On Star Dagger Bi-Matrix and Bi-Star Bi-Dagger Bi-Matrix

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Abstract: ***The concept of star dagger bi-matrix and bi-star bi-dagger bi-matrix as a generalization of star dagger matrices are introduced in this paper. Also, some properties of star dagger bi-matrices and bi-star bi-dagger bi-matrices are obtained.***

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1. Introduction

The study of matrices have got a predominant role in linear algebra. In matrix theory, one knows that every non-singular matrix A has got unique inverse, which is the unique solution X of the matrix equations $AX = XA = I$, the identity matrix. In most of the real life situations the matrix arrived in modeling processes is rectangular (in particular singular, in the case of square matrices) and it demands a 'partial inverse' in order to suggest solutions to the original problem. This leads to the study of generalized inverses of rectangular matrices as started by E. H. Moore in 1906. In this attempt, E. H. Moore [8] published a paper in 1920. Further in 1936, Von Neuflann used generalized inverses in studies of continuous geometries and regular rings. R. Penrose [9] later in 1955, established that for every complex matrix A, there exists a unique matrix X (called Moore-Penrose inverse) satisfying the four equations (called Penrose equations) (1) AX A = A, (2) XAX = X, (3) (AX)* = AX and (4) (XA)* = XA , where * denotes the conjugate transpose. Such a unique matrix X corresponding to A is called Moore-Penrose inverse of A and is denoted by $A^{\dagger}$ (read as A dagger).

The concept of unitary (isometry) matrices for non-singular category has been extended as partial isometry to rectangular matrices, via the tool of Moore-Penrose inverses. This beginning has subsequently extended the concept of partial isometry to star-dagger matrices, which coincides with normal matrices in the case of nonsingular matrices.

It is well known that the concept of hermitian positive semi-definite *(hpsd)* matrices is a generalization of non-negative real numbers. A square matrix A is said to be hermitian positive semi-definite *ii* $A = A^*$ and $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$.

W.N.Anderson Jr and R. J. Duffin [1] were led to the concept of parallel sum of two *hpsd* matrices of same order n from the parallel connection of two n electrical networks involving only resistors. The parallel sum of two *hpsd* matrices is defined as $A(A+B)^{\dagger}B$. They have also established many interesting properties of the parallel of sum of a pair of *hpsd* matrices. In particular, they have shown that the *hpsd* matrices form a commutative partially ordered semi group under the parallel sum operation. Note that $A \geq B$ means $A - B$ *is hpsd.*

Normal matrices are generalizations of hermitian matrices. The concept of normal matrices over the complex field was introduced in 1918 by O.Toeplitz [11], who gave a necessary and sufficient condition for a complex matrix to be normal (i.e., $AA^* = A^*A$). The class of normal matrices includes skew-hermitian, hermitian and unitary matrices. Also another generalization of hermitian matrices is the range hermitian matrices called the class of *EP* matrices. A square matrix A is said to be *EP* if $R(A) = R(A^*)$ equivalents $AA^{\dagger} = A^{\dagger}A$, where $R(X)$ is the range space of the matrix X. This concept of *EP* was introduced by H.Schwerdtfeger [10].

One may note at this point that the concept of *EP* is not only a generalization of hermitian, but also a generalization of other classes viz., *hpsd*, hermitian, normal, unitary and non-singular. Recently S.Jhang and Y.Tian [4] has given a new set of characterizations for *EP* matrices.

T. S. Baskett and I. J. Katz [2] have discussed about EP_r matrices, *EP* matrices of rank r. In particular, they have studied about the product of two EP_r matrices. This leads to an open problem that when a product of two *EP* matrices is *EP?* This problem was settled by R. E. Hartwig and I. J. Katz [6] after 25 years. The concept of *EP* matrices is further generalized as bi- *EP* and studied by S. L. Campbell and C. D. Meyer [3]. A matrix A is called bi-*EP* if, $AA^{\dagger}$ commutes with $A^{\dagger}A$. As mentioned eai'lier the class of star-dagger matrices, the generahzation of partial isometry was studied by C. D. Meyer [7] and formalized as star-dagger with further properties by R. E. Hartwig and K. Spenelboek [5]. It is to be noted that the intersection of classes of star-dagger and *EP* matrices is nothing but the class of normal matrices as established by C. D. Meyer [7].

In [12, 13], W.B.Vasantha Kandasamy et. al. introduced the concept of bimatrices and analyses its properties. Bimatrices play a powerful and an advanced tool which can handle more than one linear model at a time. Bimatrices will be useful when time bound comparisons are needed in the analysis of the model.

Definition 1.1 [12]

$A_B = A_1 \cup A_2$ [$\cup$ is not operation only a symbol] is Bi matrix

Definition 1.2 [7]

A Matrix $A \in C_{n\times n}$ is Star -Dagger Matrix if $A^* A^\dagger = A^\dagger A^*$

Example 1.3

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $A^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $A^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Now to find star - dagger Matrix.

$A^* A^\dagger = A^\dagger A^*$

$A_1^* A_1^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $A_1^\dagger A_1^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$A_1^* A_1^\dagger = A_1^\dagger A_1^*$

Definition 1.4 [7]

A Matrix $A \in C_{n\times n}$ is bi-Dagger Matrix, $\left(A^2\right)^\dagger = \left(A^\dagger\right)^2$

In this paper, the concept of Star dagger Bi-matrix and bi-star bi-dagger bi-matrix are introduced and analyzed its properties.

2. s-k bi-EP bi-matrices

In this section we study the characterizations of star dagger bi-matrix and bi-star bi-dagger bi-matrix as a generalization of star dagger matrices.

Definition 2.1

A Matrix $A_B \in C_{n\times n}$ is Star -Dagger bi-matrix

if $A_B^* A_B^\dagger = A_B^\dagger A_B^*$

i.e., $\left(A_1 \cup A_2\right)^* \left(A_1 \cup A_2\right)^\dagger = \left(A_1 \cup A_2\right)^\dagger \left(A_1 \cup A_2\right)^*$

$\Rightarrow \left[\left(A_1^* A_1^\dagger\right) \cup \left(A_2^* A_2^\dagger\right)\right] = \left[\left(A_1^\dagger A_1^*\right) \cup \left(A_2^\dagger A_2^*\right)\right]$

Example 2.2

$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $A_1^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $A_2^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A_1^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $A_2^\dagger = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $A_2^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Now to find Star-dagger – Bi-matrix.

$A_B^* A_B^\dagger = A_B^\dagger A_B^*$

$\left(A_1 \cup A_2\right)^* \left(A_1 \cup A_2\right)^\dagger = \left(A_1 \cup A_2\right)^\dagger \left(A_1 \cup A_2\right)^*$

$\left(A_1^* A_1^\dagger\right) \cup \left(A_1^\dagger A_1^*\right) = \left(A_1^\dagger A_1^*\right) \cup \left(A_1^* A_1^\dagger\right)$

LHS $\left[\left(A_1^* A_1^\dagger\right) \cup \left(A_2^* A_2^\dagger\right)\right] == \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$

RHS $\left[\left(A_1^\dagger A_1^*\right) \cup \left(A_2^\dagger A_2^*\right)\right] == \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$

$\left[\left(A_1^* A_1^\dagger\right) \cup \left(A_2^* A_2^\dagger\right)\right] = \left[\left(A_1^\dagger A_1^*\right) \cup \left(A_2^\dagger A_2^*\right)\right]$

$A_B^* A_B^\dagger = A_B^\dagger A_B^*$

Definition 2.3

A Matrix $A_B \in C_{n\times n}$ is **Bi-star-Bi-dagger-bi-matrix**

$A_B^* A_B^\dagger A_B^\dagger A_B^* = A_B^\dagger A_B^* A_B^* A_B^\dagger$

i.e, $\left[\left(A_1 \cup A_2\right)^* \left(A_1 \cup A_2\right)^\dagger \left(A_1 \cup A_2\right)^\dagger \left(A_1 \cup A_2\right)^*\right]$

$= \left[\left(A_1 \cup A_2\right)^\dagger \left(A_1 \cup A_2\right)^* \left(A_1 \cup A_2\right)^* \left(A_1 \cup A_2\right)^\dagger\right]$

$\Rightarrow \left[\left(A_1^* A_1^\dagger A_1^\dagger A_1^*\right) \cup \left(A_2^* A_2^\dagger A_2^\dagger A_2^*\right)\right]$

$= \left[\left(A_1^\dagger A_1^* A_1^* A_1^\dagger\right) \cup \left(A_2^\dagger A_2^* A_2^* A_2^\dagger\right)\right]$

Example 2.4

$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $A_1^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $A_2^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A_1^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $A_2^\dagger = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $A_2^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Now to find "Bi-star - Bi-dagger – Bi-matrix."

