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Solutions of the 2D Macroscopic Motor Vehicle Traffic Flow Model

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Abstract: ***The study developed heterogeneous motor vehicle traffic flow models focusing on two dimensional heterogeneity and multi - lane road heterogeneity aspects. It established the 2 - ensemble traffic composition formulation using vehicle heterogeneity characteristics in two dimensions. Lighthill - Whitham and Richards motor vehicle traffic model, investigated how a homogeneous vehicle travels in a one dimensional, single lane road; without curvatures. The study shows that that when vehicles are few, traffic density is lower, the flux is also lower; the velocity is high as there is maximum vehicle - road interaction. As the number of vehicles increases such that many vehicles passes the fifth nodal point at a relatively high velocity and density; the flow rate is high. At maximum density called jam density, vehicles cannot move; the flux is lower. Graphical representation of the traffic flow variables with respect to the distance covered on the highway is done using MATLAB. The information obtained from the study is very useful to civil engineers, mechanical engineers, town planners, computer programmers and upcoming mathematicians not forgetting drivers and riders.***

Keywords: Macroscopic, microscopic, model, dimensional, influx, efflux, REA

Abbreviations: Two dimension (2D), Lane position (C), and Lighthill-Whitham-Richards (LWR)

1. Introduction

Traffic networks that consist of avenues, highways, lanes, streets and other roadways provide a convenient and an economical conveyance of passengers and goods. The basic activity in transportation is a trip, defined by its origin/destination, departure time/arrival time and travel route. A myriad of trips interact on a traffic network to produce a sophisticated pattern of traffic flows [4]. Nowadays traffic jam has become a major problem in society and economically as far as transportation in developed and developing countries; where our country Kenya as member is considered [10]. In the last few decades much interest has been focused on traffic flow models as the amount of traffic more especially motorcycles continues to increase exponentially [9]. Traffic congestion on motorways in the third world countries is becoming an even more pressing problem, where almost in every weekday morning and every weekday evening including weekends the capacity of many main roads is exceeded [5]. Traffic management is becoming a mounting challenge for many cities and towns across the globe as the world's population grows. Increasing attention as been devoted to the modeling, simulation and visualization of traffic to investigate causes of traffic accidents, congestion and delays, to study the effectiveness of road side hardware, signs and other barriers, to improve policies and guidelines related to traffic regulation and to assist up in development and the design of highways and road systems. The transportation is one of the major pillars supporting live in cities and regions in many large cities. The potentialities of extensive development of the transportation network were exhausted over the last five decades or are approaching completion. That is why optimal planning of transportation, improvement of traffic organization and optimization of the roots of public conveyance take on special importance. Solution of these problems cannot do without mathematical modeling of the transportation systems. In this study the work of [6, 8] is extended into a two lane change model. The derivation of the multi – lane road model is presented as well as the lane shifting model for a road with two lanes. The lane changing model is consists of a system of partial differential equations (PDE). It is almost impossible to find the exact solution of the model as an initial value (IVP). That is why there is demand to find the numerical solution of the model as an initial boundary value problem (IBVP). For numerical solution of the model we describe the derivation of the numerical schemes. In order to implement the numerical scheme. The study develops a computer programming codes and performs computer simulation of the lane changing model using MATLAB with respect to various flow parameters.

2. Literature Review

[2] demonstrated that the Payne model, as well as several other second-order models available in the literature, produced flawed behavior for some traffic conditions. Specifically, it was noted that traffic arriving at the end of a densely-packed queue would result in vehicles traveling backwards in space, which is physically unreasonable. This is due to the isotropic nature of the models, as the behavior of vehicles is influenced by vehicles behind them due to diffusive effects, [1] were able to produce an anisotropic second-order model that averted the flaws noted by [2], [2] obtained a traffic flow model of the first order by considering the convective derivative of some pressure-type function of the traffic density.

3. 2D Macroscopic Motor Vehicle Traffic Flow Model

Motor vehicle traffic flow is surface transport; which occurs on a given area of the road on the earth's surface, which is in two dimensional. The old models in the literature review be it macroscopic or microscopic describe traffic flow in a linearly way, thus, in a one dimensional aspect and in one lane, straight without curvatures. Road as well as vehicle heterogeneity are not modeled. There was need to develop a two dimensional heterogeneous model to take care of the two dimensionality nature of vehicle transportation.

