International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

ZETA & ETA Functions' Calculation in Form of Matrix

Dibyajyoti Dutta

Abstract: ***Zeta and ETA Functions are shown in same form of a quadratic equation while treated as Matrix and the two roots of that quadratic equation are one for zeta function and another for Eta function.***

Keywords: Zeta and eta functions in form of Matrix

By the definition of 'ZETA' (ζ) function we know that,

$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\propto} \frac{1}{n^s}$

Similarly by the definition of 'ETA' (η) function we know that,

$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots = \sum_{n=1}^{\propto} \frac{(-1)^{(n-1)}}{n^s}$

Now we can write $(a+1)^s$ *where 'a' is an integer varies from 1 to ∞ in an expression of series form as given below,*

$$(a+1)^s = 1 + \frac{sa}{1!} + \frac{s(s-1)a^2}{2!} + \cdots = \sum_{k=0}^{s} \binom{s}{k} a^k$$

Now if we subtract a^s *in both side of this equation we will get the following expression,*

So, $(a+1)^s - a^s = \sum_{k=0}^{s} \binom{s}{k} a^k - a^s = \sum_{k=0}^{(s-1)} (a+1)\{s-1-k\}ak$

Or, $(a+1)^s - a^s = (a+1)^s \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} ak$ ……(I)

Now by dividing both sides by $a^s(a+1)^s$ *we will get as follows*

$\frac{1}{a^s} - \frac{1}{(a+1)^s} = \frac{1}{a^s} \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^k$ ……(II)

So, by putting the value of 'a' as 1 in above equation we will get,

$\frac{1}{1^s} - \frac{1}{2^s} = \frac{1}{1^s} \sum_{k=0}^{(s-1)} (2)^{-(k+1)} 1^k$

And if a=2,

$\frac{1}{2^s} - \frac{1}{3^s} = \frac{1}{2^s} \sum_{k=0}^{(s-1)} (3)^{-(k+1)} 2^k$

And so on up to ∞

Now by subtracting ζ (s)from itself but after shifting one number ahead we will get,

$\zeta(s) = \quad 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$

$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$

(-)________________________________

$0 = -1 + \left(\frac{1}{1^s} - \frac{1}{2^s}\right) + \left(\frac{1}{2^s} - \frac{1}{3^s}\right) + \cdots$

$0 = -1 + \frac{1}{1^s} \sum_{k=0}^{(s-1)} (2)^{-(k+1)} 1^k + \frac{1}{2^s} \sum_{k=0}^{(s-1)} (3)^{-(k+1)} 2^k + \cdots$

$1 = \sum_{a=1}^{\propto} [\frac{1}{a^s}] \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^k$

$1 = [\zeta(s)] \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^k$]a=1 to ∞

$\frac{1}{[\zeta(s)]} = \sum_{k=0}^{(s-1)} (\frac{1}{a+1})(\frac{a}{a+1})^k$]a=1 to ∞……… *(A)*

$\frac{1}{[\zeta(s+1)]} = \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^{(k+1)}$]a=1 to ∞*[multiplying both side by a]*

$\frac{1}{[\zeta(s+x)]} = \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^{(k+x)}$]a=1 to ∞*[multiplying both side by* a^x*]*

Now putting s=1,

$\frac{1}{[\zeta(1+x)]} = \sum_{k=0}^{(0)} (a+1)^{-(k+1)} a^{(k+x)}$]a=1 to ∞

$\frac{1}{[\zeta(1+x)]} = \sum_{a=1}^{\propto} (a+1)^{-1} a^x$ *[as k=0]*

$\frac{1}{[\zeta(s)]} = \sum_{a=1}^{\propto} (a+1)^{-1} a^{(s-1)}$ *[putting (1+x) = s]*

$\frac{1}{[\zeta(s)]} = \sum_{a=1}^{\propto} [a^s] \frac{1}{a(a+1)}$

$\frac{1}{[\zeta(s)]} = [\zeta(-s)] \sum_{a=1}^{\propto} [\frac{1}{a(a+1)}]$

$\frac{1}{[\zeta(s)][\zeta(-s)]} = \sum_{a=1}^{\propto} [\frac{1}{a(a+1)}]$ *[Note:* $\sum_{a=1}^{\propto} [\frac{1}{a(a+1)}] = 1$ *]*

