On Regular Beta (rβ) - Closed Sets in Topological Spaces

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Abstract: ***In this paper, we introduce a new class of closed sets called regular β closed sets (briefly rβ – closed sets) in topological Spaces. Here we study the relation of these sets with some of the existing closed and generalized closed sets. Also we study some of the characteristics of rβ closed sets.***

Keywords: Topological spaces, rβ - closed set, β - open, regular closure

1. Introduction

In 1983, M.E. Abd El- Monsef [1] introduced β – open set and β – closed set in a Topological spaces. In 1986 D. Andrijevic [3] introduced semi-pre open set and semi pre closed set. By the definitions both β open– sets and semi-pre open sets are the same. Regular open sets are introduced and investigated by M.H. Stone [24]. In this paper we recall some of the existing closed and open sets and introduce the concept of rβ - closed sets. Here the relation of rβ - closed sets with some of the existing closed and generalized closed sets are studied and its outcome is shown in the form of a diagram. Moreover in this paper, some of the characteristics of rβ - closed sets are studied, analysed and proved.

2. Preliminaries

Throughout this paper (X, τ) represent the topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, the closure of A and interior of A are denoted by cl (A) and int (A) respectively.

Definition 2.1:

A subset A of a topological space (X, τ) is called a

(1) Semi - open set [13] if $A \subset cl\ (int\ (A))$ and a semi - closed set if $int\ (cl\ (A)) \subset A$.
(2) Pre - open set [18] if $A \subset int\ (cl\ (A))$ and pre - closed if $cl\ (int\ (A)) \subset A$.
(3) β (or semi pre [3]) open set [1] if $A \subset cl\ (int\ (cl\ (A)))$ and β (or semi pre [3]) closed set [1] if $int\ (cl\ (int\ (A))) \subset A$
(4) Regular open set [24] if $A = int\ (cl\ (A))$ and a regular closed set if $cl\ (int\ (A)) = A$.
(5) Regular semi open [7] if there is a regular open set U such that $U \subset A \subset cl\ (U)$ and its compliment is a regular semi closed set.
(6) α- open set [19] if $A \subset int\ (cl\ (int\ (A)))$ and α - closed set if $cl\ (int\ (cl\ (A))) \subset A$.
(7) Regular α- open set [25] if there is a regular open set U such that $U \subset A \subset \alpha\ cl\ (U)$ and its compliment is a regular α – closed set.
(8) t -set [11] if and only if int (A) = int (cl (A)

Definition 2.2:

A subset A of a topological space (X, τ) is called a

(1) generalized closed set (briefly g-closed) [14] ifcl $(A) \subset U$ whenever $A \subset U$ and U is open in X
(2) generalized pre - closed set (briefly gp - closed) [17] if pcl $(A) \subset U$ whenever $A \subset U$ and U is open in X
(3) generalized semi - closed set (briefly gs - closed) [4] if scl $(A) \subset U$ whenever $A \subset U$ and U is open in X
(4) generalized semi - pre closed set (briefly gsp - closed) [9] if spcl $(A) \subset U$ whenever $A \subset U$ and U is open in X.
(5) generalized regular closed set (briefly gr - closed) [6] if rcl $(A) \subset U$ whenever $A \subset U$ and U is open in X.
(6) generalized pre - regular closed set (briefly gpr-closed) [10] if pcl $(A) \subset U$ whenever $A \subset U$ and U is regular open in X
(7) generalized semi - pre regular - closed set (briefly gspr - closed) [20] if spcl $(A) \subset U$ whenever $A \subset U$ and U is regular open in X.
(8) generalized star pre closed set (briefly g*p - closed) [26] if pcl $(A) \subset U$ whenever $A \subset U$ and U is g -open in X
(9) generalized star semi closed set (briefly g*s - closed) [22] if scl $(A) \subset U$ whenever $A \subset U$ and U is gs -open in X
(10) generalized regular star closed set (briefly gr*-closed) [12] if rcl $(A) \subset U$ whenever $A \subset U$ and U is g -open in X.
(11) generalized α- closed set (briefly gα - closed) [15] if αcl $(A) \subset U$ whenever $A \subset U$ and U is α- open in X.
(12) Semi - generalized closed set (briefly sg - closed) [5] if scl $(A) \subset U$ whenever $A \subset U$ and U is semi open in X.
(13) α-generalized closed set (briefly αg - closed) [16]if αcl $(A) \subset U$ whenever $A \subset U$ and U is open in X
(14) regular generalized closed set (briefly rg - closed) [21] if cl $(A) \subset U$ whenever $A \subset U$ and U is regular - open in X.
(15) βg*- closed set [8] if gcl $(A) \subset U$ whenever $A \subset U$ and U is β - open in X
(16) r ^ g - closed set [23] if gcl $(A) \subset U$ whenever $A \subset U$ and U is regular - open in X.
(17) tgr closed set [2] if rcl $(A) \subset U$ whenever $A \subset U$ and U is a t-set in X

The complements of the above open sets are their closed sets.