3.1 One Dimensional LWR, Macroscopic Motor Vehicle Traffic Flow Model

Considering a small portion of the road between $x = x_1$ and $x = x_2$, where x_1 and x_2 are the node positions at the entry and exit points respectively. Let the width of the road be y, and hence the lane width is $\frac{y}{2}$. It's worth noting that most roads in developing countries are two way traffic with 2 lanes, the left and the right lane. In Kenya the basic law of traffic is "keep left". The designated portion of the left lane between X_1 and X_2 is called the control area or Representative Elementally Area (REA). We assume that the number of vehicles in the control area is conserved; such that at any point in time t, the change in the number of vehicles within the control area is given by the difference between the number of traffic entering at x_1 and the number of vehicles leaving the control area at x_2

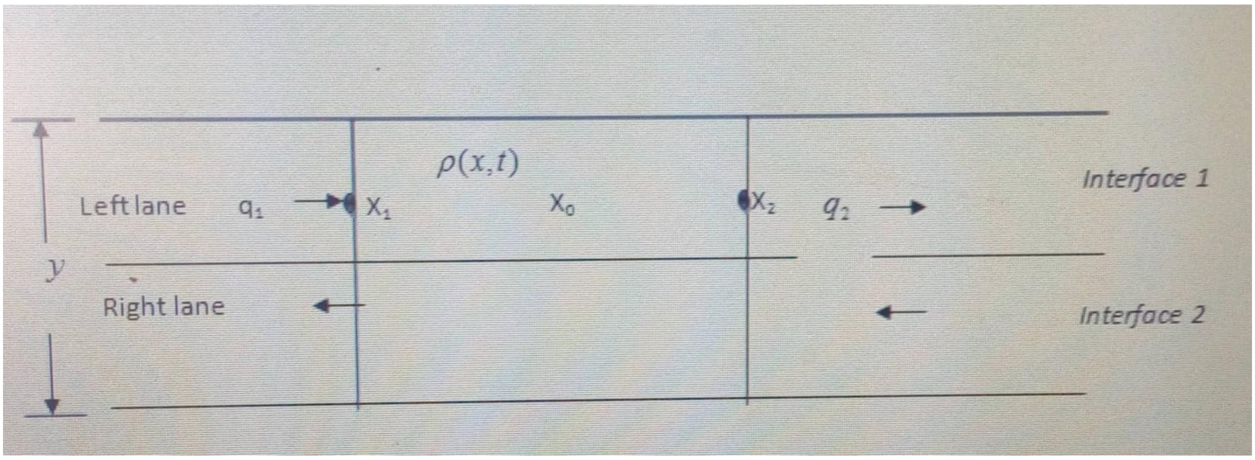

Figure 1: Nodes in the control area

The flow rate into the control area at x_1 is given by $q(x_1,t) = q_1$ the flow rate out the control area at x_2 is given by $q(x_2,t) = q_2$. Where-as ρ(x, t) is the vehicle density in the control area. The change in the number of vehicles in the control area is given as

$$\frac{\partial}{dt}\int \rho(x,t)dx = q_1 - q_2 \tag{1}$$

This is the integral-differential form of the conservation equation; as the interval reduces, implying that $X_2 \to X_1$, then $\Delta x = (x_2 - x_1) \to 0$

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{2}$$

This is the one dimensional traffic flow model.

It is clear that the integral becomes a partial differential equation that governs traffic flow systems; this can also be given by the following equations. Since density is a function of one temporal variable time, t and one spatial extent, x; then it is exact differential is obtained as;

$$\rho = \rho(x,t) \tag{3}$$

Implying that the full derivative of traffic density with respect to the space interval, x and time interval, t is given as

$$d\rho = \frac{\partial \rho}{\partial t}dt + \frac{\partial \rho}{\partial x}dx = 0 \tag{4}$$

On dividing throughout by dt, the full derivative with respect to time becomes

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} = 0 \tag{5}$$

$$\frac{dx}{dt} = \text{velocity of the incoming vehicle} = \frac{dq}{d\rho} = q'(\rho) \tag{6}$$

In order to formulate the equation completely, we require some relationship between flow rate, q and traffic density, ρ

$$q = q(\rho) = u\rho \tag{7}$$

Thus, using equation (6) equation (5) can be re-written as

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{8}$$

Again, this is the 1D traffic flow model; where the flux is expressed as a full derivative of traffic density. This relation is used in configuration of traffic lights in order to ensure that the traffic does not ''bank up" or worse still "crowd up" indefinitely; more especially at junctions and round about.