Now from equation (A) mentioned above,

$\frac{1}{[\zeta(s)]} = \sum_{k=0}^{(s-1)} (\frac{1}{a+1})(\frac{a}{a+1})^k$]a=1 to ∞

$\frac{1}{[\zeta(s)]} = \sum_{a=0}^{\propto} [1 - (\frac{a}{a+1})^s]$ *[by the property of geometric progression series summation formula up to s'th term]*

$\frac{1}{[\zeta(s)]} = \lim_{m\to\infty}\{m - \sum_{a=0}^{m} [(\frac{a}{a+1})^s]\}$ ……….(B)

$\frac{-1}{[\zeta(s)]^2} = \lim_{m\to\infty}\{0 - s\sum_{a=0}^{m} [(\frac{a}{a+1})^{(s-1)} \frac{1}{(a+1)^2}]\}$ [by taking derivative in both sides.]

$\frac{1}{[\zeta(s)]^2} = s\sum_{a=0}^{\propto} [(\frac{a}{a+1})^s (\frac{a+1}{a}) \frac{1}{(a+1)^2}]$

$\frac{1}{[\zeta(s)]^2} = s\sum_{a=0}^{\propto} [(\frac{a}{a+1})^s \frac{1}{a(a+1)}]$

$\frac{1}{[\zeta(s)]^2} = s[\zeta(-s)][\zeta(s) - 1] \frac{1}{[\zeta(s)][\zeta(-s)]}$[from the given definitions above]

$\frac{1}{[\zeta(s)]} = s[\zeta(s) - 1]$

1 $= s[\zeta(s)][\zeta(s) - 1]$so, from here we obtain the following quadratic equation

$$s[\zeta(s)]^2 - s[\zeta(s)] - 1 = 0$$

Thus by the formula of quadratic equations we can write the following expression,

$[\zeta(s)] = \frac{s \pm \sqrt{s^2+4s}}{2s} = \frac{s \pm s\sqrt{1+4/s}}{2s} = \frac{1 \pm \sqrt{1+4/s}}{2}$

Now by the definition of 'ZETA' (ζ) function we know that,

$\zeta(-s) = 1 + \frac{1}{2^{-s}} + \frac{1}{3^{-s}} + \cdots = \sum_{n=1}^{\propto} \frac{1}{n^{-s}}$

$\zeta(-s) = 1 + 2^s + 3^s + \cdots = \sum_{n=1}^{\propto} n^s$

From previously shown equation no. (I),

$$(a+1)^s - a^s = (a+1)^s \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^k$$

Now by putting a=1 in this equation,

$2^s - 1^s = 2^s \sum_{k=0}^{(s-1)} 2^{-(k+1)} 1^k$ *and so on up to ∞*

ζ (-s) = $1 + 2^s + 3^s + \cdots$ *[Now by subtracting ζ (-s)from itself but after shifting ζ (-s) =* $1 + 2^s + 3^s + \cdots$ *one number behind we will get,]*

*(-)*____

$0 = 1 + (2^s - 1^s) + (3^s - 2^s) + \cdots$

$0 = 1 + 2^s \sum_{k=0}^{(s-1)} 2^{-(k+1)} 1^k + 3^s \sum_{k=0}^{(s-1)} (3)^{-(k+1)} 2^k + \cdots$

$-1 = \sum_{a=1}^{\infty} [(a+1)^s] \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^k$

$-1 = [\zeta(-s) - 1] \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^k$]a=1 to ∞

$\frac{1}{[1-\zeta(-s)]} = \sum_{k=0}^{(s-1)} (\frac{1}{a+1})(\frac{a}{a+1})^k$]a=1 to ∞*[Now from, equation (A)given above,*

$\frac{1}{[1-\zeta(-s)]} = \frac{1}{[\zeta(s)]}$*Note:*$[\zeta(s)] = [1-\zeta(-s)], [\zeta(s) - 1] = -[\zeta(-s)]$

Or,$[\zeta(s)] + [\zeta(-s)] = 1$

Now similarly as equation (B) we can write,

$$\frac{1}{[1-\zeta(-s)]} = \lim_{m\to\infty}\{m - \sum_{a=0}^{m} [(\frac{a}{a+1})^s]\}$$

$\frac{1}{[1-\zeta(-s)]^{2}} = \lim_{m\to\infty}\{0 - s\sum_{a=0}^{m} [(\frac{a}{a+1})^{(s-1)} \frac{1}{(a+1)^2}]$ } [by taking derivative in both sides.]

