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Transient Solution for Unreliable C Servers under Balking, & Catastrophes

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Abstract: ***This paper proposed a transient modeling with balking, catastrophes, server failure and repair, all the servers having same service rate. The customers arriving the system under poisson process and the service rate follows an exponential distribution. Using the suitable generating function and laplace transforms are obtaining the transient solution of our model.***

Keywords: Homogeneous, Balking, Catastrophes, Server failure, Repair, Bessel function.

1. Introduction

Transient solution is depending upon time. More literature is available for the transient solution of various models. Dharmaraja [3] explained about the methodology to find the transient solution with balking and catastrophes. Ahmed Tarabia [1] analyzed the transient and steady state analysis of an M/M/1 model with balking, catastrophes, server failure and repair. Arul Freeda and Vidhya [2] considered the transient solution of M/M/c queue with catastrophes and impatient behavior of the customers. Rakesh Kumar and Sapna [4] discussed the balking and reneging effect in the transient model with c servers. Shanmugasundaram and Chitra [6] described the transient retrial queue with catastrophes and also, he verified the result with M/M/1 model.Shakir Majid and Manoharan[5] analyzed the M/M/c queue with single and multiple vacations.Kavitha.G and Julia Rose Mary.K developed thetransient solution of m/m/c queueing model for heterogeneous server with catastrophes and balking. Thus with the aid of available literature motivate us to develop our model with balking, catastrophes, server failure and repair of homogenous servers. None of the work described in literature explaining the criteria under homogenous multiserver queueing systems.

2. Model Description

We considered the multi-server queueing system with balking, catastrophes, server failure and repair. Here the arrival rate occurs a poisson process with the rate λ. There are c servers with same service rate μ is an exponential distribution. Let $Q(t)$ be the probability that the server is under repair at the transient state. The repair follows an exponential distribution with the rate η.All the customers in the system may leave immediately and it is activated momentarily and this system follows a FCFS queue discipline. The customer due to his impatient he balks the queue with the rate β.

Based on the criteria the difference differential equations are,

$$Q'(t) = -\eta Q(t) + \xi(1 - Q(t)) \_\_\_\_\_ (1)$$

$$P_{i0}'(t) = \mu P_{i1}(t) - (\lambda + \xi)P_{i0}(t) + \eta Q(t) \_\_ (2)$$

$$P_{in}'(t) = \lambda P_{in-1}(t) - (\lambda + \xi + \eta\mu)P_{in}(t) + (n + 1)\mu P_{in+1}(t), \quad 1 \leq n < c \_\_\_ (3)$$

$$P_{in}'(t) = P_{in-1}(t) - (\lambda\beta + \xi + c\mu)P_{in}(t) + c\mu P_{in+1}(t), \quad n = c\_ (4)$$

$$P_{in}'(t) = \lambda\beta P_{in-1}(t) - (\lambda\beta + \xi + c\mu)P_{in}(t) + c\mu P_{in+1}(t), \quad n > c\_\_\_\_\_\_\_\_\_\_\_\_ (5)$$

Define the probability generating function $H(s,t)$ for the transient state probabilities $P_{in}(t)$ by

$$H(s,t) = Q(t) + r_{ic}(t) + \sum_{n=1}^{\infty} P_{ic+n}(t)s^n, \quad |s| \leq 1 \text{--}(6)$$

$$r_{ic}(t) = \sum_{n=0}^{c} P_{in}(t) \text{ and } H(s,0) = \sum_{m=0}^{\infty} P_{im} s^{\tau(m)}, \tau(m) = (m-k)(1 - \sum_{n=0}^{k} \delta_{m,n})$$

Adding the eqn (1) to (4) we get,

$$Q'(t) + r_{ic}'(t) = \xi(1 - Q(t)) - \xi r_{ic}(t) - \lambda\beta P_{ic}(t) + c\mu P_{ic+1}(t)\_\_\_\_\_\_\_\_\_\_\_\_ (7)$$