$A_B^* A_B^\dagger A_B^\dagger A_B^* = A_B^\dagger A_B^* A_B^* A_B^\dagger$

i.e., $\left[\left(A_1 \cup A_2\right)^* \left(A_1 \cup A_2\right)^\dagger \left(A_1 \cup A_2\right)^\dagger \left(A_1 \cup A_2\right)^*\right] =$

$\left[\left(A_1 \cup A_2\right)^\dagger \left(A_1 \cup A_2\right)^* \left(A_1 \cup A_2\right)^* \left(A_1 \cup A_2\right)^\dagger\right]$

$\therefore \left[\left(A_1^* A_1^\dagger A_1^\dagger A_1^*\right) \cup \left(A_2^* A_2^\dagger A_2^\dagger A_2^*\right)\right] =$

$\left[\left(A_1^\dagger A_1^* A_1^* A_1^\dagger\right) \cup \left(A_2^\dagger A_2^* A_2^* A_2^\dagger\right)\right]$

LHS

$\left[\left(A_1^* A_1^\dagger A_1^\dagger A_1^*\right) \cup \left(A_2^* A_2^\dagger A_2^\dagger A_2^*\right)\right] == \left[\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right]$

RHS $\left[\left(A_1^\dagger A_1^* A_1^* A_1^\dagger\right) \cup \left(A_2^\dagger A_2^* A_2^* A_2^\dagger\right)\right] == \left[\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$

$$\left[\left(A_1^*A_1^\dagger A_1^\dagger A_1^*\right)\cup\left(A_2^*A_2^\dagger A_2^\dagger A_2^*\right)\right]=\left[\left(A_1^\dagger A_1^*A_1^*A_1^\dagger\right)\cup\left(A_2^\dagger A_2^*A_2^*A_2^\dagger\right)\right]$$

Hence $A_B^*A_B^\dagger A_B^\dagger A_B^* = A_B^\dagger A_B^* A_B^* A_B^\dagger$

Bi-star - Bi-dagger – Bi-matrix

Theorem 2.5

If A_B is a **Star -Dagger** Bi-matrix then iA_B is a **Star - Dagger Bi-matrix**.

Proof

Given A_B is a **Star -Dagger** Bi-matrix that is

$$A_B^*A_B^\dagger = A_B^\dagger A_B^*$$

i.e., $\left[(A_1\cup A_2)^*(A_1\cup A_2)^\dagger\right]=\left[(A_1\cup A_2)^\dagger(A_1\cup A_2)^*\right]$

$$\Rightarrow\left(A_1^*A_1^\dagger\right)\cup\left(A_2^*A_2^\dagger\right)=\left(A_1^\dagger A_1^*\right)\cup\left(A_2^\dagger A_2^*\right)$$

Consider

$$[iA_B]^*[iA_B]^\dagger=[i(A_1\cup A_2)]^*\ [i(A_1\cup A_2)]^\dagger$$

$$[\because A_B = A_1\cup A_2]$$

$$=[(iA_1)\cup(iA_2)]^*.[(iA_1)\cup(iA_2)]^\dagger$$

$$=\left[(iA_1)^*\cup(iA_2)\right]^*.\left[(iA_1)^\dagger\cup(iA_2)\right]^\dagger$$

$$=\left[\left(i^*A_1^*\right)\cup\left(i^*A_2^*\right)\right]\quad\left[\left(i^\dagger A_1^\dagger\right)\cup\left(i^\dagger A_2^\dagger\right)\right]$$

$$=\left[\left(i^*A_1^*\right)\left(i^\dagger A_1^\dagger\right)\cup\left(i^*A_2^*\right)\left(i^\dagger A_2^\dagger\right)\right]$$

$$=\left[(-i)A_1^*\left(A_1^\dagger i^\dagger\right)\cup(-i)A_2^*\left(A_2^\dagger i^\dagger\right)\right]$$

$$=\left[\left(i^\dagger A_1^\dagger\right)A_1^*(-i)\cup\left(i^\dagger A_2^\dagger\right)A_2^*(-i)\right]$$

$$=\left[\left(i^\dagger A_1^\dagger\right)\left(i^*A_1^*\right)\cup\left(i^\dagger A_2^\dagger\right)\left(i^*A_2^*\right)\right]$$

$$=\left[(A_1 i)^\dagger(iA_1)^*\cup(A_2 i)^\dagger(iA_2)^*\right]$$

$$=\left[(iA_1)^\dagger\cup(iA_2)^\dagger\right].\left[(iA_1)^*\cup(iA_2)^*\right]$$

$$=\left[(iA_1\cup iA_2)^\dagger.(iA_1\cup iA_2)^*\right]$$

$$=\left[i(A_1\cup A_2)\right]^\dagger.\left[i(A_1\cup A_2)\right]^*$$

$$=(iA_B)^\dagger.(iA_B)^*$$

$$\therefore (iA_B)^*(iA_B)^\dagger=(iA_B)^\dagger(iA_B)^*$$

Hence iA_B is **Star -Dagger** – Bi-matrix.

Theorem 2.6

If A_B is a Bi-star - Bi-dagger – Bi-matrix, Then iA_B is a bistor Bi-dagger – Bi-matrix.

Proof

Given,

A_B is a Bi-star - Bi-dagger – Bi-matrix.

i.e. $A_B^*A_B^\dagger A_B^\dagger A_B^* = A_B^\dagger A_B^* A_B^* A_B^\dagger$

Consider

$$[iA_B]^*[iA_B]^\dagger[iA_B]^\dagger[iA_B]^*=\left\{\left[i(A_1\cup A_2)\right]^*\left[i(A_1\cup A_2)\right]^\dagger\left[i(A_1\cup A_2)\right]^\dagger\left[i(A_1\cup A_2)\right]^*\right\}$$

$$=\left\{\left[(iA_1\cup iA_2)\right]^*\left[(iA_1\cup iA_2)\right]^\dagger\left[(iA_1\cup iA_2)\right]^\dagger\left[(iA_1\cup iA_2)\right]^*\right\}$$

$$=\left\{\left[(iA_1)^*\cup(iA_2)^*\right]\left[(iA_1)^\dagger\cup(iA_2)\right]^\dagger\left[(iA_1)^\dagger\cup(iA_2)\right]^\dagger\left[(iA_1)^*\cup(iA_2)\right]^*\right\}$$

$$=\left\{\left[i^*A_1^*\cup i^*A_2^*\right]\left[i^\dagger A_1^\dagger\cup i^\dagger A_2^\dagger\right]\left[i^\dagger A_1^\dagger\cup i^\dagger A_2^\dagger\right]\left[i^*A_1^*\cup i^*A_2^*\right]\right\}$$

$$=\left\{\left[\left(i^*A_1^*\right)\left(i^\dagger A_1^\dagger\right)\left(i^\dagger A_1^\dagger\right)\left(i^*A_1^*\right)\right]\cup\left[\left(i^*A_2^*\right)\left(i^\dagger A_2^\dagger\right)\left(i^\dagger A_2^\dagger\right)\left(i^*A_2^*\right)\right]\right\}$$

$$=\left\{\left[(-i)A_1^*\left(i^\dagger A_1^\dagger\right)\left(i^\dagger A_1^\dagger\right)(-i)A_1^*\right]\cup\left[(-i)A_2^*\left(i^\dagger A_2^\dagger\right)\left(i^\dagger A_2^\dagger\right)(-i)A_2^*\right]\right\}$$

$$=\left\{\left[\left(i^\dagger A_1^\dagger\right)A_1^*(-i)A_1^*(-i)\left(i^\dagger A_1^\dagger\right)\right]\cup\left[\left(i^\dagger A_2^\dagger\right)A_2^*(-i)A_2^*(-i)\left(i^\dagger A_2^\dagger\right)\right]\right\}$$

$$=\left\{\left[\left(i^\dagger A_1^\dagger\right)\left(i^*A_1^*\right)\left(i^*A_1^*\right)\left(i^\dagger A_1^\dagger\right)\right]\cup\left[\left(i^\dagger A_2^\dagger\right)\left(i^*A_2^*\right)\left(i^*A_2^*\right)\left(i^\dagger A_2^\dagger\right)\right]\right\}$$

$$=\left\{\left[(A_1 i)^\dagger(iA_1)^*(iA_1)^*(A_1 i)^\dagger\right]\cup\left[(A_2 i)^\dagger(iA_2)^*(iA_2)^*(A_2 i)^\dagger\right]\right\}$$

$$=\left\{\left[(A_1 i)^\dagger\cup(A_2 i)^\dagger\right]\left[(iA_1)^*\cup(iA_2)^*\right]\left[(iA_1)^*\cup(iA_2)^*\right]\left[(A_1 i)^\dagger\cup(A_2 i)^\dagger\right]\right\}$$

$$=\left\{[iA_1\cup iA_2]^\dagger[iA_1\cup iA_2]^*[iA_1\cup iA_2]^*[iA_1\cup iA_2]^\dagger\right\}$$

$$=\left\{\left[i(A_1\cup A_2)\right]^\dagger\left[i(A_1\cup A_2)\right]^*\left[i(A_1\cup A_2)\right]^*\left[i(A_1\cup A_2)\right]^\dagger\right\}$$

$$=\left\{(iA_B)^\dagger(iA_B)^*(iA_B)^*(iA_B)^\dagger\right\}$$

$$\therefore\left[(iA_B)^*(iA_B)^\dagger(iA_B)^\dagger(iA_B)^*\right]=\left[(iA_B)^\dagger(iA_B)^*(iA_B)^*(iA_B)^\dagger\right]$$

Hence, iA_B is a Bi-star - Bi-dagger – Bi-matrix.

Theorem 2.7

If A_B and B_B are **Star -Dagger** – Bi-matrices then so is $A_B \otimes B_B$.