3.2 Two Dimensional, Macroscopic Motor Vehicle Traffic Flow Model

Just as in the one dimensional case, traffic density;

$$\rho = \rho(x,t) \tag{9}$$

Is a one dimensional (1D) density or linear density.

In 2D, two dimensional traffic density or planer density is expressed as;

$$\rho = \rho(x,y,t) \tag{10}$$

which is a function of a temporal variable time, t and two spatial variables x and y, hence two dimensional. From the nature conservation laws, if we set S to be 2D region. Further still, if we let the vehicle concentration to be in the boundary of S, then the total density in S is obtained by integrating motor vehicle density over the control area S; with respect to 'x' and 'y' respectively;

$$\int \int \rho(x,y,t)dxdy \tag{11}$$

The rate at which the motor vehicle density is changing in S is given by taking the derivative of the above integral with the respect to time, t;

$$\iint \frac{\partial \rho(x,y,t)}{\partial t} dx\,dy \tag{12}$$

From the natural law of conservation of vehicles, the motor vehicle density is conserved in S, then the rate at which the vehicle density is changing in S is equal to the number vehicles flowing into S less the number vehicles flowing out of S at the same time, t. Thus, influx less efflux.

$$\iint \frac{\partial \rho(x,y,t)}{\partial t} dx\,dy = q_1(x_1,y_1,t) - q_2(x_2,y_2,t) \tag{13}$$

From the Gauss Divergence Theorem for a quantity flowing from one point to another point in a given curve on the surface then;

$$q_1(x_1,y_1,t) - q_2(x_2,y_2,t) = -\int \int \nabla q\, dx\, dy = -\int \int \{\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\} dx\, dy \tag{14}$$

Hence equation (11) can be re-written as;

$$\frac{\partial}{\partial t}\iint \rho(x,y,t)dxdy = -\iint \{\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\}dxdy \tag{15}$$

Since the derivative of an integral is equal to the integral of a derivative then, equation (13) is expressed as;

$$\iint \frac{\partial \rho(x,y,t)}{\partial t}dxdy = -\iint \{\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\}dxdy \tag{16}$$

After transposition we obtain;

$$\iint \{\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\}dxdy = 0 \tag{17}$$

In this way, for the equal signs to hold, the integrand is identically zero; that is to say;

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} = 0 \tag{18}$$

Equation (18) is the multi - variate, two dimensionally heterogeneous, motor vehicle traffic flow and transport equation

4. Solution of the 2D Motor Vehicle Traffic Flow Model

4.1 The 2D Motor Vehicle traffic Flow Model take the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} + \frac{\partial (v\rho)}{\partial y} = 0 \tag{19}$$

In obtaining the numerical solutions of the 2D Motor Vehicle traffic Flow Model the study took the model as an initial boundary value problem (IBVP) by inserting initial and boundary conditions. Finite difference method was used in finding the solution. Density is a function of time, t, space, x and space, y. In order to develop the numerical scheme; we discretize the time derivatives using the forward difference in time; and the space derivatives by the central difference in space.

The discretization in time and in space;

$$\frac{\rho_{i,j}^{n+1}-\rho_{i,j}^{n}}{\Delta t^{n}}+\frac{u(\rho_{i,j}^{n}-\rho_{i-1,j}^{n})}{\Delta x_{i}}+\frac{v(\rho_{i,j}^{n}-\rho_{i,j-1}^{n})}{\Delta y_{j}}=0 \tag{20}$$

At a higher time level and when the time and spatial intervals are unit

$$\left(\rho_{i,j}^{n+1}\right)=\left(\rho_{i,j}^{n}\right)-\frac{u(\rho_{i,j}^{n}-\rho_{i-1,j}^{n})}{1}-\frac{v(\rho_{i,j}^{n}-\rho_{i,j-1}^{n})}{1} \tag{21}$$

These equations are Lax Friedrich Finite difference scheme for our IBVP equation.