Now eqn(5) is multiplied by$s^n$and summing over the respective ranges of n, we obtain

$$\sum_{n=1}^{\infty} P_{ic+n}'(t)s^n = \left[\lambda\beta s - (\lambda\beta + \xi + c\mu) + \frac{c\mu}{s}\right] \sum_{n=1}^{\infty} P_{ic+1}(t)s^n + \lambda\beta s\, P_{ic}(t) - c\mu P_{ic+1}(t)\_\_\_\_ (8)$$

Adding the eqn (7) & (8) and using $H(s,t)$, we get

$$\frac{\partial H(s,t)}{\partial t} - \left[\lambda\beta s - (\lambda\beta + \xi + c\mu) + \frac{c\mu}{s}\right] H(s,t) = -\left[\lambda\beta s - (\lambda\beta + c\mu) + \frac{c\mu}{s}\right][Q(t) + r_{ic}(t)] + \lambda\beta(s - 1)P_{ic}(t) + \xi\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (9)$$

Using the initial conditions $H(s,0)$ and $Q(0) = 0$. Solving the eqn (9) in first order differential condition with the help of integrating factor$e^{-\left[\lambda\beta s - (\lambda\beta + \xi + c\mu) + \frac{c\mu}{s}\right]t}$, we have

$$H(s,t) = \int_0^t e^{[\lambda\beta s-(\lambda\beta+\xi+c\mu)+\frac{c\mu}{s}](t-u)}\Big[-\Big(\lambda\beta s - (\lambda\beta+\xi+c\mu)+\frac{c\mu}{s} + \xi\Big)[Q(u)+r_{ic}(u)] + \lambda\beta(s-1)P_{ic}(u) + \xi\Big]du + H(s,0)e^{[\lambda\beta s-(\lambda\beta+\xi+c\mu)+\frac{c\mu}{s}](t)}$$

And $\lambda\beta s-(\lambda\beta+\xi+c\mu)+\frac{c\mu}{s} = -p+qs+\frac{r}{s}$

On account of $p=\lambda\beta+\xi+c\mu, q=\lambda\beta$ and $r=c\mu$ where $\alpha = 2\sqrt{qr}$ $v=\sqrt{\frac{q}{r}}$ by using the Bessel properties, we get $e^{[-p+qs+\frac{r}{s}]t} = e^{-pt}\sum_{n=-\infty}^{\infty}(vs)^n I_n(\alpha t)$

$$H(s,t) = \int_0^t e^{-p(t-u)}\Big[-\Big(qs+\frac{r}{s}-p+\xi\Big)[Q(u)+r_{ic}(u)] + \lambda\beta(s-1)P_{ic}(u)+\xi\Big]$$

$\sum_{n=-\infty}^{\infty}(vs)^n I_n\alpha(t-u)du +$ m=0∞PimSτ(m)e−ptn=−∞∞vsnInαt__________ (10)

Comparing the coefficient of $s^n, n=1,2,3\ldots$ we obtain,

$$P_{ic+n}(t) = -\int_0^t e^{-p(t-u)}[qv^{n-1}I_{n-1}\alpha(t-u) + rv^{n+1}I_{n+1}\alpha(t-u) + (\xi-p)v^nI_n\alpha(t-u)[Q(u)+r_{ic}(u)] - \xi v^nI_n\alpha(t-u)]du + \lambda\beta\int_0^t e^{-p(t-u)}[v^{n-1}I_{n-1}\alpha(t-u) - v^nI_n(\alpha)(t-u)]P_{ic}(u)du$$

$+\sum_{m=0}^{\infty}P_{im}v^{n-\tau(m)}e^{-pt}I_{n-\tau(m)}(\alpha t)$______________________________ (11)

Comparing the constant terms with the help of Bessel property, $I_{-n}(x)=I_n(x)$, we get

$$Q(t)+r_{ic}(t) = -\int_0^t e^{-p(t-u)}[qv^{-1}I_1\alpha(t-u) + rvI_1\alpha(t-u) + (\xi-p)I_0\alpha(t-u)[Q(u)+r_{ic}(u)] - \xi I_0\alpha(t-u)]du$$

$$+\lambda\beta\int_0^t e^{-p(t-u)}[v^{-1}I_1\alpha(t-u) - I_0\alpha(t-u)]P_{ic}(u)du$$