Proof

Given that A_B and B_B are **Star -Dagger** – Bi-matrices. To prove that $A_B \otimes B_B$ is a **Star -Dagger** – Bi-matrix.

i.e, To prove that

$$[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{*} = [A_B \otimes B_B]^{*}[A_B \otimes B_B]^{\dagger}$$

To take LHS

$$[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{*} =$$

$$\left\{\left[(A_1 \cup A_2)\otimes(B_1 \cup B_2)\right]^{\dagger}\left[(A_1 \cup A_2)\otimes(B_1 \cup B_2)\right]^{*}\right\}$$

$$=\left\{\left[(A_1 \cup A_2)^{\dagger}\otimes(B_1 \cup B_2)^{\dagger}\right]\left[(A_1 \cup A_2)^{*}\otimes(B_1 \cup B_2)^{*}\right]\right\}$$

$$=\left\{\left[(A_1^{\dagger} \cup A_2^{\dagger})(A_1^{*} \cup A_2^{*})\right]\otimes\left[(B_1^{\dagger} \cup B_2^{\dagger})(B_1^{*} \cup B_2^{*})\right]\right\}$$

$$=\left\{\left[A_1^{\dagger}A_1^{*} \cup A_2^{\dagger}A_2^{*}\right]\otimes\left[B_1^{\dagger}B_1^{*} \cup B_2^{\dagger}B_2^{*}\right]\right\}$$

$$=\left\{\left[(A_1^{*} \cup A_2^{*})(A_1^{\dagger} \cup A_2^{\dagger})\right]\otimes\left[(B_1^{*} \cup B_2^{*})(B_1^{\dagger} \cup B_2^{\dagger})\right]\right\}$$

$$=\left\{\left[A_B^{*}A_B^{\dagger}\right]\otimes\left[B_B^{*}B_B^{\dagger}\right]\right\}$$

$$=\left\{\left[A_B^{*}\otimes B_B^{*}\right]\left[A_B^{\dagger}\otimes B_B^{\dagger}\right]\right\}$$

$$=\left\{\left[A_B \otimes B_B\right]^{*}\left[A_B \otimes B_B\right]^{\dagger}\right\}$$

$$\therefore \quad [A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{*} = [A_B \otimes B_B]^{*}[A_B \otimes B_B]^{\dagger}$$

Hence, $A_B \otimes B_B$ is Bi-star Bi-dagger – bi-matrix.

Theorem 2.8

If A_B and B_B are Bi-star - Bi-dagger – Bi-matrices then so is $A_B \otimes B_B$.

Proof

Given that AB and BB are Bi-star - Bi-dagger – Bi-matrices. To prove that $A_B \otimes B_B$ is a Bi-star - Bi-dagger – Bi-matrix.

i.e, To prove that

$$[A_B \otimes B_B]^{*}[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{*} = [A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{*}[A_B \otimes B_B]^{*}[A_B \otimes B_B]^{\dagger}$$

To taken LHS

$$[A_B \otimes B_B]^{*}[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{*}$$

$$=\left\{\left[(A_1 \cup A_2)\otimes(B_1 \cup B_2)\right]^{*}\left[(A_1 \cup A_2)\otimes(B_1 \cup B_2)\right]^{\dagger}\left[(A_1 \cup A_2)\otimes(B_1 \cup B_2)\right]^{\dagger}\left[(A_1 \cup A_2)\otimes(B_1 \cup B_2)\right]^{*}\right\}$$

$$=\left\{\left[(A_1 \cup A_2)^{*}\otimes(B_1 \cup B_2)^{*}\right]\left[(A_1 \cup A_2)^{\dagger}\otimes(B_1 \cup B_2)^{\dagger}\right]\left[(A_1 \cup A_2)^{\dagger}\otimes(B_1 \cup B_2)^{\dagger}\right]\left[(A_1 \cup A_2)^{*}\otimes(B_1 \cup B_2)^{*}\right]\right\}$$

$$=\left\{\left[(A_1^{*} \cup A_2^{*})\otimes(B_1^{*} \cup B_2^{*})\right]\left[(A_1^{\dagger} \cup A_2^{\dagger})\otimes(B_1^{\dagger} \cup B_2^{\dagger})\right]\left[(A_1^{\dagger} \cup A_2^{\dagger})\otimes(B_1^{\dagger} \cup B_2^{\dagger})\right]\left[(A_1^{*} \cup A_2^{*})\otimes(B_1^{*} \cup B_2^{*})\right]\right\}$$

$$=\left\{\left[(A_1^{*} \cup A_2^{*})(A_1^{\dagger} \cup A_2^{\dagger})(A_1^{\dagger} \cup A_2^{\dagger})(A_1^{*} \cup A_2^{*})\right]\otimes\left[(B_1^{*} \cup B_2^{*})(B_1^{\dagger} \cup B_2^{\dagger})(B_1^{\dagger} \cup B_2^{\dagger})(B_1^{*} \cup B_2^{*})\right]\right\}$$

$$=\left\{\left[A_1^{*}A_1^{\dagger}A_1^{\dagger}A_1^{*} \cup A_2^{*}A_2^{\dagger}A_2^{\dagger}A_2^{*}\right]\otimes\left[B_1^{*}B_1^{\dagger}B_1^{\dagger}B_1^{*} \cup B_2^{*}B_2^{\dagger}B_2^{\dagger}B_2^{*}\right]\right\}$$

$$=\left\{\left[A_1^{\dagger}A_1^{*}A_1^{*}A_1^{\dagger} \cup A_2^{\dagger}A_2^{*}A_2^{*}A_2^{\dagger}\right]\otimes\left[B_1^{\dagger}B_1^{*}B_1^{*}B_1^{\dagger} \cup B_2^{\dagger}B_2^{*}B_2^{*}B_2^{\dagger}\right]\right\}$$

$$=\left\{\left[(A_1^{\dagger} \cup A_2^{\dagger})(A_1^{*} \cup A_2^{*})(A_1^{*} \cup A_2^{*})(A_1^{\dagger} \cup A_2^{\dagger})\right]\otimes\left[(B_1^{\dagger} \cup B_2^{\dagger})(B_1^{*} \cup B_2^{*})(B_1^{*} \cup B_2^{*})(B_1^{\dagger} \cup B_2^{\dagger})\right]\right\}$$

$$=\left\{\left[A_B^{\dagger}A_B^{*}A_B^{*}A_B^{\dagger}\right]\otimes\left[B_B^{\dagger}B_B^{*}B_B^{*}B_B^{\dagger}\right]\right\}$$

$$=\left\{\left[A_B^{\dagger}\otimes B_B^{\dagger}\right]\otimes\left[A_B^{*}\cup B_B^{*}\right]\left[A_B^{*}\cup B_B^{*}\right]\left[A_B^{\dagger}\otimes B_B^{\dagger}\right]\right\}$$

$$=\left\{[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{*}[A_B \otimes B_B]^{*}[A_B \otimes B_B]^{\dagger}\right\}$$

$$\therefore \left\{[A_B \otimes B_B]^{*}[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{*}\right\} = \left\{[A_B \otimes B_B]^{\dagger}[A_B \otimes B_B]^{*}[A_B \otimes B_B]^{*}[A_B \otimes B_B]^{\dagger}\right\}$$

Hence, $A_B \otimes B_B$ is a Bi-star - Bi-dagger – Bi-matrix.

Theorem 2.9

Let $A_B, B_B \in C_{n\times n}$ be a **Star -Dagger** – Bi-matrices and that $A_B^{\dagger}B_B^{*} = B_B^{*}A_B^{\dagger}$ and $A_B^{*}B_B^{\dagger} = B_B^{\dagger}A_B^{*}$ then the bi-matrices $A_B + B_B$ and $A_B B_B$ are also **Star -Dagger** – **Bi-matrix**.

Proof

Let $A_B^{\dagger}, B_B^{\dagger} \in C_{n\times n}$ be a **Star -Dagger** – Bi-matrices and that

$A_B^{\dagger}B_B^{*} = B_B^{*}A_B^{\dagger}$

(i) consider

$$\left[(A_B+B_B)^{\dagger}(A_B+B_B)^{*}\right] = \left\{\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{\dagger}\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{*}\right\}$$

$$= \left\{\left[(A_1+B_1)\cup(A_2+B_2)\right]^{\dagger}\left[(A_1+B_1)\cup(A_2+B_2)\right]^{*}\right\}$$

$$= \left\{\left[(A_1+B_1)^{\dagger}\cup(A_2+B_2)^{\dagger}\right]\left[(A_1+B_1)^{*}\cup(A_2+B_2)^{*}\right]\right\}$$

$$= \left\{\left[(A_1+B_1)^{\dagger}(A_1+B_1)^{*}\right]\cup\left[(A_2+B_2)^{\dagger}(A_2+B_2)^{*}\right]\right\}$$

$$= \left\{\left[(A_1^{\dagger}+B_1^{\dagger})(A_1^{*}+B_1^{*})\right]\cup\left[(A_2^{\dagger}+B_2^{\dagger})(A_2^{*}+B_2^{*})\right]\right\}$$

$$= \left\{(A_1^{\dagger}A_1^{*}+B_1^{\dagger}A_1^{*}+A_1^{\dagger}B_1^{*}+B_1^{\dagger}B_1^{*})\cup(A_2^{\dagger}A_2^{*}+B_2^{\dagger}B_2^{*}+B_2^{\dagger}A_2^{*}+A_2^{\dagger}B_2^{*})\right\}$$

$$= \left\{\left[A_1^{*}(A_1^{\dagger}+B_1^{\dagger})+B_1^{*}(A_1^{\dagger}+B_1^{\dagger})\right]\cup\left[A_2^{*}(A_2^{\dagger}+B_2^{\dagger})+B_2^{*}(A_2^{\dagger}+B_2^{\dagger})\right]\right\}$$

$$= \left\{\left[(A_1+B_1)^{*}(A_1+B_1)^{\dagger}\right]\cup\left[(A_2+B_2)^{*}(A_2+B_2)^{\dagger}\right]\right\}$$

$$= \left\{\left[(A_1+B_1)^{*}\cup(A_2+B_2)^{*}\right]\left[(A_1+B_1)^{\dagger}\cup(A_2+B_2)^{\dagger}\right]\right\}$$

$$= \left\{\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{*}\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{+}\right\}$$

$$= \left\{(A_B+B_B)^{*}(A_B+B_B)^{\dagger}\right\}$$

$$\therefore\ (A_B+B_B)^{\dagger}(A_B+B_B)^{*} = (A_B+B_B)^{*}(A_B+B_B)^{\dagger}$$

Hence A_B+B_B is **Star -Dagger** – Bi-matrix.