Taking j = 1; u = 60km/hr, v = 20km/hr and the nodal points in the x – direction as i = 1, 2, 3, …, 5. The numerical scheme for the 2D Motor Vehicle traffic Flow Model becomes;

4.1 Solution of the 2D Motor Vehicle Traffic Flow Model at n = 0

$$\begin{aligned}
\left(\rho_{11}^{1}\right)&=-79\rho_{11}^{0}+60\rho_{01}^{0}+20\rho_{10}^{0}\\
\left(\rho_{21}^{1}\right)&=-79\rho_{21}^{0}+60\rho_{11}^{0}+20\rho_{20}^{0}\\
\left(\rho_{31}^{1}\right)&=-79\rho_{31}^{0}+60\rho_{21}^{0}+20\rho_{30}^{0}\\
\left(\rho_{41}^{1}\right)&=-79\rho_{41}^{0}+60\rho_{31}^{0}+20\rho_{40}^{0}\\
\left(\rho_{51}^{1}\right)&=-79\rho_{51}^{0}+60\rho_{41}^{0}+20\rho_{50}^{0}
\end{aligned} \tag{22}$$

The system of equations (22) can be written in matrix – vector form and solved at four time levels that is n = 0, n = 1, n = 2, n = 3. Taking u = 60km/hr, v = 20km/hr into the above scheme, at the nodal points in the x - direction as i = 1, 2, 3,..., 5. Systems of algebraic equations are obtained. Using the initial condition ρ (x, y, 0) = exponential x, the system of equations generated from the numerical scheme above is solved by matrix laboratory. The solutions obtained are as given in table 1 for the 2D LWR extended model scheme; at varying time levels

Table 1: The 2D Extended LWR Model Solutions Values at Varying Time Levels

Grid point(i, j, n)	n = 0	n = 1	n = 2	n = 3
(1, 1, n)	12.000	15.0000	19.00000	24.10000
(2, 1, n)	42.000	65.8000	96.00000	140.8000
(3, 1, n)	141.000	67.0000	158.0000	307.6000
(4, 1, n)	186.000	287.3000	484.0000	845.0000
(5, 1, n)	499.000	742.0000	1306.000	12998.374

The results in table 1 are represented graphically in 2D in figure 1.

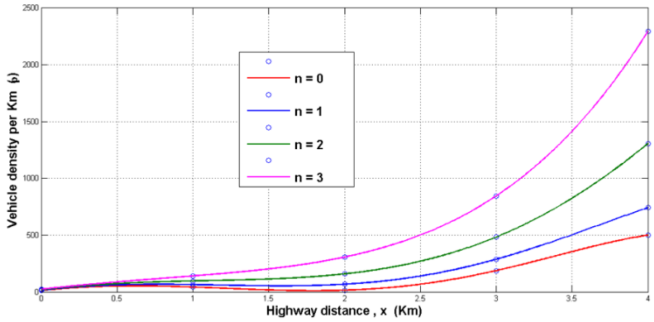

Figure 1: Variation of traffic density with time

It is seen from the results in table 1 that for a given value of i, $\rho_{1,i}^{n}$ values tends to increases to infinity as n increases to infinity. Also for a given value of n, $\rho_{1,i}^{n}$ values increases to infinity as i increases to infinity.

4.2 Effects of Vehicle Velocity on Vehicle Density

We use the scheme (22) to determine the effects of vehicle velocity on density. Taking v = 20km/hr, j = 1 into the scheme and nodal points in the x – direction as i = 1, 2, 3, …, 5. The following five algebraic equations are obtained;

$$\begin{aligned}
\left(\rho_{11}^{1}\right)&=(-19-u)\rho_{11}^{0}+u\rho_{01}^{0}+20\rho_{10}^{0}\\
\left(\rho_{21}^{1}\right)&=(-19-u)\rho_{21}^{0}+u\rho_{11}^{0}+20\rho_{20}^{0}\\
\left(\rho_{31}^{1}\right)&=(-19-u)\rho_{31}^{0}+u\rho_{21}^{0}+20\rho_{30}^{0}\\
\left(\rho_{41}^{1}\right)&=(-19-u)\rho_{41}^{0}+u\rho_{31}^{0}+20\rho_{40}^{0}\\
\left(\rho_{51}^{1}\right)&=(-19-u)\rho_{51}^{0}+u\rho_{41}^{0}+20\rho_{50}^{0}
\end{aligned} \tag{23}$$

The system of equations (23) can be written in matrix – vector form. Using the initial condition ρ (x, y, 0) = 21/0.1km and boundary conditions as ρ (0, y, t) = 0 and ρ (x,0, t) = 0, and the initial density function as ρ (x, y, 0) = e^{x} We solve the matrix - vector equation (23) at different values of velocity, that is u = 60Km/hr, u = 80Km/hr, u = 100Km/hr and u = 120Km/hr. The solution values are presented in the table 2 below;