$+\sum_{m=0}^{\infty}P_{im}v^{-\tau(m)}e^{-pt}I_{\tau(m)}(\alpha t)$______________________________ (12)

Eqn(10) does not contain any negative powers of s in the left hand side so equate the right hand side equate to zero and using Bessel property, Bessel identity and also using some algebra we yield,

$$P_{ic+n}(t) = nv^n\int_0^t e^{-p(t-u)}\Big(\frac{I_n\alpha(t-u)}{(t-u)}\Big)P_{ic}(u)du + \sum_{m=0}^{\infty}P_{im}v^{n-\tau(m)}e^{-pt}\big[I_{n-\tau(m)}(\alpha t) - \text{In+τm(αt)}\big]\text{__________ (13)}$$

The remaining probabilities can be found out by using the system of equations (2)-(3) of the form:

$$P'(t) = AP(t) + \eta e_0Q(t) + (n+1)\mu P_{ic}(t)e_c \text{______________ (14)}$$

where the matrix $A=(a_{m,n})_{(c\times c)}$ is given as

$$\begin{bmatrix} -(\lambda+\xi) & \mu & \cdots & 0 \\ \lambda & -(\lambda+\xi+\mu) & & 0 \\ 0 & \lambda & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & -(\lambda+\xi+n\mu) \end{bmatrix}$$

$P(t) = \big(P_{i,0}(t), P_{i,1}(t)\ldots P_{i,c-1}(t)\big)^T$ $e_n = (0,0\ldots 1)^T$ and $n=0,1,2\ldots c.$

Let

$P^*(z) = (P_{i,0}{}^*(z), P_{i,1}{}^*(z), P_{i,2}{}^*(z)P_{i,3}{}^*(z)\ldots P_{i,c-1}{}^*(z))^T$ and $Q^*(z)$ denotes the Laplace transforms of $P(t)$ and $Q(t)$ respectively

$$P^*(z) = [zI-A]^{-1}[\eta e_0Q^*(z) + (n+1)\mu P_{ic}{}^*(z)e_c + P(0)]\text{______________ (15)}$$

where $[zI-A]^{-1} = (a_{m,n})_{(c\times c)}$ we get,

$$P^*{}_{im}(z) = \eta Q^*(z)b_{m,0}{}^*(z) + (n+1)\mu b_{m,c-1}{}^*(z)P_{ic}{}^*(z) + \sum_{j=0}^{k-1}P_{i,j}b_{m,j}{}^*(z), m=0,1,2\ldots c-1\text{__________(16)}$$

Taking the inversion Laplace ofeqn (1) and (16) and using the convolution theorem we get,

$$Q(t) = \frac{\xi}{(\eta+\xi)} + ce^{-(\eta+\xi)t}\text{______________________________(17)}$$

$$P_{im}(t) = \eta\int_0^t Q(u)b_{m,0}(t-u)du + (n+1)\mu\int_0^t b_{m,c-1}(t-u)P_{ic}(u)du + \sum_{j=0}^{k-1}P_{ij}b_{mj}(t)$$

$m=0,1,2\ldots c-1$______________________________ (18)

Taking the Laplace transforms for both sides of eqn (12), and solving $[Q^*(z)+r_{ic}{}^*(z)]$ by using $\lambda\beta = \frac{\alpha v}{2}$ we get,

$$(\xi+z)[Q^*(z)+r_{ic}{}^*(z)] = \sum_{m=0}^{\infty}P_{im}\Big[\frac{[z+p-\sqrt{(z+p)^2-\alpha^2}]^{\tau(m)}}{(\alpha v)^{\tau(m)}}\Big] + \frac{1}{2}\Big[z+p-\sqrt{(z+p)^2-\alpha^2} - v\alpha\Big]P_{ic}{}^*(z) + \frac{\xi}{z}\text{ ______ (19)}$$

