Compatible Uniformities on Pseudometric Spaces

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Abstract: ***In this paper we construct different uniformities on a pseudo metric space which are compatible with the topology of the pseudo metric space. It is proved that on a pseudo metric space there may be unequal uniformities compatible with the topology out of which one is uniformly continuous uniformity while the other need not be.***

Keywords: Pseudometric spaces, Completely regular space, Uniformity

1. Introduction

Let (X,d) be a pseudometric space. Then the product topology on X × X may be obtained by a pseudometric on X × X defined d((x,y),(p,q)) = d(x,p) + d(y,q).

2. Definition

Compatible uniformity on a topological space. A uniformity on a topological space X is said to be compatible with the topology of X if the later coincides with the topology induced by the uniformity.

3. Main Result

We had already constructed different uniformities $\mathcal{U}$ and $\mathcal{U}_1$ compatible with the topology of X [3] where $\mathcal{U}$ is the uniformity generated by continues real valued functions on a topological space X and $\mathcal{U}_1$ is the uniformity generated by continues bounded real valued functions on X. Here we construct a uniformity on a pseudometric space (X,d) which is also uniformly continuous uniformity compatible with the topology of (X,d). Here (X,$\mathcal{U}$) is called uniformly continuous uniform space if every $\mathcal{T}_{\mathcal{U}}$ - continuous real valued function is $\mathcal{U}$-uniformly continuous.

Theorem 01:-If (X, d) is a pseudo metric space and $\mathcal{U}$ is the set of all neighborhoods of Δ in the product topology of $X \times X$ then $\mathcal{U}$ is a uniformity on X. Here Δ is diagonal in $X \times X$.

Proof

Let $\mathcal{U}$ = {U/U is neighborhood of each point of Δ in $X \times X$ }.We show that $\mathcal{U}$ is uniformity on X.

1) If $U \in \mathcal{U}$and $V \supset U$ then for every $x \in X$,U being a neighborhood of (x, x) so is V. Thus every subset of $X \times X$ containing a set from $\mathcal{U}$ is in $\mathcal{U}$.
2) The intersection of two sets of $\mathcal{U}$ contains a set of $\mathcal{U}$.;
   Let U and V be neighborhoods of Δ in $X \times X$ and let $\in X$. Then there are $r_x > 0$ and $s_x > 0$ Such that $S_{r_x}^{(x,x)} \subset U$ and $S_{s_x}^{(x,x)} \subset V$.
   Here $S_{r_x}^{(x,x)}$ = $\{(p,q) \in X \times X \;/\; d((x,x),(p,q)) < r_x\}$.
   If t_x = min (r_x, s_x), $S_{t_x}^{(x,x)} \subset U \cap V$ i.e $U \cap V \in \mathcal{U}$
3) Every set of $\mathcal{U}$ contains the diagonal; It is obvious that every$V \in \mathcal{U}$ contains the diagonal.
4) If $V \in \mathcal{U}$ then there exists $V' \in \mathcal{U}$ such that $V' \subset V^{-1}$;
   Suppose $\in \mathcal{U}$. Then for every $\in X$, there is $r_x > 0$ such that $S_{r_x}^{(x,x)} \subset V$.
   Take $V' = \bigcup_{x\in X} S_{r_x}^{(x,x)}$. Thus obviously V' is neighborhood of each point $(x, x) \in X \times X$. Hence $V' \in \mathcal{U}$.
   Further if $(u,v) \in V'$, $(u, v) \in S_{r_x}^{(x,x)}$ for some $x \in X$.
   Since $S_{r_x}^{(x,x)} \subset V$ and $(v, u) \in S_{r_x}^{(x,x)}$,
   $(v,u) \in V$ i.e. $(u, v) \in V^{-1} \Rightarrow V' \subset V^{-1}$.
5) If $V \in \mathcal{U}$ then there exists $W \in \mathcal{U}$ such that $W \circ W \subset V$.

Let $V \in \mathcal{U}$. Then for every $\in X$. there is $r_x > 0$ such that $S_{r_x}^{(x,x)} \subset V$

Let $W = \bigcup_{x\in X} S_{r_x/2}^{(x,x)}$ Then $W \in \mathcal{U}$. We show that $W \circ W \subset V$.

Suppose $(p,q) \in W \circ W$. Then there is $z \in X$ is such that $(p, z) \in S_{\frac{r_u}{2}}^{(u,u)}$ for some $u \in X$ i.e. $d(p,u) + d(z,u) < \frac{r_u}{2}$ -------(1)

and $(z, q) \in S_{\frac{r_v}{2}}^{(v,v)}$ for some $v \in X$ i.e.