(ii) consider

$$\left[(A_B+B_B)^{\dagger}(A_B+B_B)^{*}\right] = \left\{\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{\dagger}\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{*}\right\}$$

$$= \left\{\left[A_1B_1\cup A_2B_2\right]^{\dagger}\left[A_1B_1\cup A_2B_2\right]^{*}\right\}$$

$$= \left\{\left[(A_1B_1)^{\dagger}\cup(A_2B_2)^{\dagger}\right]\left[(A_1B_1)^{*}\cup(A_2B_2)^{*}\right]\right\}$$

$$= \left\{\left[(B_1^{\dagger}A_1^{\dagger})\cup(B_2^{\dagger}A_2^{\dagger})\right]\cup\left[(B_1^{*}A_1^{*})(B_2^{*}A_2^{*})\right]\right\}$$

$$= \left\{\left[(B_1^{\dagger}A_1^{\dagger}B_1^{*}A_1^{*})\right]\cup\left[(B_2^{\dagger}A_2^{\dagger}B_2^{*}A_2^{*})\right]\right\}$$

$$= \left\{\left[(B_1^{\dagger}B_1^{*}A_1^{\dagger}A_1^{*})\right]\cup\left[(B_2^{\dagger}B_2^{*}A_2^{\dagger}A_2^{*})\right]\right\}$$

$$= \left\{\left[(B_1^{*}B_1^{\dagger}A_1^{\dagger}A_1^{*})\right]\cup\left[(B_2^{*}B_2^{\dagger}A_2^{\dagger}A_2^{*})\right]\right\}$$

$$= \left\{\left[B_1^{*}B_1^{\dagger}A_1^{*}A_1^{\dagger}\right]\cup\left[B_2^{*}B_2^{\dagger}A_2^{*}A_2^{\dagger}\right]\right\}$$

$$= \left\{\left[B_1^{*}A_1^{*}B_1^{\dagger}A_1^{\dagger}\right]\cup\left[B_2^{*}A_2^{*}B_2^{\dagger}A_2^{\dagger}\right]\right\}$$

$$= \left\{\left[(A_1B_1)^{*}(A_1B_1)^{\dagger}\right]\cup\left[(A_2B_2)^{*}(A_2B_2)^{\dagger}\right]\right\}$$

$$= \left\{\left[(A_1B_1)^{*}\cup(A_2B_2)^{*}\right]\left[(A_1B_1)^{\dagger}\cup(A_2B_2)^{\dagger}\right]\right\}$$

$$= \left\{\left[(A_1B_1)\cup(A_2B_2)\right]^{*}\left[(A_1B_1)\cup(A_2B_2)\right]^{\dagger}\right\}$$

$$= \left\{\left[(A_1\cup A_2)(B_1\cup B_2)\right]^{*}\left[(A_1\cup A_2)(B_1\cup B_2)\right]^{\dagger}\right\}$$

$$= \left\{\left[A_BB_B\right]^{*}\left[A_BB_B\right]^{\dagger}\right\}$$

Hence A_B+B_B is a **Star -Dagger** – Bi-matrix.

Theorem 2.10

Let $A_B, B_B \in C_{n\times n}$ be a **Bi-star -Bi-dagger** Bi-matrices and that $A_B^{\dagger}B_B^{*}B_B^{*}A_B^{\dagger} = B_B^{*}A_B^{\dagger}A_B^{\dagger}B_B^{*}$ and $A_B^{*}B_B^{\dagger}B_B^{\dagger}A_B^{*} = B_B^{\dagger}A_B^{*}A_B^{*}B_B^{\dagger}$ then the bi-matrices A_B+B_B and A_BB_B are also Bi-star - Bi-dagger – Bi-matrix.

(i) consider

$$\left[(A_B+B_B)^{\dagger}(A_B+B_B)^{*}(A_B+B_B)^{*}(A_B+B_B)^{\dagger}\right]$$

$$= \left\{\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{\dagger}\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{*}\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{*}\left[(A_1\cup A_2)+(B_1\cup B_2)\right]^{\dagger}\right\}$$

$$= \left\{\left[(A_1+B_1)\cup(A_2+B_2)\right]^{\dagger}\left[(A_1+B_1)\cup(A_2+B_2)\right]^{*}\left[(A_1+B_1)\cup(A_2+B_2)\right]^{*}\left[(A_1+B_1)\cup(A_2+B_2)\right]^{\dagger}\right\}$$

$$= \left\{\left[(A_1+B_1)^{\dagger}\cup(A_2+B_2)^{\dagger}\right]\left[(A_1+B_1)^{*}\cup(A_2+B_2)^{*}\right]\left[(A_1+B_1)^{*}\cup(A_2+B_2)^{*}\right]\left[(A_1+B_1)^{\dagger}\cup(A_2+B_2)^{\dagger}\right]\right\}$$

$$= \left\{\left[(A_1+B_1)^{\dagger}(A_1+B_1)^{*}(A_1+B_1)^{*}(A_1+B_1)^{\dagger}\right]\cup\left[(A_2+B_2)^{\dagger}(A_2+B_2)^{*}(A_2+B_2)^{*}(A_2+B_2)^{\dagger}\right]\right\}$$

$$= \left\{\left[(A_1^{\dagger}+B_1^{\dagger})(A_1^{*}+B_1^{*})(A_1^{*}+B_1^{*})(A_1^{\dagger}+B_1^{\dagger})\right]\cup\left[(A_2^{\dagger}+B_2^{\dagger})(A_2^{*}+B_2^{*})(A_2^{*}+B_2^{*})(A_2^{\dagger}+B_2^{\dagger})\right]\right\}$$

$$= \left\{\left[(A_1^{\dagger}A_1^{*}+B_1^{\dagger}A_1^{*}+A_1^{\dagger}B_1^{*}+B_1^{\dagger}B_1^{*})\right]\left[(A_1^{*}A_1^{\dagger}+A_1^{*}B_1^{\dagger}+B_1^{*}A_1^{\dagger}+B_1^{\dagger}B_1^{*})\right]\right.$$

$$\left.\cup\left[(A_2^{*}A_2^{\dagger}+B_2^{\dagger}B_2^{*}+A_2^{\dagger}B_2^{*}+B_2^{\dagger}B_2^{*})\right]\left[(A_2^{*}A_2^{\dagger}+A_2^{*}B_2^{\dagger}+B_2^{*}A_2^{\dagger}+B_2^{*}B_2^{\dagger})\right]\right\}$$

$$=\{[A_1^*(A_1^\dagger+B_1^\dagger)+B_1^*(A_1^\dagger+B_1^\dagger)+A_1^\dagger(A_1^*+B_1^*)+B_1^\dagger(A_1^*+B_1^*)]$$
$$\cup[A_2^*(A_2^\dagger+B_2^\dagger)+B_2^*(A_2^\dagger+B_2^\dagger)+A_2^\dagger(A_2^*+B_2^*)+B_2^\dagger(A_2^*+B_2^*)]\}$$
$$=\{[(A_1^*+B_1^*)(A_1^\dagger+B_1^\dagger)(A_1^\dagger+B_1^\dagger)(A_1^*+B_1^*)]\cup[(A_2^*+B_2^*)(A_2^\dagger+B_2^\dagger)(A_2^\dagger+B_2^\dagger)(A_2^*+B_2^*)]\}$$
$$=\{[(A_1+B_1)^*(A_1+B_1)^\dagger(A_1+B_1)^\dagger(A_1+B_1)^*]\cup[(A_2+B_2)^*(A_2+B_2)^\dagger(A_2+B_2)^\dagger(A_2+B_2)^*]\}$$
$$=\{[(A_1+B_1)^*\cup(A_2+B_2)^*][(A_1+B_1)^\dagger\cup(A_2+B_2)^\dagger][(A_1+B_1)^\dagger\cup(A_2+B_2)^\dagger][(A_1+B_1)^*\cup(A_2+B_2)^*]\}$$
$$=\{[(A_1+B_1)\cup(A_2+B_2)]^*[(A_1+B_1)\cup(A_2+B_2)]^\dagger[(A_1+B_1)\cup(A_2+B_2)]^\dagger[(A_1+B_1)\cup(A_2+B_2)]^*\}$$
$$=\{[(A_1\cup A_2)+(B_1\cup B_2)]^*[(A_1\cup A_2)+(B_1\cup B_2)]^\dagger[(A_1\cup A_2)+(B_1\cup B_2)]^\dagger[(A_1\cup A_2)+(B_1\cup B_2)]^*\}$$
$$=\{(A_B+B_B)^*(A_B+B_B)^\dagger(A_B+B_B)^\dagger(A_B+B_B)^*\}$$
$\therefore\ [(A_B+B_B)^\dagger(A_B+B_B)^*(A_B+B_B)^*(A_B+B_B)^\dagger]=[(A_B+B_B)^*(A_B+B_B)^\dagger(A_B+B_B)^\dagger(A_B+B_B)^*]$ Hence A_B+B_B is a **Bi-star -Bi-dagger** - Bi-matrix.

