Coloring of Field

Akshaya .G, Sudha Lakshmi .G

II M.Sc., Mathematics
Department of Mathematics, Thassim Beevi Abdul Kadher College for Women, Kilakarai, Tamilnadu, India

Abstract: ***The purpose of this article is to present the idea of coloring of a field. This idea establishes a connection between graph theory and field theory which hopefully will turn out to be mutually beneficial for these two branches of mathematics. In this introductory paper we shall mainly be interested in characterizing and discussing the field which are finitely colorable, leaving aside, for the moment, possible applications to graph theory.***

Keywords: Field, coloring, finite coloring, graph theory, field theory

1. Introduction

Let F be a field. We consider F as a simple graph whose vertices are the elements of F, such that two different element x and y are adjacent if and only if yx =0. We let χ (F) denote the chromatic number of the graph, i.e., the minimal number of colors which can be assigned to the elements of F in such a way that every two adjacent elements have different colors.

A subset T = {y_1,….,y_n}is called clique provided $y_i y_j$ = 0 for all i≠j. if F contains a clique with n elements, and every clique has at most n element, We say that the clique number of F is n and write clique F= n. If the size of the cliques in F are not bounded we define clique F =∞. We shall show that clique F =∞ actually entails the existence of an infinite clique.

Obviously χ(F) ≥ clique F and for general graph G we certainly may have χ(G) > clique G. However, in the case of commutative rings we have not found any example where χ(F)> clique F. The lack of such counter example together with the fact that we have been able to establish the equality χ(F) =clique f for certain (rather wide) classes of field like reduced and principal ideal field motivates the following conjecture.

2. Preliminaries

2.1 Definition

A field is a set with two operation called addition & multiplication which satisfy the following axioms

Axiom for addition
- If x∈F, y∈F then x+y∈F
- x+y = y+x
- Addition is associative
- F contains an element zero such that 0+x = x for every x∈F
- To every x∈F corresponds an element -x∈F such that x+ (-x) = 0

2.2 Definition

If an element x of a ring R is called *nilpotent* if there exist some positive integer n such that x^n = 0

2.3 Definition

The *nil radical* of a commutative ring is the ideal consisting of the nilpotent element of the ring

2.4 Definition

A subset I is called the ideal of the ring if it satisfy the following condition
a) (I, +) is a subgroup of (R, +)
b) for every r∈R and x∈I then rx∈I.

2.5 Definition

A *clique* which is a subset of vertices of an undirected graph such that every two distinct vertices are adjacent, its induced subgraph is complete.

2.6 Definition

A graph *coloring* is an assignment of label called colors to the vertices of the graph such that no two vertices (adjacent) share the same color.

2.7 Definition

The *chromatic number* χ (G) of a graph G is the minimal number of colors for which such an assignment it possible.

2.8 Annihilator

Let R be a ring & let M be a left R-module choose a non-empty subset S of M, it is set of all elements r in R such that for all s in S, rs =0
Ann_R(S) = {r∈R\forall s∈S:rs = 0}

Presumption 1. χ(R) = Clique R.
Presumption 2.χ (R) = 2if and only if R is an integral domain, R≅Z_4, or R≅ $Z_2[X]/(X^2)$.
Presumption 3. Ifp_1,…..p_k, q_1,…..$q_r$be different prime numbers and N= $p_1^{2n_1}$,….,$p_k^{2n_k} q_1^{2m_1+1} \dots q_t^{2m_r+1}$.Then χ($Z_N$) =clique($Z_N$) =$p_1^{n_1}$,….,$p_k^{n_k} q_1^{m_1} \dots q_r^{m_r}$ + r.

Presumption 4. Suppose that R has an infinite number of finite elements then R contains an infinite clique.

Presumption 5. If I be a finite ideal in R. The ring R contains an infinite cliqueif and only if R\I has an infinite clique.

Presumption 6. If the nilradical of R is infinite then R has an infinite clique.

3. Main Result

3.1 Definition

A field F is called a coloring provide that χ(F) is finite.

3.2 Definition

The *nilradical* of the field is the ideal consist of nilpotent element of the field.

3.3 Theorem

Let x and y be element in F such that Ann x and Ann y are different prime ideals. Then xy=0.