Definition 2.3 [25]:

The regular closure of a subset A ⊂ X is defined as the set rcl (A) = ∩ {B ⊂ X: B is regular closed and A ⊂ B}

3. Regular Beta (rβ) - Closed Sets

Definition 3.1:

A subset A of a topological space (X, τ) is called regular beta closed set (briefly rβ- closed) if rcl (A) ⊂ U whenever A ⊂ U and U is β - open and its compliment is called a regular beta open set (briefly rβ- open set).

Example 3.2:

Let X= {a, b, c, d}, and
τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}
β - open = {Φ, {a}, {b}, {a, b}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}, X}
rβ - closed = {Φ, {d}, {a, d}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}, X}
rβ – open = {Φ, {a}, {b}, {d}, {a, b}, {b, c}, {a, b, c}, X}

Proposition 3.3:

Every regular closed set is rβ - closed.

Proof:

Let A be a regular closed set in X such that A ⊂ U and U is β - open. Then rcl (A) = A. Hence rcl (A) ⊂ U. Therefore A is rβ- closed.

Remark 3.4:

The converse of the above proposition need not be true as shown in the following example.

Example 3.5: Let X ={a, b, c, d}, and τ ={Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}.Then the set {d} is rβ - closed but not regular closed.

Proposition 3.6:

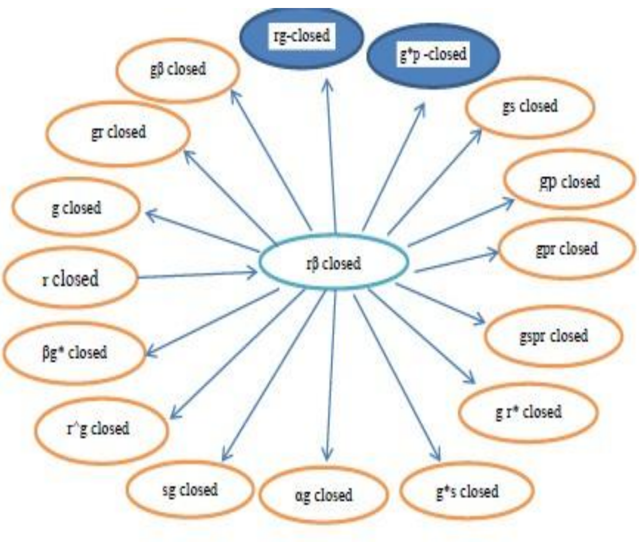
1. Every rβ - closed set is g - closed.
2. Every rβ - closed set is rg - closed.
3. Every rβ - closed set is gr - closed.
4. Every rβ - closed set is gβ- closed
5. Every rβ - closed set is gs - closed.
6. Every rβ - closed set is gp - closed.
7. Every rβ - closed set is gpr – closed.
8. Every rβ - closed set is gspr-closed
9. Every rβ - closed set is gr*-closed.
10. Every rβ - closed set is g*p- closed
11. Every rβ - closed set is g*s- closed
12. Every rβ - closed set is αg - closed
13. Every rβ - closed set is sg – closed
14. Every rβ - closed set is βg* - closed
15. Every rβ - closed set is r ^ g - closed

Proof:

1. Let A be a rβ - closed set in X. Let U be an open set in X such that A ⊂ U. Since every open set is β-open and since A is rβ - closed, rcl (A) ⊂ U. But we have cl (A) ⊂ rcl (A) ⊂ U. Hence A is g-closed.
2. Let A be a rβ -closed set in X. Let U be a regular open set in X such that A ⊂ U. Since every regular open set is β - open and A is rβ - closed, rcl (A) ⊂ U. But we have cl (A) ⊂ rcl (A) ⊂ U. Hence A is rg-closed.
3. Let A be a rβ - closed set in X. Let U be an open set in X such that A ⊂ U. Since every open set is β- open and since A is rβ-closed, rcl (A) ⊂ U. Hence A is gr- closed.
4. Let A be a rβ - closed set in X. Let U be an open set in X such that A ⊂ U. Since every open set is β - open and since A is rβ-closed, rcl (A) ⊂ U. But we have spcl (A) ⊂ rcl (A) ⊂ U. Hence A is gβ- closed.
5. Let A be a rβ - closed set in X. Let U be an open set in X such that A ⊂ U. Since every open set is β -open and since A is rβ-closed, rcl (A) ⊆ U. But we have scl (A) ⊂ rcl (A) ⊂ U. Hence A is gs-closed.
6. Let A be a rβ - closed set in X. Let U be an open set in X such that A ⊂U. Since every open set is β- open and since A is rβ-closed, rcl (A) ⊂ U. But we have pcl (A) ⊂ rcl (A) ⊂ U. Hence A is gp-closed.
7. Let A be a rβ - closed set in X. Let U be a regular open set in X such that A ⊂ U. Since every regular open set is β - open and A is rβ-closed, rcl (A) ⊂ U. But we have pcl (A) ⊂ rcl (A) ⊂ U. Hence A is gpr-closed.
8. Let A be a rβ - closed set in X. Let U be a regular open set in X such that A ⊂ U. Since every regular open set is β - open and A is rβ-closed, rcl (A) ⊂ U. But we have spcl (A) ⊂ rcl (A) ⊂ U. Hence A is gspr-closed.
9. Let A be a rβ - closed set in X. Let U be a g- open set in X such that A ⊂ U. Since every g-open set is β- open and since A is rβ-closed, rcl (A) ⊂ U. Hence A is gr*- closed
10. Let A be a rβ - closed set in X. Let U be a g - open set in X such that A ⊂ U. Since every g - open set is β - open and A is rβ-closed, rcl (A) ⊂ U. But we have pcl (A) ⊂ rcl (A) ⊂ U. Hence A is g*p- closed.
11. Let A be a rβ - closed set in X. Let U be a gs open set in X such that A ⊂ U. Since every gsopen set is β -open and since A is rβ-closed, rcl (A) ⊂ U. But we have scl (A) ⊂ rcl (A) ⊂ U. Hence scl (A) ⊂ U. Hence A is g *s – closed.
12. Let A be a rβ - closed set in X. Let U be an open set in X such that A ⊂ U. Since every open set is β -open and since A is rβ-closed, rcl (A) ⊂ U. But we have αcl (A) ⊂ rcl (A) ⊂ U. Hence αcl (A) ⊂ U. Hence A is αg-closed.
13. Let A be a rβ - closed set in X. Let U be a semi open set in X such that A ⊂ U. Since every semi open set is β - open and since A is rβ-closed, rcl (A) ⊂ U. But we have scl (A) ⊂ rcl (A) ⊂ U. Hence scl (A) ⊂ U. Hence A is sg-closed..
14. Let A be a rβ - closed set in X. Let U be a β open set in X such that A ⊂ U. Since A is rβ-closed, rcl (A) ⊂U but we have gcl (A) ⊂ rcl (A) ⊂ U. Hence gcl (A) ⊂ U. Hence A is β g* - closed.
15. Let A be a rβ - closed set in X. Let U be a regular open set in X such that A ⊂ U. Since every regular open set is β -open and since A is rβ-closed, rcl (A) ⊂ U. But we have gcl (A) ⊂ rcl (A) ⊂ U. Hence gcl (A) ⊂ U. Hence A is r^ g – closed.

Remark 3.7:

From the above discussions and known results the relationship between rβ-closed sets and other existing generalizations of closed sets are implemented in the following figure



In the above figure, A B means the set A implies the set B

Remark 3.8:

The converse of the above propositions need not be true as shown in the following example.

Example 3.9:

1 Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}.Then{c} is g - closed but not rβ-closed.
2. Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}.Then{a, c} is rg- closed but not rβ-closed
3. Let X = {a, b, c, d}, and τ = {Φ, {c}, {d}, {c, d}, {a, c, d} X}.Then {b, d} is gr-closed but not rβ-closed.
4. Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X. Then {a, c} is gβ-closed but not rβ-closed.
5. Let X = {a, b, c, d}, and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then {b, c} is gs-closed but not rβ-closed.
6. Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then {c} isgp-closed but not rβ-closed.
7. Let X = {a, b, c, d} and τ ={Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then {a, b, c} is gpr-closed but not rβ-closed.
8. Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then {a, b, d} is gspr-closed but not rβ-closed.
9. Let X = {a, b, c, d}, and τ = {Φ, {c}, {d}, {c, d}, {a, c, d}, X}. Then {b} is gr*-closed but not rβ-closed.
10. Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then {c} is g*p- closed but not rβ-closed.
11. Let X = {a, b, c, d} and τ ={Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then {b, c} isg*s - closed but not rβ-closed.
12. Let X = {a, b, c, d} and. τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X} Then {c} is αg-closed but not rβ-closed.
13. Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then {b, c} is sg-closed but not rβ-closed.
14. Let X = {a, b, c, d} and τ = {Φ, {c}, {d}, {c, d}, {a, c, d}, X}. Then {b, d} is βg* - closed, but not rβ-closed.
15. Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then {a, b, d} is r^ g -closed but not rβ-closed.