(ii) Consider

$$[(A_BB_B)^\dagger(A_BB_B)^*(A_BB_B)^*(A_BB_B)^\dagger]$$
$$=\{[(A_1\cup A_2)(B_1\cup B_2)]^\dagger[(A_1\cup A_2)(B_1\cup B_2)]^*[(A_1\cup A_2)(B_1\cup B_2)]^*[(A_1\cup A_2)(B_1\cup B_2)]^\dagger\}$$
$$=\{[A_1B_1\cup A_2B_2]^\dagger[A_1B_1\cup A_2B_2]^*[A_1B_1\cup A_2B_2]^*[A_1B_1\cup A_2B_2]^\dagger\}$$
$$=\{[(A_1B_1)^\dagger\cup(A_2B_2)^\dagger][(A_1B_1)^*\cup(A_2B_2)^*][(A_1B_1)^*\cup(A_2B_2)^*][(A_1B_1)^\dagger\cup(A_2B_2)^\dagger]\}$$
$$=\{[(B_1^\dagger A_1^\dagger)\cup(B_2^\dagger A_2^\dagger)][(B_1^*A_1^*)\cup(B_2^*A_2^*)][(B_1^*A_1^*)\cup(B_2^*A_2^*)][(B_1^\dagger A_1^\dagger)\cup(B_2^\dagger A_2^\dagger)]\}$$
$$=\{[(B_1^\dagger A_1^\dagger)(B_1^*A_1^*)(B_1^*A_1^*)(B_1^\dagger A_1^\dagger)]\cup(B_2^\dagger A_2^\dagger)(B_2^*A_2^*)(B_2^*A_2^*)(B_2^\dagger A_2^\dagger)\}$$
$$=\{[(B_1^\dagger B_1^*A_1^\dagger A_1^*)(B_1^*B_1^\dagger A_1^*A_1^\dagger)]\cup[(B_2^\dagger B_2^*A_2^\dagger A_2^*)(B_2^*B_2^\dagger A_2^*A_2^\dagger)]\}$$
$$=\{[(B_1^*B_1^\dagger A_1^\dagger A_1^*)(B_1^\dagger B_1^*A_1^*A_1^\dagger)]\cup[(B_2^*B_2^\dagger A_2^\dagger A_2^*)(B_2^\dagger B_2^*A_2^*A_2^\dagger)]\}$$
$$=\{[(B_1^*B_1^\dagger A_1^*A_1^\dagger)(B_1^\dagger B_1^*A_1^\dagger A_1^*)]\cup[(B_2^*B_2^\dagger A_2^*A_2^\dagger)(B_2^\dagger B_2^*A_2^\dagger A_2^*)]\}$$
$$=\{[(B_1^*A_1^*B_1^\dagger A_1^\dagger)(B_1^\dagger A_1^\dagger B_1^*A_1^*)]\cup[(B_2^*A_2^*B_2^\dagger A_2^\dagger)(B_2^\dagger A_2^\dagger B_2^*A_2^*)]\}$$
$$=\{[(A_1B_1)^*(A_1B_1)^\dagger(A_1B_1)^\dagger(A_1B_1)^*]\cup[(A_2B_2)^*(A_2B_2)^\dagger(A_2B_2)^\dagger(A_2B_2)^*]\}$$
$$=\{[(A_1B_1)^*\cup(A_2B_2)^*][(A_1B_1)^\dagger\cup(A_2B_2)^\dagger][(A_1B_1)^\dagger\cup(A_2B_2)^\dagger][(A_1B_1)^*\cup(A_2B_2)^*]\}$$
$$=\{[A_1B_1\cup A_2B_2]^*[A_1B_1\cup A_2B_2]^\dagger[A_1B_1\cup A_2B_2]^\dagger[A_1B_1\cup A_2B_2]^*\}$$
$$=\{[(A_1\cup A_2)(B_1\cup B_2)]^*[(A_1\cup A_2)(B_1\cup B_2)]^\dagger[(A_1\cup A_2)(B_1\cup B_2)]^\dagger[(A_1\cup A_2)(B_1\cup B_2)]^*\}$$
$$=\{[A_BB_B]^*[A_BB_B]^\dagger[A_BB_B]^\dagger[A_BB_B]^*\}$$
$$\therefore[(A_BB_B)^\dagger(A_BB_B)^*(A_BB_B)^*(A_BB_B)^\dagger]=[(A_BB_B)^*(A_BB_B)^\dagger(A_BB_B)^\dagger(A_BB_B)^*]$$

Hence A_BB_B is a Bi-star - Bi-dagger – Bimatix.

Theorem 2.11

If A_B is a **Star -Dagger** Bi-matrix and a is a complex number then

(i) $A_B+\lambda I_B$ is a **Star -Dagger** Bi-matrix

(ii) $A_B - \lambda I_B$ is a **Star -Dagger** Bi-matrix.

Proof of (i)

Let A_B be a **Star -Dagger** – Bi-matrix.

$\therefore A_B^{\dagger} A_B^{*} = A_B^{\dagger} A_B^{*}$

i.e., $(A_1 \cup A_2)^{\dagger}(A_1 \cup A_2)^{*} = (A_1 \cup A_2)^{*}(A_1 \cup A_2)^{\dagger}$

$\therefore (A_1^{\dagger}A_1^{*}) \cup (A_2^{\dagger}A_2^{*}) = (A_1^{*}A_1^{\dagger}) \cup (A_2^{*}A_2^{\dagger})$

Now $(A_B + \lambda I_B)^{\dagger}(A_B + \lambda I_B)^{*}$

$$= \left\{\left[(A_1 \cup A_2) + \lambda(I_1 \cup I_2)\right]^{\dagger}\left[(A_1 \cup A_2) + \lambda(I_1 \cup I_2)\right]^{*}\right\}$$

$$= \left\{\left[(A_1 \cup A_2) + (\lambda I_1 \cup I_2)\right]^{\dagger}\left[(A_1 \cup A_2) + (\lambda I_1 \cup I_2)^{*}\right]\right\}$$

$$= \left\{\left[(A_1 + \lambda I_1) \cup (A_2 + \lambda I_2)\right]^{\dagger}\left[(A_1 + \lambda I_1) \cup (A_2 + \lambda I_2)\right]^{*}\right\}$$

$$= \left\{\left[(A_1 + \lambda I_1)^{\dagger} \cup (A_2 + \lambda I_2)^{\dagger}\right]\left[(A_1 + \lambda I_1)^{*} \cup (A_2 + \lambda I_2)^{*}\right]\right\}$$

$$= \left\{\left[(A_1^{\dagger} + \lambda^{\dagger} I_1) \cup (A_2^{\dagger} + \lambda^{\dagger} I_2)\right]\left[(A_1^{*} + \lambda^{*} I_1) \cup (A_2^{*} + \lambda^{*} I_2^{*})\right]\right\}$$

$$= \left\{\left[(A_1^{\dagger} + \lambda^{\dagger} I_1)(A_1^{*} + \bar{\lambda} I_1) \cup (A_2^{\dagger} + \lambda^{\dagger} I_2)(A_2^{*} + \bar{\lambda} I_2)\right]\right\}$$

$$= \left\{\left[(A_1^{\dagger}A_1^{*} + \lambda^{\dagger}A_1^{*} + \bar{\lambda}A_1^{\dagger} + \lambda^{\dagger}\bar{\lambda})\right] \cup \left[(A_2^{\dagger}A_2^{*} + \lambda^{\dagger}A_2^{*} + \bar{\lambda}A_2^{\dagger} + \lambda^{\dagger}\bar{\lambda})\right]\right\}$$

$$= \left\{\left[(A_1^{*}A_1^{\dagger} + \lambda^{\dagger}A_1^{*} + \bar{\lambda}A_1^{\dagger} + \lambda^{\dagger}\bar{\lambda})\right] \cup \left[(A_2^{*}A_2^{\dagger} + \lambda^{\dagger}A_2^{*} + \bar{\lambda}A_2^{\dagger} + \lambda^{\dagger}\bar{\lambda})\right]\right\}$$

$$= \left\{\left[A_1^{*}(A_1^{\dagger} + \lambda^{\dagger} I_1) + \bar{\lambda}(A_1^{\dagger} + \lambda^{\dagger} I_1)\right] \cup \left[A_2^{*}(A_2^{\dagger} + \lambda^{\dagger} I_2) + \bar{\lambda}(A_2^{\dagger} + \lambda^{\dagger} I_2)\right]\right\}$$

$$= \left\{\left[(A_1^{*} + \bar{\lambda} I_1)(A_1^{\dagger} + \lambda^{\dagger} I_1) \cup (A_2^{*} + \bar{\lambda} I_2)(A_2^{\dagger} + \lambda^{\dagger} I_2)\right]\right\}$$

$$= \left\{\left[(A_1^{*} + \bar{\lambda} I_1) \cup (A_2^{*} + \bar{\lambda} I_2)\right]\left[(A_1^{\dagger} + \lambda^{\dagger} I_1) \cup (A_2^{\dagger} + \lambda^{\dagger} I_2)\right]\right\}$$

$$= \left\{\left[(A_1 + \lambda I_1)^{*} \cup (A_2 + \lambda I_2)^{*}\right]\left[(A_1 + \lambda I_1)^{\dagger} \cup (A_2 + \lambda I_2)^{\dagger}\right]\right\}$$

$$= \left\{\left[(A_1 + \lambda I_1) \cup (A_2 + \lambda I_2)\right]^{*}\left[(A_1 + \lambda I_1) \cup (A_2 + \lambda I_2)\right]^{\dagger}\right\}$$

$$= \left\{\left[(A_1 \cup A_2) + (\lambda I_1 \cup \lambda I_2)\right]^{*}\left[(A_1 \cup A_2) + (\lambda I_1 \cup \lambda I_2)\right]^{\dagger}\right\}$$

$$= \left\{\left[(A_1 \cup A_2) + \lambda(I_1 \cup I_2)\right]^{*}\left[(A_1 \cup A_2) + \lambda(I_1 \cup I_2)\right]^{\dagger}\right\}$$

$$= (A_B + \lambda I_B)^{*}(A_B + \lambda I_B)^{\dagger}$$

$$\therefore (A_B + \lambda I_B)^{\dagger}(A_B + \lambda I_B)^{*} = (A_B + \lambda I_B)^{*}(A_B + \lambda I_B)^{\dagger}$$

Hence $A_B + \lambda I_B$ is a **Star –Dagger-** Bi-matrix.