$d(z,v) + d(q,v) < \frac{r_v}{2}$ -------(2)

Adding (1) and (2) ,

$d(p,u) + d(z,u) + d(z,v) + d(q,v) < \frac{r_u}{2}+\frac{r_v}{2}$ -- (3)

If $r_v \leq r_u$ then $d(p,u) + d(q,u)$

$< d(p,u) + d(q,v) + d(v,z) + d(z,u)$

$< \frac{r_u}{2} + \frac{r_v}{2} \leq r_u \Longrightarrow (p,q) \in S_{r_u}^{(u,u)}$

But $S_{r_u}^{(u,u)} \subset V$.

$\therefore (p,q) \in V$

Similarly if $r_u \leq r_v$ then $(p,q) \in S_{r_v}^{(v,v)} \subset V$,

$\therefore (p,q) \in V$. Thus $W \circ W \subset V$.

Theorem 02:- If $(X, \mathcal{T})$ be a topological space such that neighborhoods of Δ form a uniformity $\mathcal{U}$ on X then $\mathcal{T}_{\mathcal{U}} \subset \mathcal{T}$.

Proof: - Let $A \in \mathcal{T}_{\mathcal{U}}$. Then we show that $A \in \mathcal{T}$.

Let $\in A$. Choose symmetric $U \in \mathcal{U}$ such that $x \in U[x] \subset$ A.

Since U is neighborhood of Δ , U is neighborhood of (y, y) for each $y \in X$.

Hence U is neighborhood of (x,x).Choose open sets $G_1, G_2 \in \mathcal{T}$ such that
$(x,x) \in G_1 \times G_2 \subset U$
Now we show that $x \in G_1 \subset U[x]$. Let $y \in G_1$.
Then $(y,x) \in G_1 \times G_2 \subset \cup$ i,e. $y \in U[x]$
i,e. $G_1 \subset U[x] \subset A$.
i.e A is neighborhood of each point of Δ.
Thus $A \in \mathcal{T}$. This proves that $\mathcal{T}_{\mathcal{U}} \subset \mathcal{T}$.

Theorem 03:- Let $(X,\mathcal{T})$ be a completely regular space such that neighborhoods of Δ form a uniformity $\mathcal{U}$ on X .Then $\mathcal{T} \subset \mathcal{T}_{\mathcal{U}}$.

Proof: Suppose $G \in \mathcal{T}$ and $x \in G$. Since X is completely regular there exists a $\mathcal{T}$- continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and
$f(X - G) = 0$.
Take $U = (G \times G) \cup (G_1 \times G_2)$ where $G_1 = f^{-1}(\frac{-1}{2},\frac{1}{2}) \in \mathcal{T}$.
We show that $U \in \mathcal{U}$ and $U[x] \subset$ G. If $y \in G$
then $(x,y) \in G \times G \subset U$.
If $y \notin G$ then $f(y) = 0 \therefore y \in G_1$.
$\therefore (y,y) \in G_1 \times G_1 \subset U$.
Thus U is neighborhood of each point of Δ.

Now we show that $U[x] \subset G$. For $y \in U[x]$
$(x,y) \in$ U $\Rightarrow (x,y) \in G \times G$ or $(x,y) \in G_1 \times G_1$.
If $(x,y) \in G \times G$,then $x \in G$ and $y \in G$. i.e. $y \in G$.
If $(x,y) \in G_1 \times G_1$then $x \in G_1$and $y \in G_1$.
Since $G_1 = f^{-1}(\frac{-1}{2},\frac{1}{2})$ and $f(x) = 1$, $x \notin G_1$.
Thus $(x,y) \in G_1 \times G_1$is not possible.
$\therefore (x,y) \in G \times G$ and $U[x] \subset G$ i.e. $G \in \mathcal{T}_{\mathcal{U}}$.
Thus $\mathcal{T} \subset \mathcal{T}_{\mathcal{U}}$.

Theorem 04:- If (X,d) is a pseudo metric space and $\mathcal{U}$ is the set of all neighborhoods of Δ in the product topology of $X \times X$ then topology of uniformity $\mathcal{U}$ coincides with the original topology of (X,d).

Proof: By Theorem 1 $\mathcal{U}$ becomes a uniformity on X.
By Theorem 2 topology generated by $\mathcal{U}$ is coarser than original topology of (X,d).
Since X is a pseudo metric space it is completely regular. Hence by theorem 4.1.4,
$\mathcal{T} \subset \mathcal{T}_{\mathcal{U}}$. Thus $\mathcal{T}_{\mathcal{U}} = \mathcal{T}$.

