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A Markov-Bernstein Type Inequality for Generalized Polynomial of Generalized Degree

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Abstract

Let $0 < p < \infty$ *and* $0 \leq \alpha < \beta \leq 2\pi$. *Let* $P(z) := \omega \prod_{j=1}^{n} (z - z_j)^{r_j}$, $r_j(\geq 1)$ *is rational,* $\omega \in \mathbb{C}$, *be a nonnegative generalized polynomial with generalized degree* $N := \sum_{j=1}^{n} r_j$. *Let*

$$\varepsilon_N(z) := \left[\frac{|z - e^{i\alpha}||z - e^{i\beta}| + \left(\frac{\beta - \alpha}{N}\right)^2}{\left(z + e^{i\frac{\alpha+\beta}{2}}\right)^2 + \left(\frac{1}{N}\right)^2} \right]^{1/2}$$

We show that the following result which we proved in [5] holds for this special case

$$\int_\alpha^\beta \left|(P'\varepsilon_N)(e^{i\theta})\right|^p d\theta \leq C \int_\alpha^\beta \left|(P)(e^{i\theta})\right|^p d\theta$$

Keywords: Large sieve, Generalized Polynomial, Arcs of the Circle

Introduction and Results

The classical Markov-Bernstein Inequality for trigonometric polynomials

$$S_n(\theta) := \sum_{j=0}^{n} \left(c_j \cos j\theta + d_j \sin j\theta\right)$$

of degree $\leq n$ is

$$\|S'_n\|_{L_\infty[0,2\pi]} \leq n\|S\|_{L_\infty[0,2\pi]} \quad (1)$$

As an L_p analogue of (1), D S Lubinsky [7] proved the following inequality

$$\int_\alpha^\beta |S'_n(\theta)|^p \left[\left|\sin\left(\frac{\theta - \alpha}{2}\right)\right| \left|\sin\left(\frac{\theta - \beta}{2}\right)\right| + \left(\frac{\beta - \alpha}{n}\right)^2 \right]^{p/2} d\theta \leq C n^p \int_\alpha^\beta |S_n(\theta)|^p d\theta \quad (2)$$

for trigonometric polynomials S_n of degree$\leq n$ $(n \geq 1)$, $0 < p < \infty$, and $0 \leq \alpha < \beta \leq 2\pi$, where C

is a constant independent of α, β, n, S_n. This inequality is almost certainly sharp with respect to the growth in n when $[\alpha, \beta]$ is fixed proper subinterval of $(0, \pi)$, and most specially when $[\alpha, \beta]$ is small. But it is not sharp when $[\alpha, \beta]$ approaches $[0, 2\pi]$. In [5], we established the following inequality which is almost certainly sharp for $[\alpha, \beta]$ close to $[0, 2\pi]$.

$$\int_{\alpha}^{\beta} |S'_n(\theta)|^p \left[\frac{\left|\sin\left(\frac{\theta-\alpha}{2}\right)\right| \left|\sin\left(\frac{\theta-\beta}{2}\right)\right| + \left(\frac{\beta-\alpha}{n}\right)^2}{\left(\cos\frac{\theta-\frac{\alpha+\beta}{2}}{2}\right)^2 + \left(\frac{1}{n}\right)^2} \right]^{p/2} d\theta \le Cn^p \int_{\alpha}^{\beta} |S_n(\theta)|^p d\theta \quad (3)$$

with the same conditions as in (2). We deduced (3) from the following theorem which we proved using the proofs of some technical lemmas.

Theorem 1 (CKK, DSL)

Let $0 < p < \infty$, and $0 \le \alpha < \beta \le 2\pi$ and let

$$\varepsilon_n(z) := \left[\frac{|z - e^{i\alpha}||z - e^{i\beta}| + \left(\frac{\beta-\alpha}{n}\right)^2}{\left(z + e^{i\frac{\alpha+\beta}{2}}\right)^2 + \left(\frac{1}{n}\right)^2} \right]^{p/2}$$

Then for algebraic polynomials P of degree $\le n$ $(n \ge 1)$, we have

$$\int_{\alpha}^{\beta} \left|(P'\varepsilon_n)(e^{i\theta})\right|^p d\theta \le C \int_{\alpha}^{\beta} \left|(P)(e^{i\theta})\right|^p d\theta$$

where C is a constant independent of α, β, n, S_n.

In this paper, we prove that the Theorem 1 still hold for generalized polynomial of generalized degree $N := \sum_{j=1}^{n} r_j$ where $r_j (\ge 1)$ is rational.