Remark 3.10:

rβ-closed set is independent of r α – closed sets
rβ-closed set is independent of r s – closed sets
rβ-closed set is independent of tgr – closed sets

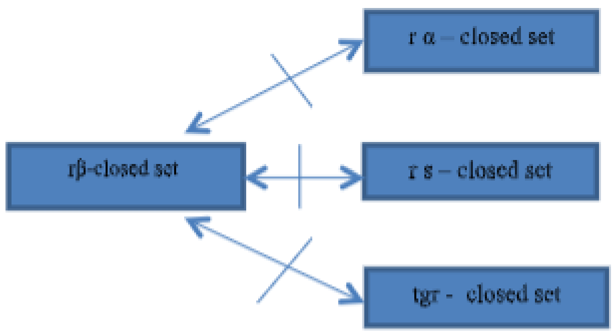
Example 3.11:

Let X = {a, b, c, d} and τ = {Φ, {a}, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}, X}. Then

1. {d} is rβ-closed but not r α – closed and {a} is r α – closed but not rβ-closed
2. {a, c, d} is rβ-closed but not r s – closed and {b, c} is r s – closed but not rβ-closed
3. {c, d} is rβ-closed but not tgr –closed and {a, b, d} is tgr-closed but not rβ-closed.

Remark 3.12:

The independency of rβ-closed sets with some of the existing closed sets are implemented in the following figure.



In the above figure A B means the set A and B are independent of each other.

4. Characteristics of Regular Beta (rβ) - Closed Sets

Theorem 4.1:

The finite union of any two rβ closed sets of X is rβ closed.

Proof:

Let A and B be the rβ -closed sets in X. Let U be a β -open set in X such that A∪B ⊂ U. Then A ⊂ U and B ⊂ U. Since A and B are rβ -closed sets in X, rcl (A) ⊂ U and rcl (B) ⊂ U. We have by [12], rcl (A∪B) = rcl (A) ∪rcl (B) ⊂ U. This implies rcl (A∪B) ⊂ U. Hence A∪ B is rβ -closed.

Remark 4.2:

Intersection of two rβ -closed sets need not be rβ -closed as shown in the following example.

Example 4.3:

Let X= {a, b, c, d} with topology τ = {Φ, {a}, {b, d}, {a, b, d}, X}. Let A = {a, b, c} and B = {b, c, d} be two rβ -closed sets in X. Then A∩B = {b, c} is not a rβ- closed set.

Theorem 4.4:

If a subset A of X is rβ -closed set in X, then rcl (A) – A contains no non- empty β - closed set.

Proof:

Let F be a non-empty β-closed set in X such that F ⊂rcl (A) – A. That is F ⊂ rcl (A) ∩ A^c. Therefore F ⊂ rcl (A) and F ⊂ A^c and so A ⊂ F^c. Now since A is rβ –closed, and F^c is β open, rcl (A) ⊂ F^c. This implies F ⊂ $[\text{rcl (A)}]^c$. Also we have F ⊂rcl (A). Therefore F ⊂ rcl (A) ∩ $[\text{rcl (A)}]^c$= Φ. This is a contradiction. Therefore rcl (A) – A contains no non-empty β -closed set.

Remark 4.5:

If rcl (A) – A contains no non-empty β - closed set in X, then A need not be rβ -closed set in X.

Example 4.6:

Let X= {a, b, c, d} with topology τ = {Φ, {a}, {b, d}, {a, b, d}, X} and A = {b}. Here rcl (A) – A contains no non-empty β - closed set in X, but A is not a rβ -closed set in X.

Theorem 4.7:

Let A ⊂ B ⊂ rcl (A) and A is rβ -closed set in X, then B is also rβ -closed.

Proof:

Let U be a β-open set in X such that A ⊂ U. Now if A ⊂ B ⊂ rcl (A) then rcl (A) ⊂ rcl (B) ⊂ rcl (A). Therefore rcl (B) = rcl (A). Since A is rβ –closed rcl (A) ⊂ U. Therefore rcl (B) = rcl (A) ⊂ U. Hence B is rβ -closed.

Theorem 4.8:

If A is β-open subset of X and rβ -closed set in X. Then A is a regular closed set.

Proof:

Since A is β-open subset of X and rβ - closed, rcl (A) ⊂A. But A ⊂ rcl (A). Therefore

A= rcl (A). Hence A is regular closed.

Theorem 4.9:

For every point x of the space X the set X – {x} is either rβ -closed (or) β -open.

Proof:

Suppose that X – {x} is not β -open. Then X is the only β -open set containing X–{x}. That is X– {x} ⊂ X. This implies rcl (X–{x}) ⊂ rcl (X) ⊂ X. Therefore X – {x} is arβ - closed set in X.

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