On Topological Transitivity

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Abstract: ***Topological transitivity is an important property in the setting of discrete dynamical systems, as it represents a type of complex global behavior, which is equivalent to some kind of chaos. This necessitates the study of conditions ensuring transitivity for various systems. Such conditions, for maps on intervals, have been in [1], [2], [3] etc. Since it is not always possible to exhibit dense orbits, or to construct a topological Conjugacy with known topologically transitive maps, these results have acquired practical value. In the present work, we provide elementary proofs for some known results in this context, and also improve them. A common idea in these results is roughly this: If the graph of f is sufficiently steep everywhere, then f is leo (see the definition below). The sufficient steepness is expressed in terms of the number of critical points in intervals J whose images f(J) almost cover the whole interval. Let $f^n = f \circ f^{n-1}$ for $n \geq 1$and $f^0 =$identity on [0, 1]. 1) A self-map f on a topological space X is said to be topologically transitive, if for every non-empty open sets U and V in X, there exists x in U and a natural number n such that $f^n(x) \in V$ i.e $f^n(U) \cap V$ is non empty. 2) A point x in X is said to be a c-point for a self-map f on X, if either f is not continuous at x or in every neighbourhood of x, f fails to be one-one. 3) Let J be an interval, and let J be partitioned into n intervals $Z_1, Z_2, \ldots\ldots, Z_n$, given by n-1 points $c_1 < , \ldots\ldots, < c_{n-1}$, and let $f: J \to J$ be a map such that the restriction of f to each Z_i is continuous and strictly monotonic. Then f is said to be piecewise monotonic on J. 4) Let f, J, $Z_1, Z_2, \ldots\ldots, Z_n$ be as above; then each Z_j is called lap of f. 5) Let f, J be as above and let f have only finitely many points of nondifferentiability. Then f is said to be piecewise differentiable. 6) Let f, J be as above such that for every nonempty open set U there exists $k \in N$ with $J - f^k(U)$ finite. Then f is said to be locally eventually onto (abbreviated leo) on J.We note that f being leo implies that f is topologically transitive on J. Here, throughout this work, C is the set of all c-points for f and D is the set of all points of differentiability of f. Further we obtain the following theorems.***

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Theorem 1: Let $n \in N$, $n \geq 2$. Suppose that $f:[0,1] \to [0,1]$ is a piecewise monotonic, piecewise differentiable map such that $\inf_{x \in D} \left| f^1(x) \right| > n$. Moreover, assume that $[0,1] - f(J)$ is finite for any subinterval J of [0, 1] containing at least nc-points. Then f is leo.

Proof: Let $\inf_{x \in D} \left| f^1(x) \right| = n + \delta$, where δ >0.If J is an interval contained in a lap, then $J = J_1 \cup J_2 \cup \ldots\ldots \cup J_t$ where each $\mathrm{int}(J_i)$ is contained in D. Also f(J) is connected and by an easy application of the mean value theorem,

$$l(f(J)) = \sum_{i=1}^{t} l(f(J_i)) \geq (n+\delta) \sum_{i=1}^{t} l(J_i) \geq (n+\delta) l(J),$$

where $l(A)$ denotes the length of the subinterval A of [0, 1].

We claim that $\left| f(J) \cap C \right| \geq n$ or f (J) contains some interval L which is contained in some lap and satisfies $l(L) \geq (1+\frac{\delta}{n}) l(J)$. This is because, if $\left| f(J) \cap C \right| \leq n-1$ and if $l(f(J) \cap L) < (1+\frac{\delta}{n}) l(J)$ for every interval L contained in the lap, then $l(f(J)) < n(1+\frac{\delta}{n}) l(J) = (n+\delta) l(J)$. a contradiction.

Next, we claim that for each k there exists $i \leq k$ with $\left| I \cap C \right| \geq n$ for some interval $I \subset f^i(J)$ or $f^k(J)$ contains some interval L_k which is contained in some lap and satisfies $l(L_k) \geq (1+\frac{\delta}{n})^k l(J)$. If $\left| I \cap C \right| \geq n$ for some $i \leq k$ and some interval $I \subset f^i(J)$, this is obvious. Otherwise, contains some interval L_{k-1} which is contained in some lap and satisfies $l(L_{k-1}) \geq (1+\frac{\delta}{n})^k l(J)$. Then $\left| f(L_{k-1}) \cap C \right| \geq n$ or $f(L_{k-1})$ contains some interval L_k which is contained in some lap and satisfies $l(L_k) \geq (1+\frac{\delta}{n})\ l(L_{k-1})) \geq (1+\frac{\delta}{n})^k l(J)$. As $f(L_{k-1}) \subset f^k(J)$ and $f(L_{k-1})$ is an interval. This finishes the induction.

Since $(1+\frac{\delta}{n})^k \to \infty\, as\, k \to \infty$, and since the length of subintervals of $f^k(J)$ cannot exceed 1, we have $\left| I \cap C \right| \geq n$ for some k and some interval $I \subset f^k(J)$. Then by our hypothesis $[0,1] - f^{k+1}(J)$ is finite. Thus f is leo.