Theorem 2

Let $0 < p < \infty$ and $0 \le \alpha < \beta \le 2\pi$. Let $P(z) := \omega \prod_{j=1}^{n} (z - z_j)^{r_j}$, $r_j (\ge 1)$ is rational, $\omega \in \mathbb{C}$, be a nonnegative generalized polynomial with generalized degree $N := \sum_{j=1}^{n} r_j$. Let

$$\varepsilon_N(z) := \left[\frac{|z - e^{i\alpha}||z - e^{i\beta}| + \left(\frac{\beta-\alpha}{N}\right)^2}{\left(z + e^{i\frac{\alpha+\beta}{2}}\right)^2 + \left(\frac{1}{N}\right)^2} \right]^{1/2}$$

Assume $\|P\| \le C|P(z)|$ for sufficient C. Then we have

$$\int_{\alpha}^{\beta} \left|(P'\varepsilon_N)(e^{i\theta})\right|^p d\theta \le C \int_{\alpha}^{\beta} \left|(P)(e^{i\theta})\right|^p d\theta$$

Proof

Let $r_j = \frac{q_j}{q}$ with $q_j, q \in \mathbb{N}$. Then $f := P^{2q}$ is an algebraic polynomial of degree $\leq 2qN$. Suppose that f has all its zeros outside the unit circle. Then (by Erdos, Lax) we have

$$\|f'\| \leq \frac{2qN}{2}\|f\| = qN\|f\| \quad (4)$$

Now from $f = P^{2q}$, we have

$$f' = 2qP^{2q-1}P' = 2qP^{2q}\frac{P'}{P} \quad (5)$$

Then

$$2q|P^{2q}(z)|\frac{|P'(z)|}{|P(z)|} = 2q|f(z)|\frac{|P'(z)|}{|P(z)|} = |f'(z)|$$
$$\leq \|f'\|$$
$$\leq qN\|f\| \text{ by (4)}$$

Thus

$$\frac{|P'(z)|}{|P(z)|} \leq \frac{N}{2}\frac{\|f\|}{|f(z)|} \leq \frac{C_1}{2}N \quad (6)$$

since by our assumption on the theorem that $\|P\| \leq C|P(z)|$ for sufficient C.

Now, applying the Theorem 1 to $f := P^{2q}$, we get

$$C\int_\alpha^\beta \left|P^{2q}(e^{i\theta})\right|^p d\theta \geq \int_\alpha^\beta \left|2qP^{2q-1}P'\varepsilon_N(e^{i\theta})\right|^p d\theta$$
$$= \int_\alpha^\beta \left|2q\left(\frac{P}{P'}\right)^{2q-1}(P')^{2q}\varepsilon_N(e^{i\theta})\right|^p d\theta$$
$$\geq (2q)^p\left(\frac{2}{C_1N}\right)^{p(2q-1)}\int_\alpha^\beta \left|\frac{\left(P'\varepsilon_N(e^{i\theta})\right)^{2q}}{\left(\varepsilon_N(e^{i\theta})\right)^{2q-1}}\right| d\theta \quad (7)$$

since $\left|\frac{P}{P'}\right| \geq \frac{2}{C_1N}$ by (6).

From Lemma 1.3 in [5], we have shown that

$$\left|\varepsilon_N(e^{i\theta})\right| \leq \frac{6\cos\left(\frac{\alpha}{2}\right)}{2Nq} \leq \frac{3}{Nq}$$
$$\Rightarrow \frac{1}{\left(\varepsilon_N(e^{i\theta})\right)^{(2q-1)p}} \geq \left(\frac{Nq}{3}\right)^{p(2q-1)}$$

Now (7) gives

$$C\int_\alpha^\beta \left|P^{2q}(e^{i\theta})\right|^p d\theta \geq (2q)^p\left(\frac{2}{\underline{C_1N}}\right)^{p(2q-1)}\int_\alpha^\beta \left|\left(P'\varepsilon_N(e^{i\theta})\right)^{2q}\right|^p d\theta$$

This implies

$$\int_{\alpha}^{\beta}\left|\left(P'\varepsilon_N\left(e^{i\theta}\right)\right)^{2q}\right|^{p}d\theta \le C\left(\frac{3C_1}{2q}\right)^{p(2q-1)}\frac{1}{(2q)^p}\int_{\alpha}^{\beta}\left|P^{2q}\left(e^{i\theta}\right)\right|^{p}d\theta$$

$$\le C(3C_1)^{p(2q-1)}\left(\frac{1}{2q}\right)^{2pq}\int_{\alpha}^{\beta}\left|P^{2q}\left(e^{i\theta}\right)\right|^{p}d\theta$$

Now replacing p by $\frac{p}{2q}$

$$\int_{\alpha}^{\beta}\left|P'\varepsilon_N\left(e^{i\theta}\right)\right|^{p}d\theta \le \int_{\alpha}^{\beta}\left|P\left(e^{i\theta}\right)\right|^{p}d\theta$$

where C_p is a constant depending only on p, not on $\alpha, \beta. P,$ and N ■

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