Table 2: Effects of Vehicle Velocity on Vehicle Density

Grid point(i, j, n)	u = 60Km/hr	u = 80Km/hr	u = 100Km/hr	u = 120Km/hr
(1, 1, n)	237.000	237.000	237.0000	237.0000
(2, 1, n)	1340.000	1320.000	1280.0000	1220.0000
(3, 1, n)	3066.000	2666.000	2266.0000	1866.0000
(4, 1, n)	8452.000	7372.000	6292.0000	5212.0000
(5, 1, n)	22957.000	19997.000	17037.000	14077.000

The results in Table 2 are represented graphically in 2D. It is seen from the results in Table 2 that for a given value of i, density values tends to decrease as u values increases. Also for a given value of u, density values decreases to initial density as i tends to zero, in its initial position.

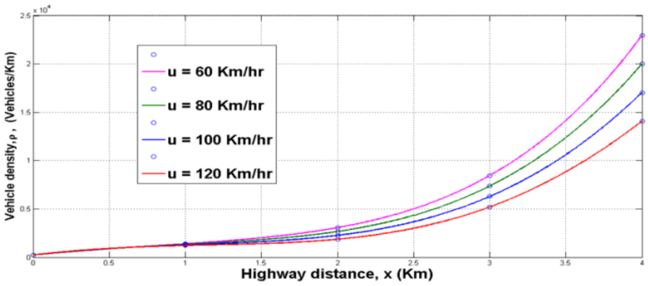

Figure 2: Effects of vehicle velocity on vehicle density

Table 3: Variation of Density, Velocity and Flux with Time, t

Grid point(i, lane 2, n)	Time, t	Density, ρ	Velocity, u	Flux, q
(5, 2, 0)	0.000	1.08000	79.22000	85.536
(5, 2, 1)	6.000	1.92500	61.80000	119.00
(5, 2, 2)	12.000	4.12000	34.50000	142.14
(5, 2, 3)	18.000	7.80000	16.65600	130.00
(5, 2, 4)	24.000	16.0000	7.500000	120.00

Table 3 shows that when vehicles are few, traffic density is lower, the flux is also lower; the velocity is high as there is maximum vehicle - road interaction. As the number of vehicles increases such that many vehicles passes the fifth nodal point at a relatively high velocity and density; the flow rate is high. At maximum density called jam density, vehicles cannot move; the flux is lower. The results from table 3 can be represented graphically in 2D in figure 3 to show the variation of density, ρ with velocity, u and with flux, q, respectively.

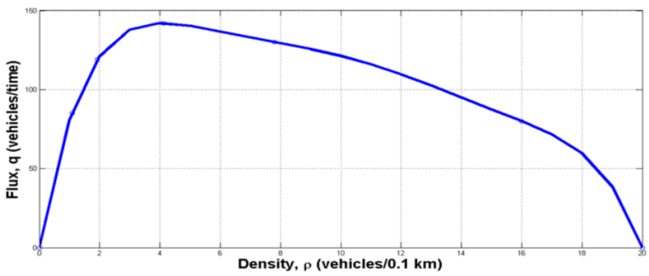

Figure 3: Variation of flux with vehicle density

5. Conclusion and Recommendations

5.1 Conclusion

Based on the objectives, the study developed heterogeneous motor vehicle traffic flow models focussing on two dimensional heterogeneity and multi - lane road heterogeneity aspects. It established the 2 - ensemble traffic composition formulation using vehicle heterogeneity characteristics in two dimensions. It determined numerical schemes founded on finite difference method to solve the model equations. The method obtained a finite system of linear algebraic equations from the PDE by discretizing the generated PDEs and coming up with the numerical schemes analogous to the equations. The thesis solved the equations subject to the given initial and boundary conditions as per the geometry of the traffic problems as an IBVP. MATLAB software was used to generate solution values in this study and produce 2D graphical representation of the traffic flow variables at different points on the highway. The finite difference technique, used as a solution method in the study, basically involves replacing the partial derivatives occurring in the partial differential equation as well as in the boundary and initial conditions by their corresponding finite difference approximations and then solving the resulting linear algebraic system of equations by a standard iterative procedure. The numerical values of the dependent variables are obtained at the points of intersection of the parallel lines, called mesh points or nodal points.

5.2 Recommendations

The traffic flow system is not just a mechanical system but a system where human decisions are involved. The inclusion of the driver's reaction time to traffic conditions downstream will make our model more realistic, although the mathematical computations will become more rigorous.

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