Theorem05: Suppose (X,d) is a pseudo metric space. $\mathcal{U}_d$ is uniformity generated by d whose base is given by$\{(x,y) / d(x,y) < r\}, r >$o and $\mathcal{U}$ is a uniformity on X consisting of all neighborhoods of Δ .Then $\mathcal{U}_d \subset \mathcal{U}$, but in general $\mathcal{U}_d \neq \mathcal{U}$.

Proof: Let $V \in \mathcal{U}_d$ .Then there is $r >$o Such that $V \supset \{(x,y) /$ d$(x,y) < r\}$.
Put $U = \{(p,q) / d(p,q) < r\}$.

We show that U is neighborhood of (x,x) for each $x \in X$.
Suppose $x \in X$. We show that $S_r^{(x,x)}$ which is a neighborhood of (x,x) is contained in U. Suppose $(u,v) \in S_r^{(x,x)}$. Then $d(u,x) + d(v,x) < r$
i.e. $d(u,v) \le d(u,x) + d(x,v) < r$ i.e $d(u,v) < r$. i.e $(u,v) \in U$. Thus $S_r^{(x,x)} \subset U$ i.e U is neighborhood of each point of Δ i.e, $U \in \mathcal{U}$.

Since $U \subset V$ and $\mathcal{U}$ is uniformity $\therefore V \in \mathcal{U}$.Thus $\mathcal{U}_d \subset \mathcal{U}$.
Now to show that in general $\mathcal{U}_d \neq \mathcal{U}$,
We construct a pseudo metric space (X,d) where $\mathcal{U} \not\subset \mathcal{U}\, d$.
Take $X = \boldsymbol{R}$ and $d(x,y) = |x - y|\ x,y \in \boldsymbol{R}$.
Take $U = \{(x,y) / |x^2 - y^2| \le 1\}$.
Then we show that U is neighborhood of each point of Δ . So that $U \in \mathcal{U}$.
Suppose $x \in \boldsymbol{R}$ .Take $r = \frac{1}{2\,(1+2|x|)}$ .We claim that $S_r^{(x,x)} \subset U$
Suppose $(u,v) \in S_r^{(x,x)}$. Then $|u - x| + |v - x| < r$
$\Rightarrow |u - x| < r$ and $|v - x| < r$

Consider $|u^2 - v^2| = |u^2 - x^2 + x^2 - v^2|$
$\le |u^2 - x^2| + |v^2 - x^2|$
$\le |u - x|\,|u + x| + |v - x|\,|v + x|$
Now $|u + x| = |u - x + 2x|$
$\le |u - x| + 2|x|$
$< r + 2|x|$
Since $r < 1$, $|u + x| < 1 + 2|x|$
Similarly $|v + x| \le 1 + 2|x|$
Thus $|u^2 - v^2| < \frac{1}{2(1+2|x|)}(1 + 2|x|) + \frac{1}{2(1+2|x|)}(1 + 2|x|) \le \frac{1}{2}+\frac{1}{2} \le 1$
i.e $(u,v) \in U$ $\therefore U$ is neighborhood of each point of Δ.
Now we show that $U \notin \mathcal{U}_d$
i.e There does not exist $s > o$ such that$\{(x,y) / |x - y| < s\} \subset U$.
Thus we have to show that for every $s > o$
there exists (x_s , y_s) such that $|x_s - y_s| < s$
But $(x_s , y_s) \notin U$. For any $s > o$,
take $x_s = \frac{1}{s}$ and $y_s = \frac{1}{s} + \frac{s}{2}$.
Then $|x_s - y_s| = \left|\frac{1}{s} - (\frac{1}{s} + \frac{s}{2})\right| < \left|\frac{s}{2}\right| < s$.
Also $|x_s^2 - y_s^2| = |(x_s + y_s)(x_s - y_s)|$
$= |x_s + y_s|\,|x_s - y_s|$
$= \left|\frac{1}{s} + (\frac{1}{s} + \frac{s}{2})\right| \left|\frac{1}{s} - (\frac{1}{s} + \frac{s}{2})\right|$
$= (\frac{2}{s} + \frac{s}{2})\,(\frac{s}{2}) = 1 + \frac{s^2}{4} > 1$
$\therefore |x_s^2 - y_s^2| > 1$ i.e $(x_s , y_s) \notin U$.
This proves that $\mathcal{U} \not\subseteq \mathcal{U}_d$.

