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Zero-Divisor Graph in Power Set Ring and Matrices Ring of 2 X 2

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Abstract: ***The first article about graph theory was written by Leonhard Euler the famous Swiss mathematician which was published in 1736. Primarily, the idea of graph was not important as point of mathematics because it most deals with recreational puzzles. But the recent improvement in mathematics specially its application brought a strong revolution in graph theory. Therefore, this article is written under the tile of (Zero-divisor graph in power set ring and matrices ring of 2X2). In this article we first introduce the zero-divisor graph set of alternative in R ring. Then we present the zero-divisor graph in power set ring and matrices ring of 2X2. Whereas the vertex is denoted with K_1 and the elements of an optional ring which are not zero-divisors they are the vertices without edges of that ring in zero-divisor graph. Next, we will study when a graph is planar in power set ring and under which circumstances a graph is complete and complete bipartite in matrices set ring of 2X2. After the research we found out: If the element numbers of $X \neq \varnothing$ set be minor than 4, the zero-divisor graph ring of $P(X)$ is planar and if the numbers of X set elements be maximum than 4, the zero-divisor graph ring is not planar. Suppose that $R = M_2(F)$ matrix ring of $2x2$ is in F field with elements .If $U = \left\{ \begin{pmatrix} a & \circ \\ \circ & b \end{pmatrix} : a, b \in F \right\}$, The sub inductive graph on U is complete bipartite of zero-divisor graph in ring R and if $V = \left\{ \begin{pmatrix} \circ & a \\ \circ & \circ \end{pmatrix} : a \in F \right\}$ or $V = \left\{ \begin{pmatrix} \circ & \circ \\ a & \circ \end{pmatrix} : a \in F \right\}$, in this case the zero-divisor graph in ring R of sub inductive graph on V is complete. The zero-divisor graphs in ring $R = M_2(F)$ are planar if and only if*** $|F| \leq 3$.

Keywords: Graph, zero-divisors and zero-divisors' graph

1. Introduction

The graph theory is a basic issue in recent mathematics for research and the concept of graph in mathematics from past till now is arguable. For example, it has most usage in graph relations and functions. The graph theory back in 18 century. Later on as its application on different sections it discussed which has a research area at this time. From 1988 to now studies have been done and mathematicians released their research results in 2005, 2006 and 2007. The topic that is going to be discussed is how a power set ring is a planar graph and matrices set ring of 2×2 is a complete graph and complete bipartite.

First, in this article we will define graph, subgraph, inductive subgraph, regular graph of r-, complete graph, bipartite graph, complete bipartite graph, star graph, path, close path, circuit, connected graph, connected graph diameter, planar graph and zero-divisor graph of R in alternative ring. Then, we will introduce zero-divisor graph in power set ring and matrices ring of 2×2. Later on we present how power set ring in a graph is planar? and matrix ring set of 2×2 a graph is complete and complete bipartite.

Definition 1: Suppose that $V(\Gamma) \neq \varnothing$ is a finite set and $E(\Gamma)$ is set which each element of it be an ordered pair of $V(\Gamma) \times V(\Gamma)$ elements, in this case ordered pair of $(V(\Gamma), E(\Gamma))$ is called undirected graph and we denote with $\Gamma = (V(\Gamma), E(\Gamma))$ notation in which $V(\Gamma)$ set shows vertices and $E(\Gamma)$ set denotes edges of Γ (Kung-chug & Robert, 2008).

Definition 2: Assume that Γ is a graph with $V(\Gamma)$ vertices and $E(\Gamma)$ edges, in this case:

a) If v_i and v_j be two vertices of Γ and (v_i, v_j) is an edge of it, thus v_i and v_j are adjacent in which v_i is initial vertex and v_j is end vertex.
b) The numbers of edges which pass from v is called v degree that shows with $\deg(v)$. also the vertex which its degree is zero, is called sole vertex and denotes with K_1. K_1 means a sole vertex (Vadis, 2009).
c) A S subgraph of Γ is a graph which its edges of sub set of $E(\Gamma)$ with vertices be connected to this edges. So every graph is a subgraph of itself.

d) A U inductive graph of Γ is a graph which its sub set vertices of $V(\Gamma)$ with its all edges of Γ in which two vertices of each edges include $V(U)$ (Kung-chug & Robert, 2008).