Proof of (ii)

$(A_B - \lambda I_B)^{\dagger}(A_B - \lambda I_B)^{*}$

$$= \left\{\left[(A_1 \cup A_2) - \lambda(I_1 \cup I_2)\right]^{\dagger}\left[(A_1 \cup A_2) - \lambda(I_1 \cup I_2)\right]^{*}\right\}$$

$$= \left\{\left[(A_1 \cup A_2) - (\lambda I_1 \cup \lambda I_2)\right]^{\dagger}\left[(A_1 \cup A_2) - (\lambda I_1 \cup \lambda I_2)\right]^{*}\right\}$$

$$= \left\{\left[(A_1 - \lambda I_1) \cup (A_2 - \lambda I_2)\right]^{\dagger}\left[(A_1 - \lambda I_1) \cup (A_2 - \lambda I_2)\right]^{*}\right\}$$

$$= \left\{\left[(A_1 - \lambda I_1)^{\dagger} \cup (A_2 - \lambda I_2)^{\dagger}\right]\left[(A_1 - \lambda I_1)^{*} \cup (A_2 - \lambda I_2)^{*}\right]\right\}$$

$$= \left\{\left[(A_1^{\dagger} - \lambda^{\dagger} I_1) \cup (A_2^{\dagger} - \lambda^{\dagger} I_2)\right]\left[(A_1^{*} - \lambda^{*} I_1) \cup (A_2^{*} - \lambda^{*} I_2)\right]\right\}$$

$$= \left\{\left[(A_1^{\dagger} - \lambda^{\dagger} I_1) \cup (A_2^{\dagger} - \lambda^{\dagger} I_2)\right]\left[(A_1^{*} - \bar{\lambda} I_1) \cup (A_2^{*} - \bar{\lambda} I_2)\right]\right\}$$

$$=\left\{\left[\left(A_1^\dagger-\lambda^\dagger I_1\right)\left(A_1^*-\bar{\lambda}I_1\right)\right]\cup\left[\left(A_2^\dagger-\lambda^\dagger I_2\right)\left(A_2^*-\bar{\lambda}I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^\dagger A_1^*-\lambda^\dagger A_1^*-\bar{\lambda}A_1^\dagger+\lambda^\dagger\bar{\lambda}\right)\right]\cup\left[\left(A_2^\dagger A_2^*-\lambda^\dagger A_2^*-\bar{\lambda}A_2^\dagger+\lambda^\dagger\bar{\lambda}\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^* A_1^\dagger-\lambda^\dagger A_1^*-\bar{\lambda}A_1^\dagger+\lambda^\dagger\bar{\lambda}\right)\right]\cup\left[\left(A_2^* A_2^\dagger-\lambda^\dagger A_2^*-\bar{\lambda}A_2^\dagger+\lambda^\dagger\bar{\lambda}\right)\right]\right\}$$

$$=\left\{\left[A_1^*\left(A_1^\dagger-\lambda^\dagger I_1\right)-\bar{\lambda}\left(A_1^\dagger-\lambda^\dagger I_1\right)\right]\cup\left[A_2^*\left(A_2^\dagger-\lambda^\dagger I_2\right)-\bar{\lambda}\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^*-\bar{\lambda}I_1\right)\left(A_1^\dagger-\lambda^\dagger I_1\right)\right]\cup\left[\left(A_2^*-\bar{\lambda}I_1\right)\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^*-\bar{\lambda}I_1\right)\cup\left(A_2^*-\bar{\lambda}I_1\right)\right]\left[\left(A_1^\dagger-\lambda^\dagger I_1\right)\cup\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^*-\lambda^* I_1\right)\cup\left(A_2^*-\lambda^* I_2\right)\right]\left[\left(A_1^\dagger-\lambda^\dagger I_1\right)\cup\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1-\lambda I_1\right)^*\cup\left(A_2-\lambda I_2\right)^*\right]\left[\left(A_1-\lambda I_1\right)^\dagger\cup\left(A_2-\lambda I_2\right)^\dagger\right]\right\}$$

$$=\left\{\left[\left(A_1-\lambda I_1\right)\cup\left(A_2-\lambda I_2\right)\right]^*\left[\left(A_1-\lambda I_1\right)\cup\left(A_2-\lambda I_2\right)\right]^\dagger\right\}$$

$$=\left\{\left[\left(A_1\cup A_2\right)-\left(\lambda I_1\cup\lambda I_2\right)\right]^*\left[\left(A_1\cup A_2\right)-\left(\lambda I_1\cup\lambda I_2\right)\right]^\dagger\right\}$$

$$=\left\{\left[\left(A_1\cup A_2\right)-\lambda\left(I_1\cup I_2\right)\right]^*\left[\left(A_1\cup A_2\right)-\lambda\left(I_1\cup I_2\right)\right]^\dagger\right\}$$

$$=\left\{\left(A_B-\lambda I_B\right)^*\left(A_B-\lambda I_B\right)^\dagger\right\}$$

$$\therefore\left[\left(A_B-\lambda I_B\right)^\dagger\left(A_B-\lambda I_B\right)^*\right]=\left[\left(A_B-\lambda I_B\right)^*\left(A_B-\lambda I_B\right)^\dagger\right]$$

Hence $A_B-\lambda I_B$ is **Star -Dagger** –Bi-matrix.

Theorem 2.12

If A_B is a Bi-star - Bi-dagger – Bi-matrix and λ is a complex number then.

(i) $A_B+\lambda I_B$ is a Bi-star - Bi-dagger – Bi-matrix.

(ii) $A_B-\lambda I_B$ is a Bi-star - Bi-dagger – Bi-matrix.

Proof

Let A_B be a Bi-star - Bi-dagger – Bi-matrix.

i.e., $A_B^\dagger A_B^* A_B^* A_B^\dagger = A_B^* A_B^\dagger A_B^\dagger A_B^*$

$$\therefore\left[\left(A_1\cup A_2\right)^\dagger\left(A_1\cup A_2\right)^*\left(A_1\cup A_2\right)^*\left(A_1\cup A_2\right)^\dagger\right]=\left[\left(A_1\cup A_2\right)^*\left(A_1\cup A_2\right)^\dagger\left(A_1\cup A_2\right)^\dagger\left(A_1\cup A_2\right)^*\right]$$

i.e., $\left(A_1^\dagger A_1^* A_1^* A_1^\dagger\right)\cup\left(A_2^\dagger A_2^* A_2^* A_2^\dagger\right)=\left(A_1^* A_1^\dagger A_1^\dagger A_1^*\right)\cup\left(A_2^* A_2^\dagger A_2^\dagger A_2^*\right)$

Proof of (i)

Now, $\left[\left(A_B+\lambda I_B\right)^\dagger\left(A_B+\lambda I_B\right)^*\left(A_B+\lambda I_B\right)^*\left(A_B+\lambda I_B\right)^\dagger\right]$

$$=\left\{\left[\left(A_1\cup A_2\right)+\lambda\left(I_1\cup I_2\right)\right]^\dagger\left[\left(A_1\cup A_2\right)+\lambda\left(I_1\cup I_2\right)\right]^*\left[\left(A_1\cup A_2\right)+\lambda\left(I_1\cup I_2\right)\right]^*\left[\left(A_1\cup A_2\right)+\lambda\left(I_1\cup I_2\right)\right]^\dagger\right\}$$

$$=\left\{\left[\left(A_1\cup A_2\right)+\left(\lambda I_1\cup\lambda I_2\right)\right]^\dagger\left[\left(A_1\cup A_2\right)+\left(\lambda I_1\cup\lambda I_2\right)\right]^*\left[\left(A_1\cup A_2\right)+\left(\lambda I_1\cup\lambda I_2\right)\right]^*\left[\left(A_1\cup A_2\right)+\left(\lambda I_1\cup\lambda I_2\right)\right]^\dagger\right\}$$

$$=\left\{\left[\left(A_1+\lambda I_1\right)\cup\left(A_2+\lambda I_2\right)\right]^\dagger\left[\left(A_1+\lambda I_1\right)\cup\left(A_2+\lambda I_2\right)\right]^*\left[\left(A_1+\lambda I_1\right)\cup\left(A_2+\lambda I_2\right)\right]^*\left[\left(A_1+\lambda I_1\right)\cup\left(A_2+\lambda I_2\right)\right]^\dagger\right\}$$

$$=\left\{\left[\left(A_1+\lambda I_1\right)^\dagger\cup\left(A_2+\lambda I_2\right)^\dagger\right]\left[\left(A_1+\lambda I_1\right)^*\cup\left(A_2+\lambda I_2\right)^*\right]\left[\left(A_1+\lambda I_1\right)^*\cup\left(A_2+\lambda I_2\right)^*\right]\left[\left(A_1+\lambda I_1\right)^\dagger\cup\left(A_2+\lambda I_2\right)^\dagger\right]\right\}$$

