,y,0)}{y_0'} = \frac{ty_0' + y_0'^2 - 0}{y_0'} = \frac{ty_0' + y_0'^2}{y_0'} = t + y_0'\cdot$$

Since, we have

$$D_y f(t,y,y_0') = 0$$

Then, we get

$$D_t(D_{y_0'}f(t,y,y_0')) = 0$$

Therefore, we get

$$D_{y_0'}f(t,y,y_0') = c_1\cdot$$

This gives

$$t + y_0' = c_1$$

Which is equivalent to

$$y_0' = -t + c_1\cdot$$

This implies that

$$\frac{y(t)-y(0)}{t} = -t + c_1\cdot$$

Then, we get

$$y(t) - y(0) = -t^2 + c_1 t$$

This gives

$$y(t) = -t^2 + c_1 t + y(0)$$

Which implies that

$$y(t) = -t^2 + c_1 t + c_2$$

Where

$$c_2 = y(0).$$

This completes the proof.

Remark 1. *In this study we introduced the free-Euler-Lagrange equation. A free analogue of some nuclear algebras of operators acting on space of holomorphic functions on a free analogue complexification of real nuclear space can be studied and we expect to develop a new quantum white noise analogue of free-Euler-Lagrange equation .(seeRef. [3], [14], [15], [16], [17], [18] and [19]).*

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