Remark 1: Actually, we can strengthen the conclusion of

Theorem 1 as follows: Suppose that f satisfies the assumptions of Theorem 1, and let α be smaller than or equal to the minimum lap length. Let J be any interval of length α. Assume that *k* is an integer greater than or equal to $(\log(\frac{2\mu}{\alpha}))/(\log(1+\frac{\delta}{n}))+1$, where μ is the maximum lap length and $\inf_{x\in D}\left|f^1(x)\right| = n+\delta$, then $[0,1]-f^k(J)$ is finiteIt is proved in the following way. First note that J contains a subinterval J_1 which is contained in a lap and satisfies $l(J_1)\geq\frac{\alpha}{2}$. The proof of Theorem 1 shows that there exists $i\leq k-2$ with $\left|I\cap C\right|\geq n$ for some interval $I\subset f^i(J_1)$ or $f^{k-2}(J_1)$ contains some interval *K* which is contained in some lap and satisfies $l(K)\geq(1+\frac{\delta}{n})^{k-2}l(J_1)$. in the first case $[0,1]/f^{i+1}(J)$ is finite and therefore $[0,1]-f^k(J)$ is finite. Otherwise, $l(f(K))\geq n((1+\frac{\delta}{n})^{k-1})(\frac{\alpha}{2})\geq n\mu$. This implies that the closure of *f*(*K*) contains at least nc-points, and therefore $[0,1]-f^k(J)$ is finite. Obviously our Theorem 1 implies Theorem 1 of [2]. We give an example where Theorem 1 of [2] cannot be applied (because no lap is mapped onto [0, 1]), but our Theorem 1 is applicable. Let $f:[0,1]\to[0,1]$ be such that,

$$f(0)=f(\tfrac{1}{2})=f(1)=0,\ f(\tfrac{1}{4})=f(\tfrac{3}{4})=1,\ f(\tfrac{1}{12})=f(\tfrac{5}{12})=f(\tfrac{7}{12})=f(\tfrac{11}{12})=\tfrac{5}{6},$$

$$f(\tfrac{1}{6})=f(\tfrac{1}{3})=f(\tfrac{2}{3})=f(\tfrac{5}{6})=\tfrac{1}{6},$$

be linear on $[\frac{k}{12},\frac{k+1}{12}]$ for all $k\in\{0,1,2,\ldots\ldots..,11\}$. No lap is mapped onto [0, 1].

Every interval containing at least 6 *c*-points is mapped onto [0, 1]. Also, $\left|f^1(x)\right|\geq 8$ whenever it exists. By Theorem 1, f is leo and therefore topologically transitive.

The proof of the following theorem is analogous to the proof of Theorem 2 of [2]. Only a few modifications have to be done. Moreover, using the same techniques as in our proof of Theorem 1 we can simplify this proof avoiding the Markov diagram. As the resulting proof is only a small variation of the proof of Theorem 2 of [2] we omit it.

Theorem 2: Let $f:[0,1]\to[0,1]$ be a piecewise monotonic, piecewise differentiable map. Suppose that $f(c+)$ and $f(c-)\in\{0,1\}$ for each c-point c. Also, let $\left|f^1(x)\right|\geq 2$ for all points x of differentiability of f. Then f is leo, or else the image of the first lap or the image of the last lap or their union is invariant.
Note that we need neither the restriction on the number of laps nor the additional conditions of Theorem 2 of [2].

Remark 2: A particular case of Theorem 2, when interpreted for maps on the circle, becomes more elegant.

Let $f:S^1\to S^1$ be a continuous piecewise differentiable map with finitely many c-points (i.e. critical points). As above denote by C the set of all c-points, and by D be the set of points, where f is differentiable. Let there exist $c\in S^1$ that $C\subset f^{-1}(C)\cup\{c\}$. If $\left|f^1(x)\right|>2$ for all $x\in D$, then f is leo. Moreover, if $\left|f^1(x)\right|\geq 2$ for all $x\in D$, then f is leo or $f(c,c_1)$ or $f(c_{n-1},c\ )$ or their union is invariant, where c_1 and c_n are critical points nearest to c, one on each side.

Finally, we give another formulation of Theorem 3 in [2] and indicate how its proof can be simplified avoiding the Markov diagram.

Theorem 3: Let $f:[0,1]\to[0,1]$ be a piecewise monotonic, piecewise differentiable map. Suppose that $f(c+)$=0 for all points c that *f*(0+)=0, and that *f* is onto. If $\inf_{x\in D}\left|f^{'}(x)\right|$ is strictly greater than the number of *c*-points in $[0,\alpha]$, where α is the smallest c-point such that $f(\alpha-)=1$, then *f* is leo.

Proof: If *J* is an interval containing 0, we claim that *f*(*J*) is an interval (though *f* may not be continuous on *J*). This is because *f* takes every interval which is contained in a lap to an interval containing 0 in its closure, and the union of any collection of intervals containing 0, is again an interval.This observation, together with the ideas of proofs of earlier theorems, will prove Theorem 3. Dually, we have another result when *f* satisfies *f*(*c*-)=1, for all c-points, with corresponding modifications in the statement.

Remark 3: The results in [2] rely heavily on the results in [1], and use the complicated tools of Markov diagram; whereas the proofs in this paper are elementary.

References

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