$$=\left\{\left[\left(A_1^\dagger+\lambda^\dagger I_1\right)\cup\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\left[\left(A_1^*+\lambda^* I_1\right)\cup\left(A_2^*+\lambda^* I_2\right)\right]\left[\left(A_1^*+\lambda^* I_1\right)\cup\left(A_2^*+\lambda^* I_2\right)\right]\left[\left(A_1^\dagger+\lambda^\dagger I_1\right)\cup\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^\dagger+\lambda^\dagger I_1\right)\cup\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\left[\left(A_1^*+\bar{\lambda}I_1\right)\cup\left(A_2^*+\bar{\lambda}I_2\right)\right]\left[\left(A_1^*+\bar{\lambda}I_1\right)\cup\left(A_2^*+\bar{\lambda}I_2\right)\right]\left[\left(A_1^\dagger+\lambda^\dagger I_1\right)\cup\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^\dagger+\lambda^\dagger I_1\right)\left(A_1^*+\bar{\lambda}I_1\right)\left(A_1^*+\bar{\lambda}I_1\right)\left(A_1^\dagger+\lambda^\dagger I_1\right)\right]\cup\left[\left(A_2^\dagger+\lambda^\dagger I_2\right)\left(A_2^*+\bar{\lambda}I_2\right)\left(A_2^*+\bar{\lambda}I_2\right)\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\left[A_1^\dagger A_1^*+\lambda^\dagger A_1^*+\bar{\lambda}A_1^\dagger+\lambda^\dagger\bar{\lambda}\right]\left[A_1^* A_1^\dagger+\bar{\lambda}A_1^\dagger+A_1^*\lambda^\dagger+\bar{\lambda}\lambda^\dagger\right]\right\}\cup$$

$$\left\{\left[A_2^\dagger A_2^*+\lambda^\dagger A_2^*+\bar{\lambda}A_2^\dagger+\lambda^\dagger\bar{\lambda}\right]\left[A_2^* A_2^\dagger+\bar{\lambda}A_2^\dagger+A_2^*\lambda^\dagger+\bar{\lambda}\lambda^\dagger\right]\right\}$$

$$=\left\{\left[\left(A_1^* A_1^\dagger+\lambda^\dagger A_1^*+\bar{\lambda}A_1^\dagger+\bar{\lambda}\lambda^\dagger\right)\right]\left[A_1^\dagger A_1^*+\bar{\lambda}A_1^\dagger+A_1^*\lambda^\dagger+\bar{\lambda}\lambda^\dagger\right]\right\}$$

$$\cup\left\{\left[\left(A_2^* A_2^\dagger+\lambda^\dagger A_2^*+\bar{\lambda}A_2^\dagger+\bar{\lambda}\lambda^\dagger\right)\right]\left[A_2^\dagger A_2^*+\bar{\lambda}A_2^\dagger+A_2^*\lambda^\dagger+\bar{\lambda}\lambda^\dagger\right]\right\}$$

$$=\left\{\left[A_1^*\left(A_1^\dagger+\lambda^\dagger I_1\right)+\bar{\lambda}\left(A_1^\dagger+\lambda^\dagger I_1\right)\right]\left[A_1^\dagger\left(A_1^*+\bar{\lambda}I_1\right)+\lambda^\dagger\left(A_1^*+\bar{\lambda}I_1\right)\right]\right\}$$

$$\cup\left\{\left[A_2^*\left(A_2^\dagger+\lambda^\dagger I_2\right)+\bar{\lambda}\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\left[A_2^\dagger\left(A_2^*+\bar{\lambda}I_2\right)+\lambda^\dagger\left(A_2^*+\bar{\lambda}I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^*+\bar{\lambda}I_1\right)\left(A_1^\dagger+\lambda^\dagger I_1\right)\left(A_1^\dagger+\lambda^\dagger I_1\right)\left(A_1^*+\bar{\lambda}I_1\right)\right]\cup\left[\left(A_2^*+\bar{\lambda}I_2\right)\left(A_2^\dagger+\lambda^\dagger I_2\right)\left(A_2^\dagger+\lambda^\dagger I_2\right)\left(A_2^*+\bar{\lambda}I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^*+\bar{\lambda}I_1\right)\cup\left(A_2^*+\bar{\lambda}I_2\right)\right]\left[\left(A_1^\dagger+\lambda^\dagger I_1\right)\cup\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\left[\left(A_1^\dagger+\lambda^\dagger I_1\right)\cup\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\left[\left(A_1^*+\bar{\lambda}I_1\right)\cup\left(A_2^*+\bar{\lambda}I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^*+\lambda^* I_1\right)\cup\left(A_2^*+\lambda^* I_2\right)\right]\left[\left(A_1^\dagger+\lambda^\dagger I_1\right)\cup\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\left[\left(A_1^\dagger+\lambda^\dagger I_1\right)\cup\left(A_2^\dagger+\lambda^\dagger I_2\right)\right]\left[\left(A_1^*+\lambda^* I_1\right)\cup\left(A_2^*+\lambda^* I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1+\lambda I_1\right)^*\cup\left(A_2+\lambda I_2\right)^*\right]\left[\left(A_1+\lambda I_1\right)^\dagger\cup\left(A_2+\lambda I_2\right)^\dagger\right]\left[\left(A_1+\lambda I_1\right)^\dagger\cup\left(A_2+\lambda I_2\right)^\dagger\right]\left[\left(A_1+\lambda I_1\right)^*\cup\left(A_2+\lambda I_2\right)^*\right]\right\}$$

$$=\left\{\left[\left(A_1+\lambda I_1\right)\cup\left(A_2+\lambda I_2\right)\right]^*\left[\left(A_1+\lambda I_1\right)\cup\left(A_2+\lambda I_2\right)\right]^\dagger\left[\left(A_1+\lambda I_1\right)\cup\left(A_2+\lambda I_2\right)\right]^\dagger\left[\left(A_1+\lambda I_1\right)\cup\left(A_2+\lambda I_2\right)\right]^*\right\}$$

$$=\left\{\left[\left(A_1\cup A_2\right)+\left(\lambda I_1\cup\lambda I_2\right)\right]^*\left[\left(A_1\cup A_2\right)+\left(\lambda I_1\cup\lambda I_2\right)\right]^\dagger\left[\left(A_1\cup A_2\right)+\left(\lambda I_1\cup\lambda I_2\right)\right]^\dagger\left[\left(A_1\cup A_2\right)+\left(\lambda I_1\cup\lambda I_2\right)\right]^*\right\}$$

$$=\left\{\left[\left(A_1\cup A_2\right)+\lambda\left(I_1\cup I_2\right)\right]^*\left[\left(A_1\cup A_2\right)+\lambda\left(I_1\cup I_2\right)\right]^\dagger\left[\left(A_1\cup A_2\right)+\lambda\left(I_1\cup I_2\right)\right]^\dagger\left[\left(A_1\cup A_2\right)+\lambda\left(I_1\cup I_2\right)\right]^*\right\}$$

$$=\left\{\left[A_B+\lambda I_B\right]^*\left[A_B+\lambda I_B\right]^\dagger\left[A_B+\lambda I_B\right]^\dagger\left[A_B+\lambda I_B\right]^*\right\}$$

$\therefore\left\{\left(A_B+\lambda I_B\right)^\dagger\left(A_B+\lambda I_B\right)^*\left(A_B+\lambda I_B\right)^*\left(A_B+\lambda I_B\right)^\dagger\right\}=\left\{\left(A_B+\lambda I_B\right)^*\left(A_B+\lambda I_B\right)^\dagger\left(A_B+\lambda I_B\right)^\dagger\left(A_B+\lambda I_B\right)^*\right\}$ Hence $A_B+\lambda I_B$ is a Bi-star - Bi-dagger –Bi-matrix.

Proof of (ii)

Now, $\left[\left(A_B-\lambda I_B\right)^\dagger\left(A_B-\lambda I_B\right)^*\left(A_B-\lambda I_B\right)^*\left(A_B-\lambda I_B\right)^\dagger\right]$

$$=\left\{\left[\left(A_1\cup A_2\right)-\lambda\left(I_1\cup I_2\right)\right]^\dagger\left[\left(A_1\cup A_2\right)-\lambda\left(I_1\cup I_2\right)\right]^*\left[\left(A_1\cup A_2\right)-\lambda\left(I_1\cup I_2\right)\right]^*\left[\left(A_1\cup A_2\right)-\lambda\left(I_1\cup I_2\right)\right]^\dagger\right\}$$

$$=\left\{\left[\left(A_1\cup A_2\right)-\left(\lambda I_1\cup\lambda I_2\right)\right]^\dagger\left[\left(A_1\cup A_2\right)-\left(\lambda I_1\cup\lambda I_2\right)\right]^*\left[\left(A_1\cup A_2\right)-\left(\lambda I_1\cup\lambda I_2\right)\right]^*\left[\left(A_1\cup A_2\right)-\left(\lambda I_1\cup\lambda I_2\right)\right]^\dagger\right\}$$

$$=\left\{\left[\left(A_1-\lambda I_1\right)\cup\left(A_2-\lambda I_2\right)\right]^\dagger\left[\left(A_1-\lambda I_1\right)\cup\left(A_2-\lambda I_2\right)\right]^*\left[\left(A_1-\lambda I_1\right)\cup\left(A_2-\lambda I_2\right)\right]^*\left[\left(A_1-\lambda I_1\right)\cup\left(A_2-\lambda I_2\right)\right]^\dagger\right\}$$

$$=\left\{\left[\left(A_1-\lambda I_1\right)^\dagger\cup\left(A_2-\lambda I_2\right)^\dagger\right]\left[\left(A_1-\lambda I_1\right)^*\cup\left(A_2-\lambda I_2\right)^*\right]\left[\left(A_1-\lambda I_1\right)^*\cup\left(A_2-\lambda I_2\right)^*\right]\left[\left(A_1-\lambda I_1\right)^\dagger\cup\left(A_2-\lambda I_2\right)^\dagger\right]\right\}$$