We also have$r_{ic}{}^*(z) = e^TP^*(z) + P_{ic}{}^*(z)$

$$r_{ic}{}^*(z) = e^T[ZI-A]^{-1}P(0) + \mu e^T[ZI-A]^{-1}e_kP_{ic}{}^*(z) + \eta e^T[ZI-A]^{-1}e_0Q^*(z) + P_{ic}{}^*(z)\text{_____ (20)}$$

Substituting (20) in (19) after the simplification we get,

$$P_{ic}{}^*(z) = \frac{1}{(\xi+z) - \frac{1}{2}\big[z+p-\sqrt{(z+p)^2-\alpha^2} - v\alpha\big] + (\xi+z)\mu\sum_{m=0}^{k-1}b_{m\,k-1}{}^*(z)}$$

$\Big\{-[\xi + z +$

ηm=0c−1bm0∗(z)]Q∗(z)+m=0∞Pim[[z+P−z+p2−α2]τmαντm]−
(ξ+z)j=0c−1m=0c−1Pijbmj∗z+ξz_________ (21)

Using partial factor fraction technique we will be rewritten as

$$b_{mj}{}^{*}(z) = \sum_{n=0}^{c-1} \frac{c_{mj}{}^{n}}{z-z_n} \qquad \text{\_\_\_\_\_\_\_\_\_(22)}$$

Here n=0,1,2…k-1 are the eigen values of the matrix A which can be mentioned above

Inverting Eqn (22) implies that, $b_{mj}\,(\mathrm{t}) = \sum_{n=0}^{c-1} C_{m,j}{}^{n} e^{z_n(t)}$

Also, we have $\sum_{m=0}^{c-1}(\xi + z) b_{mj}{}^{*}(z) = 1 + \sum_{m=0}^{c-1} \frac{B_j{}^{(m)}}{z-z_m}$

$$=1+b_j{}^{*}(z)$$

Using $2(z + \xi + \mu + \alpha\nu = 2(z+p)$ the above equation (21) after some simplification it can be written as, $\left(\frac{2}{\alpha^2}\right)[z + a - \sqrt{(z+a)^2 - \alpha^2}\,] \times$

$$\left\{1 + \frac{[z+a-\sqrt{(z+a)^2-\alpha^2}\,]}{\alpha\nu} b_{c-1}{}^{*}(z)\right\}^{-1} \times$$

$$\Big\{-[(\xi + z) + \eta[1 + b_0{}^{*}(z)]]Q^{*}(z) +$$

m=0∞Pim[z+a−z+a2−α2]τmαντm−j=0c−1Pij(1+bj
z)+ξz.

By taking the inverse of the above equation we obtain the required result.

3. Conclusion

In this paper by using probability generating function technique the time dependent solution of the homogenous $\mathrm{M/M/C}$ queueing model with catastrophes, unreliable server under restricted admissibility is derived. In future, with the use of transient state probability, the various performance measures are also analyzed.

References

[1] Arul Freedavinodhini.G, &Vidhya.V, Transient solution of a multiserver queue with catastrophes and impatient customers when system is down, Applied Mathematical sciences, vol(8), 2014, pp 4585-4592.

[2] Dharmaraja.S& Rakesh kumar, Transient solution of a markovian queueing model with heterogenous servers and catastrophes, opsearch , 52(4),810-826, 2015.

[3] Rakesh Kumar &Sapnasharma, Transient analysis of an M/M/c queueing system with balking and retention of reneging customers, communications in statistics-Theory and methods, vol(47), 2018, pp 1318-1327.

[4] Shanmugasundaram&Chitra.S, Transient state solution of M/M/c retrial queue and feedback on non-retrial customers with catastrophes, Mathematical Sciences International research journal, vol(5), 2016, pp 76-82.

[5] Shakir Majid &Manogaran, Analysis of a M/M/c queue with single &multiplesynchronous working vacations, Applications and Applied Mathematics; An international journal, vol(12), 2017, pp 671-694.