$\frac{1}{[1-\zeta(-s)]^{2}} = -s\sum_{a=0}^{\infty} [(\frac{a}{a+1})^s (\frac{a+1}{a}) \frac{1}{(a+1)^2}]$

$\frac{1}{[1-\zeta(-s)]^2} = -s\sum_{a=0}^{\infty} [(\frac{a}{a+1})^s \frac{1}{a(a+1)}]$

$\frac{1}{[1-\zeta(-s)]^2} = -s[\zeta(-s)]\,[\zeta(s) - 1]\frac{1}{[\zeta(s)][\zeta(-s)]}$[From the given definitions above]

$$\frac{1}{[1-\zeta(-s)]^2} = -s\frac{1}{[\zeta(s)]}[\zeta(s) - 1]$$

$\frac{1}{[1-\zeta(-s)]^2} = -s\frac{-1}{[1-\zeta(-s)]}[\zeta(-s)]$[From the given note above]

1 = $s[\zeta(-s)][1-\zeta(-s)]$so, from here we obtain the following quadratic equation

$$s[\zeta(-s)]^2 - s[\zeta(s)] + 1 = 0$$

Thus by the formula of quadratic equations we can write the following expression,

$[\zeta(-s)] = \frac{s\pm\sqrt{s^2-4s}}{2s} = \frac{s\pm s\sqrt{1-4/s}}{2s} = \frac{1\pm\sqrt{1-4/s}}{2}$

Now 'ETA' (η) functions,

η(s) = $1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots$ *[BY ADDING THESE TWO]*

η(s) = $1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots$ 1-1/2^s+1/3^s-…

*(+)*____

2η (s) = $1 + \left(\frac{1}{1^s} - \frac{1}{2^s}\right) - \left(\frac{1}{2^s} - \frac{1}{3^s}\right) + \cdots$

2[η (s)]-1 = $\sum_{a=1}^{\infty} [\frac{(-1)^{(n-1)}}{a^s}] \sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^k$

2[η (s)]-1 = [η (s)] $\sum_{k=0}^{(s-1)} (a+1)^{-(k+1)} a^k$]a=1 to ∞

$2 - \frac{1}{[\eta(s)]} = \frac{1}{[\zeta(s)]}$ *[from, equation (A) given above]*

$1 - \frac{1}{[\zeta(s)]} = \frac{1}{[\eta(s)]} - 1$ *…… (c)*

Now similarly as equation (B) we can write

$2 - \frac{1}{[\eta(s)]} = \lim_{m\to\infty}\{m - \sum_{a=0}^{m} [(\frac{a}{a+1})^s]\}$

$\frac{1}{[\eta(s)]^2} = -s\frac{1}{[\zeta(s)]}[\zeta(s) - 1]$[by taking derivative in both sides.]

$\frac{1}{[\eta(s)]^2} = -s[1 - \frac{1}{[\zeta(s)]}]$

$\frac{1}{[\eta(s)]^2} = -s[\frac{1}{[\eta(s)]} - 1]$*[from, equation (c) given above]*

1 = $-s[\eta(s)] + s[\eta(s)]^2$

So, $s[\eta(s)]^2 - s[\eta(s)] - 1 = 0$*, this is the same quadratic equation as that of the 'ZETA' (ζ) function.So, CONCLUTION is as follows-*