Proof:
Assume that xy≠ 0. Then x ∉ Ann y and y ∉ Ann x.
Since Ann x and Ann y are prime ideals, we derive Ann x: y =Ann y:x =Ann(xy).
Thus, Ann x = Anny.
Hence proved.

3.4 Theorem

For a reduced field F the following are equivalent
(1) χ (F) is finite
(2) Clique F is finite
(3) The zero ideal in F is a finite intersection of prime ideal
(4) F does not contain an infinite clique

Proof:
The implication (1) ⇒ (2), (1) ⇒ (4), (2) ⇒ (4) are evident.
To see (3) ⇒ (4) let (0) = $P_1 \cap \ldots \cap P_K$ where $P_1, \ldots. P_k$ are prime ideals.
Define a coloring g on F byputting f (0) =0 and f(x) = min $\{i/x \notin P\}$ for x ≠0.
We note that χ(F) ≤ k+1.
It suffices now to show that (4) ⇒ (3).
We assume that F is reduced and does not contain an infinite clique.

Then F satisfies a.c.c on ideals of the form Ann a (since, if F be a reduced field which does not contain an infinite clique. Then F has a.c.c on ideals of the form Ann x).

Let Annx_i,
i∈ I be the different maximal member of the family $\{Ann\ a/a \neq 0\}$.

It is easily shown that each Annx_i is a prime ideal. (Sincex and y be element in F such that Ann x and Ann y are different prime ideal. Then xy = 0). The index set I is finite.

Pick x∈ F, x ≠ 0. The Ann x ⊂ Ann x_i for some i∈I. If xx_i = 0 then x_i ∈Ann x⊂ Ann x_i and we drive that x_i^2 = 0, which entails that x_i = 0.

We conclude that xx_i ≠ 0. And thus x∉Annx_i.
Hence $\cap_I Annx_i$= (0).
This completes the proof.

3.5 Theorem

A coloring has an ideal of the form Ann a.

Proof:
Let F be the coloring and let us assume that Ann x_1<Ann x_2<….
We have to prove that F has an ideal of the form Ann a.
Let us consider the nilradical T is finite since for the reduced field the zero ideal in F is a finite intersection of prime ideal than χ(F) is finite

Let us consider T= $P_1 \cap P_2 \cap \ldots,,,$where P_i are the prime ideals for x ∈ F we getT: x= (P_1:x)∩ (P_2:x)∩........∩ (P_n:x)

This shows that the family $\{T: x/x \in F\}$ is finite.
Consequently, there exist a subsequence $\{y_j\}$ of$\{x_i\}$ for whichT: y_1 = T: y_2=……
implies that, Anny_1<Anny_2<…………
Which contains in Ts:y_1 which contradicting the fact that $T: y_i/Ann\ y_i$is finite. This complete the proof.

3.6 Definition

In algebra a subfield of algebra over a field F is an F-subalgebrathat is also a field.

3.7 Definition

Maximal subfield that is not contained in strictly larger subfield of A

3.8 Note

A sub field of a coloring is itself a coloring.

3.9 Theorem

Let I be a finitely generated ideal in a coloring. The kn$F/Ann\ I$ is a coloring.

Proof:
Let I = (T_1,…,T_n). Then Ann I = Ann T_1 ∩ … .∩ $Ann\ T_n$.
We have an injection F/Ann I→ $R/AnnT_1 \times \ldots \times R/AnnT_n$.
Each of the rings $R/AnnT_i$ is a coloring since the subfield of a coloring is itself a coloring. and the finite product of colorings is a coloring. So that the proof is complete.

4. Detached Elements

4.1 Definition: An element y in F is detached provided that y≠ 0 and ba = 0 implies yb = 0orya = 0.

4.2 Definition: If I be an ideal. An element x∈ I is I-detached provided yI ≠ (0) and whenever ba = 0 for some elements b, a∈I henceyb = 0 or then ya = 0.