Theorem 06: Let $(X,\mathcal{T})$ be a completely regular space, such that neighborhoods of Δ form a uniformity $\mathcal{U}$. Then $\mathcal{U}$ is compatible with $\mathcal{T}$ and $(X,\mathcal{U})$ is a uniformly continuous space.

Proof: By Theorem 4, $\mathcal{U}$ is compatible with $\mathcal{T}$. To prove that f is $\mathcal{U}$-uniformly continuous real function if f is $\mathcal{T}$-continuous real function. Suppose f is $\mathcal{T}$- continuous function and $\in >$ o is given. We have to find $U \in \mathcal{U}$ such that $(x,y) \in U \Rightarrow |f(x) - f(y)| < \in$. For $x \in X$

Put $G_x = f^{-1}(f(x) - \frac{\in}{2}, f(x) + \frac{\in}{2})$.Since f is $\mathcal{T}$ continuous at x,
G_x is an open set containing x. Take $U = \cup_{x \in X} G_x \times G_x$.

Then U is neighborhood of each point (x, x) of Δ.
Now we show that $(u, v) \in U \Rightarrow |f(u) - f(v)| < \in$.
Let $(u, v) \in U$.Choose $x \in$ X such that $(u, v) \in G_x \times G_x$.
Thus $u \in G_x$ and $v \in G_x$.
Since $G_x = f^{-1}(f(x) - \in /2, f(x) + \in/2)$ and $u \in G_x$, $v \in G_x$.
$f(u) \in (f(x) - \in /2, f(x) + \in/2)$ and $f(v) \in (f(x) - \in /2, f(x) + \in/2)$.
i.e $|f(u) - f(x)| < \in/2$ and $|f(v) - f(x)| < \in/2$
Hence $|f(u) - f(v)| = |f(u) - f(x) + f(x) - f(v)|$
$\leq |f(u) - f(x)| + |f(x) - f(v)|$
$< \frac{\in}{2} + \frac{\in}{2} < \in$.
Thus there is $\boldsymbol{U} \in \boldsymbol{\mathcal{U}}$ such that $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{U} \Rightarrow |\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})| < \in$.

This proves that f is $\mathcal{U}$-uniformly continuous function.

Hence $(X, \mathcal{U})$ is a uniformly continuous uniform space.

Theorem 07:- Let (X, d) be a pseudo metric space and $\mathcal{U}$ be the uniformity consisting of all neighborhoods of Δ. Then $(X, \mathcal{U})$ is a uniformly continuous space and. $(X, \mathcal{U}_d)$ need not be a uniformly continuous space.

Proof: - From Theorem 6 it follows that,
(X, $\mathcal{U}$) is a uniformly continuous space.

Now we show that $(X, \mathcal{U}_d)$ need not be a uniformly continuous space. Consider X = **R** and function f : **R** → **R** defined by $f(x) = x^2$, $x \in$ **R**. f is continuous on **R**. We show that f is not $\mathcal{U}_d$ uniformly continuous function. For any $\delta > o$, take $x_\delta = \frac{1}{\delta}$ and $y_\delta = \frac{1}{\delta} + \frac{\delta}{2}$
Then $|x_\delta - y_\delta| = \frac{\delta}{2} < \delta$
But $|f(x_\delta) - f(y_\delta)|$

$$= \left|(\frac{1}{\delta})^2 - (\frac{1}{\delta} + \frac{\delta}{2})^2\right|$$

$$= \left|\frac{1}{\delta^2} - \frac{1}{\delta^2} - 2\frac{1}{\delta}\frac{\delta}{2} - \frac{\delta^2}{4}\right|$$

$$= \left|(1 + \frac{\delta^2}{4})\right| = 1 + \frac{\delta^2}{4} \geq 1$$

Thus for $\in = 1$, there does not exist any $\delta > o$,
$|x - y| < \delta \Rightarrow |f(x) - f(y)| < 1$ is satisfied.

Thus for $\in = 1$, there does not exist any $\delta > o$, $|x_\delta - y_\delta| < \delta$
$\Rightarrow |f(x_\delta) - f(y_\delta)| < \in$.
i.e f is not $\mathcal{U}_d$ uniformly continuous function.
$\therefore (\boldsymbol{R}, \mathcal{U}_d)$ is not uniformly continuous space.

4. Conclusion

It is proved that on a pseudo metric space there may be unequal uniformities compatible with the topology out of which one is uniformly continuous uniformity while the other need not be.

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