$$=\left\{\left[\left(A_1^\dagger-\lambda^\dagger I_1\right)\cup\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\left[\left(A_1^*-\lambda^* I_1\right)\cup\left(A_2^*-\lambda^* I_2\right)\right]\left[\left(A_1^*-\lambda^* I_1\right)\cup\left(A_2^*-\lambda^* I_2\right)\right]\left[\left(A_1^\dagger-\lambda^\dagger I_1\right)\cup\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^\dagger-\lambda^\dagger I_1\right)\cup\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\left[\left(A_1^*+\bar{\lambda}I_1\right)\cup\left(A_2^*-\bar{\lambda}I_2\right)\right]\left[\left(A_1^*+\bar{\lambda}I_1\right)\cup\left(A_2^*-\bar{\lambda}I_2\right)\right]\left[\left(A_1^\dagger-\lambda^\dagger I_1\right)\cup\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^\dagger-\lambda^\dagger I_1\right)\left(A_1^*-\bar{\lambda}I_1\right)\left(A_1^*+\bar{\lambda}I_1\right)\left(A_1^\dagger-\lambda^\dagger I_1\right)\right]\cup\left[\left(A_2^\dagger-\lambda^\dagger I_2\right)\left(A_2^*-\bar{\lambda}I_2\right)\left(A_2^*-\bar{\lambda}I_2\right)\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\right\}$$

$$=\left\{\begin{array}{l}\left[\left(A_1^\dagger A_1^*-A_1^*\lambda^\dagger-\bar{\lambda}A_1^\dagger+\bar{\lambda}\lambda^\dagger\right)\left(A_1^\dagger A_1^\dagger-A_1^\dagger\bar{\lambda}-A_1^*\lambda^\dagger+\bar{\lambda}\lambda^\dagger\right)\right] \\ \qquad\qquad \cup\left[\left(A_2^\dagger A_2^*-\lambda^\dagger A_2^*-\bar{\lambda}A_2^\dagger+\bar{\lambda}\lambda^\dagger\right)\left(A_2^*A_2^\dagger-\bar{\lambda}A_2^\dagger-\lambda^\dagger A_2^*+\bar{\lambda}\lambda^\dagger\right)\right]\end{array}\right\}$$

$$=\left\{\begin{array}{l}\left[\left(A_1^*A_1^\dagger-A_1^*\lambda^\dagger-\bar{\lambda}A_1^\dagger+\bar{\lambda}\lambda^\dagger\right)\left(A_1^\dagger A_1^*-A_1^\dagger\bar{\lambda}-A_1^*\lambda^\dagger+\bar{\lambda}\lambda^\dagger\right)\right] \\ \qquad\qquad \cup\left[\left(A_2^*A_2^\dagger-A_2^*\lambda^\dagger-\bar{\lambda}A_2^\dagger+\bar{\lambda}\lambda^\dagger\right)\left(A_2^\dagger A_2^*-A_2^\dagger\bar{\lambda}-\lambda^\dagger A_2^*+\bar{\lambda}\lambda^\dagger\right)\right]\end{array}\right\}$$

$$=\left\{\left[A_1^*\left(A_1^\dagger-\lambda^\dagger I_1\right)-\bar{\lambda}\left(A_1^\dagger-\lambda^\dagger I_1\right)\right]\left[A_1^\dagger\left(A_1^*-\bar{\lambda}I_1\right)-\lambda^\dagger\left(A_1^*-\bar{\lambda}I_1\right)\right]\right\}$$

$$\cup\left\{\left[A_2^*\left(A_2^\dagger-\lambda^\dagger I_2\right)-\bar{\lambda}\left(A_2^\dagger-\lambda^\dagger I_2\right)\right]\left[A_2^\dagger\left(A_2^*-\bar{\lambda}I_2\right)-\lambda^\dagger\left(A_2^*-\bar{\lambda}I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^*-\bar{\lambda}I_1\right)\left(A_1^\dagger-\lambda^\dagger I_1\right)\left(A_1^\dagger-\lambda^\dagger I_1\right)\left(A_1^*-\bar{\lambda}I_1\right)\right]\cup\left[\left(A_2^*-\bar{\lambda}I_2\right)\left(A_2^\dagger-\lambda^\dagger I_2\right)\left(A_2^\dagger-\lambda^\dagger I_2\right)\left(A_2^*-\bar{\lambda}I_2\right)\right]\right\}$$

$$=\left\{\left[\left(A_1^*-\lambda^* I_1\right)\left(A_1^\dagger-\lambda^\dagger I_1\right)\left(A_1^\dagger-\lambda^\dagger I_1\right)\left(A_1^*-\lambda^* I_1\right)\right]\cup\left[\left(A_2^*-\lambda^* I_2\right)\left(A_2^\dagger-\lambda^\dagger I_2\right)\left(A_2^\dagger-\lambda^\dagger I_2\right)\left(A_2^*-\lambda^* I_2\right)\right]\right\}$$

$$=\left\{\left[(A_1-\lambda I_1)^*(A_1-\lambda I_1)^\dagger(A_1-\lambda I_1)^\dagger(A_1-\lambda I_1)^*\right]\cup\left[(A_2-\lambda I_2)^*(A_2-\lambda I_2)^\dagger(A_2-\lambda I_2)^\dagger(A_2-\lambda I_2)^*\right]\right\}$$

$$=\left\{\left[(A_1-\lambda I_1)^*\cup(A_2-\lambda I_2)^*\right]\left[(A_1-\lambda I_1)^\dagger\cup(A_2-\lambda I_2)^\dagger\right]\left[(A_1-\lambda I_1)^\dagger\cup(A_2-\lambda I_2)^\dagger\right]\left[(A_1-\lambda I_1)^*\cup(A_2-\lambda I_2)^*\right]\right\}$$

$$=\left\{\left[(A_1-\lambda I_1)\cup(A_2-\lambda I_2)\right]^*\left[(A_1-\lambda I_1)\cup(A_2-\lambda I_2)\right]^\dagger\left[(A_1-\lambda I_1)\cup(A_2-\lambda I_2)\right]^\dagger\left[(A_1-\lambda I_1)\cup(A_2-\lambda I_2)\right]^*\right\}$$

$$=\left\{\left[(A_1\cup A_2)-(\lambda I_1\cup\lambda I_2)\right]^*\left[(A_1\cup A_2)-(\lambda I_1\cup\lambda I_2)\right]^\dagger\left[(A_1\cup A_2)-(\lambda I_1\cup\lambda I_2)\right]^\dagger\left[(A_1\cup A_2)-(\lambda I_1\cup\lambda I_2)\right]^*\right\}$$

$$=\left\{\left[(A_1\cup A_2)-\lambda(I_1\cup I_2)\right]^*\left[(A_1\cup A_2)-\lambda(I_1\cup I_2)\right]^\dagger\left[(A_1\cup A_2)-\lambda(I_1\cup I_2)\right]^\dagger\left[(A_1\cup A_2)-\lambda(I_1\cup I_2)\right]^*\right\}$$

$$=\left\{\left[A_B-\lambda I_B\right]^*\left[A_B-\lambda I_B\right]^\dagger\left[A_B-\lambda I_B\right]^\dagger\left[A_B-\lambda I_B\right]^*\right\}$$

$$\therefore\left\{\left[A_B-\lambda I_B\right]^\dagger\left[A_B-\lambda I_B\right]^*\left[A_B-\lambda I_B\right]^*\left[A_B-\lambda I_B\right]^\dagger\right\}=\left\{\left[\left[A_B-\lambda I_B\right]^*\left[A_B-\lambda I_B\right]^\dagger\right]\left[A_B-\lambda I_B\right]^\dagger\left[A_B-\lambda I_B\right]^*\right\}$$

Hence $A_B-\lambda I_B$
is a Bi-star Bi-dagger-Bi-matrix.

References

[1] Anderson, W.N., Jr and Duffin, R.J., Series and parallel addition of matrices, *Journal of Mathematical Analysis and Applications*, 26 (1969), 576-594.

[2] Baskett, T.S. and Katz, I.J., Theorems on product of *EPr* matrices, *Linear Algebra AppL*, 2 (1969), 87-103.

[3] Campbell, S.L. and Mayer, C.D., EP operators and generalized inverses, *Cana. Math. Bull*, 18 (1975), 327-333.

[4] Cheng, S., Tian, Y., Two'sets of new characterizations for normal and EP matrices, *Linear Algebra AppL*, 375 (2003), 181-195.

[5] Hartwig, R.E., Spindelboek, K., matrices for which *A** and *A^* Commute, *Linear and Multilinear Algebra* 14 (1984), 241-256.

[6] Hartwig, R.E., Hatz, I.J., On product of EP matrices. *Linear Algebra Appl.*, 225 (1997), 339-345.

[7] Meyer, C.D., Some remarks on *EP*,. matrices and Generahzed inverses. *Linear Algebra Appl*, 3(1970), 275-278

[8] Moore, E.H., On the reciprocal of the generalized Algebraic matrix (Abstract), *Bull. Amer. Math. Soc.* , 26(1920), 394-395.

[9] Penrose, R., A generalized inverses for matrices, *Proc. Cambridge. Philos. Soc*, 51 (1955), 406-413.

[10] Schwardtfeger, H., Some remarks on generalized inverses of matrices, *J. Lin. Alg. AppL.*, 1 (1968), 325-328.

[11] Toeplitz, O., Das Algebraiche Analogen Zu einen satz Von, Fijer-Mat/i. *Zeitschrijt. 2* (1918) 187-197.

[12] Vasantha Kandaswamy, W.B., Florentian Samarandache, and Ilanthendral, K., "Introduction to bimatrices" 2005.

[13] Vasantha Kandaswamy, W.B., Florentian Samarandache, Ilanthendral,K., "Applications of bimatrices to some Fuzzy and Neutrosophic models", Hexis. Phonix, Arizone, 2005.