4.3 Theorem: Let I be a principal ideal in a coloring. If $I^2 \neq (0)$ then I contains an I detached elements.

Proof.
Let I = Fy and suppose $y^2 \neq 0$.Since F has a.c.c on annihilators
It is easily shown that 0:y^2t is a prime ideal for some t∈F.
We claim that yt is I-detached.
Let b, a∈I and assume ba = 0. Write b = dy and a = cy.
Then dcy^2 = 0.
Hence dc is contained in the prime ideal 0 : y^2t. If for instance d∈0:y^2 we derive that (dy)(yt)=0, i.e., b(yt) =0.
Furthermore, (ty)y = t$y^2 \neq 0$ prove that (ty)I≠ 0.This complete the proof.
Note 1: Given a coloring F, thenχ(F) = clique F provided clique F≤ 2 orχ (F) ≤ 2.
Note 2: Let F be a coloring .Then clique F = 3 if and only if χ (F) = 3.

4.4 Theroem: Let F be a coloring and T an integer ≤ 4. Then χ (F) = T if and only if clique F = T. Moreover, χ(F) = 5 implies clique F = 5.

Proof
According to note 2 it suffices to show that χ(F)>4 implies that F>4.

If F is a reduced field χ (F) equals clique F. We assume therefore that the nil radical I is non-zero.
Since, let $q_1, \ldots . q_k$ be the minimal prime ideals in a coloring F, and ε(F) = # {k\ F_{q_j} is a field}.then clique F = clique L+ ε(F) and χ(F) = χ(L) + χ(F)+ε(F).

It suffices to show that in our case clique L = χ(L).
Let k = L ∩ Ann L. since Lis nilpotent and non-zero | k|≥2, note that k is a clique in K. If k=L, L is a clique and trivially χ(L) =clique L. Also if |k|>4 there is nothing to prove.

If |k|=4 let y∈ L−K. Then k∪{y} is a clique with 5 elements. If |k|=3 and χ(L)>4 at least two different elements. The only case which offers some difficulties is that in which |k|=2 and χ(L)≥5.

Let K = (0, D). since k is an ideal D+D = 0. Sinceχ(L)≥5 the set L−k requires at least three distinct colors. Hence we can pick a minimal odd cycle $B_1, \ldots . B_n$ in L−k and assume n≥5.If $B_1 B_i$=0 for some I≠ 1,2,…n the cyclic will decompose in two smaller cycles of which one is odd. Hence $B_k B_l$=0 only if B_k *and* B_l are neighbours. As in the proof of Note 2 we conclude that B_k B_l does not belongs to the cycle since B_k B_l is adjacent to at least three of the member of the cycle.

Let k≠1,2,..n. If k is even $B_1 B_k, B_2, \ldots . B_{k-1}$ is an odd cycle of length k−1< n and if k is odd$B_1 B_k, B_{k+1}, \ldots . B_n$ is an odd cycle of length n−k+1< n.

Since the odd cycle$B_{1,\ldots,} B_n$ is minimal in L−k we conclude that $B_k B_l$=0 only if B_k *and* B_l are neighbor and $B_k B_l$=D if k≠L and B_k *and* B_l are not neighbours.

We claim that $B_1^2 \neq 0. If\ B_1^2 \neq 0$ and $B_2 \neq B_1 +$ D then,B_1, B_2,B_1+D is a cycle in L−K.This implies that B_1 + D $= B_2$,but then 0=$B_2 B_3 = B_1 B_3 + DB_3$=$B_1 B_3$. This contradiction the fact thatB_1 and B_3 are not neighbour.
Hence B_k^2 ≠0 for 1≤ k≤n. consider now C =$B_1 + \cdots + B_{n-2}$. We have CB_{n-1} =$B_1 B_{n-1} + \cdots + B_{n-3} B_{n-1} + B_{n-2} B_{n-1}$ =(n−3)D = 0 since D+D = 0 and n−3 is even.

Similarly we see that CB_n = 0.

Since $B_n^2 \neq 0$ and $B_{n-1}^2 \neq 0$ we conclude that C≠ B_n *and* $C \neq B_n$.

Write n=2I+1 and consider
CB_i = $B_1 B_i + \cdots + B_{i-2} B_i + B_{i-1} B_i$+$B_i^2$+$B_{i+1} + B_i + \cdots + B_{2i-1} B_i$
= $B_i^2 + 2(I-2)D = B_i^2$ ≠0.

This proves that $C \notin$ k.
Hence {0,B_{n-1}, B_n, C, D} is a clique in L.

This completes the proof.

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