So, Values are

$$[\zeta(s)] = \frac{1+\sqrt{1+4/s}}{2}$$

$$[\zeta(-s)] = \frac{1+\sqrt{1-4/s}}{2}$$

$$[\eta(s)] = \frac{1-\sqrt{1+4/s}}{2}$$

$[\eta(-s)] = \frac{1-\sqrt{1-4/s}}{2}$

<u>EQUATIONS IN FORM OF MATRIX</u>

$\frac{1}{[\zeta(s)][\zeta(-s)]} = \sum_{a=1}^{\infty}[\frac{1}{a(a+1)}]$ *[Note:* $\sum_{a=1}^{\infty}[\frac{1}{a(a+1)}] = 1$]

$[\zeta(s)] + [\zeta(-s)] = 1$

$2 - \frac{1}{[\eta(s)]} = \frac{1}{[\zeta(s)]}$

$$\frac{1}{[\zeta(s)][\zeta(-s)]}[\zeta(s)] + [\zeta(-s)] = 1\text{x}1$$

Or, $\frac{1}{[\zeta(s)]} + \frac{1}{[\zeta(-s)]} = 1$ [Note: $\frac{1}{[\zeta(s)]} = 1 - \frac{1}{[\zeta(-s)]}$]

So, $2 - \frac{1}{[\eta(s)]} = 1 - \frac{1}{[\zeta(-s)]}$

Or, $2 - \frac{1}{[\eta(s)]} = 1 - 2 + \frac{1}{[\eta(-s)]}$

Or, $\frac{1}{[\eta(s)]} + \frac{1}{[\eta(-s)]} = 3$

Now, $\lim_{s\to\infty}[\zeta(s)] = 1$&$\lim_{s\to\infty}[\eta(s)] = 0$

<u>Ramanujan's infinite sum</u>

S = 1 + 2 + 3 + 4 + ……

¼ = 1 – 2 + 3 – 4 + ……

*(-)*____

S – ¼ = 4 + 8 + 12 + ……

S – ¼ = 4(1 + 2 + 3 + ……) [but it is not the series S but its expansion is ½ of S series.]

$\lim_{m\to\infty}\sum_{n=1}^{m}[n] = \lim_{m\to\infty}\frac{m(m+1)}{2} = \lim_{m\to\infty}\frac{m^2}{2}$

so, $\lim_{m\to\infty}\sum_{n=1}^{m/2}[n] = \lim_{m\to\infty}\frac{m/2(m/2+1)}{2} = \lim_{m\to\infty}\frac{m^2}{4.2} = \frac{S}{4}$

So, (S – ¼) = $4.\frac{S}{4}$ = S[So, S cancels out in both side and can't be determine.]

From the expression given above of$[\zeta(-s)]$

1 + 2 + 3 + 4 + …… = $\frac{1+\sqrt{1-4}}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Now, $[\eta(s)] = (1 - 2^{1-s})[\zeta(s)]$ *this expression is also stands wrong due to same assumption of the value of a ½ expand zeta series as a full zeta series function.*

Now,$[\zeta(1)] = \lim_{m\to\infty}\sum_{n=1}^{m}[\frac{1}{n}] = 0.5(1+\sqrt{5})$[it has a finite value because it's last digit $\lim_{m\to\infty}\frac{1}{m} = 0$]

Whereas,$[\zeta(0)] = \lim_{m\to\infty}\sum_{n=1}^{m} n^0$ = *1+ 1+ 1+…… up to∞* = ∞

Now from the given expression of $[\zeta(s)]$&$[\eta(s)]$ *we found that their graphs are mirror images to each other with*

respect to a straight line$[\zeta\ (s)]$or$[\eta\ (s)]$ = *0.5*s^0 *Thus as* $[\zeta\ (0)] = +\infty, [\eta\ (0)] = -\infty$ *and* $[\zeta\ (-4)] = [\eta\ (-4)]$ = *0.5*
For {-4<S<0}, $[\zeta\ (s)]$&$[\eta\ (s)]$ *are imaginary.*
For $\zeta\ (s) = \frac{1}{\zeta\ (-s)}$ *and* $(s) = \frac{1}{\eta(-s)}$, $s = \frac{1}{\sqrt{3}}$=1/√3 *i*
So, these all which are mentioned above are the CONCLUTION of this topic.