Definition 3: We call regular -r the Γ graph, if every degree of Γ vertex be equal to r (Dosti, 2011).

Definition 4: n vertex graph and the ordered $(n-1)$ are complete graph which denotes with K_n (Kung-chug & Robert, 2008).

Definition 5: Γ graph is bipartite, if $V(\Gamma)=V_1(\Gamma)\cup V_2(\Gamma)$ and $V_1(\Gamma)\cap V_2(\Gamma)=\varnothing$, as two vertices are not adjacent in $(i=1,2)$ V_i. Furthermore if each vertex of $V_1(\Gamma)$ be adjacent with $V_2(\Gamma)$ this graph is called complete bipartite graph and indicates it with $k_{m,n}$ notation. In which $|V_1(\Gamma)|=m$ and $|V_2(\Gamma)|=n$ notice that $k_{m,n}$ have as $m+n$ vertex and mn edge (Schutz, 1995).

Definition 6: Assume that Γ is a graph, v_i and v_j is its two vertices. A path of l length from v_i to v_j, finite line of vertices and Γ are as follow:

$$v_i=u_0,e_1,u_1,e_2,u_2,e_3,\ldots,e_l,u_l=v_j$$

In a way that for each $(1\le t\le l)$, the vertices u_{t-1} and u_t are adjacent. In this definition the repetition of edges and vertices are allowable (Kung-chug & Robert, 2008).

Definition 7: Close path in Γ graph is a circuit as follow:

$$v_i=u_0,e_1,u_1,e_2,u_2,e_3,\ldots,e_l,u_l=v_j$$

As $v_i=v_j$ (Dosti, 2011).

Definition 8: A path which does not have repeated edge and vertex is called circuit. We show the edge of n circuit with P_n notation. The numbers of edges in this circuit is called length of circuit. Every vertex is a zero length circuit.

Definition 9: We call Γ graph, connected if there is a path inside both vertices. The graph which is not connected we call this disconnected graph. In connected graph the degree of each edge is at least equal to one. Every n edge simple connected graph has at least $n-1$ edge (Dosti, 2011).

Definition 10: Suppose that Γ is a graph, v_i and v_j are its two vertices. The shortest length path between v_i and v_j is called space. We denote it with $\partial(v_i,v_j)$. If there is not such a path, so $\partial(v_i,v_j)=\infty$ and $\partial(v_i,v_j)$ have metric as:

a) Ever $\partial(v_i,v_j)\ge 0$ and there is equivalence if and only if $v_i=v_j$;

b) $\partial(v_i,v_j)=\partial(v_j,v_i)$

c) $\partial(v_i,u_l)\le\partial(v_i,v_j)+\partial(v_j,u_l)$ By letting each equal length, $1,\Gamma$ is a metric environment (Babinski, Kupczyk, Simi, & Zwierzynski, 2000).

d) We define the diameter of connected graph of Γ as follow:

$$diam(\Gamma)=\max\{\partial(x,y)\,|\,x,y\in V(\Gamma)\}$$

If the graph of Γ is compound of sole edge, $diam(\Gamma)=0$ (Morty, 2006).

Definition 11: A planar graph is a graph that can be drawn on plane in such a way that no edges cross each other. The complete graph can be drawn with four vertices of k_4 without intersection of edges. Therefore, k_4 is a planar graph (Vadis, 2009).

Mathematicians try to describe planar and non-planar graphs for a long time. Finally, they were solved in 1930 by Kazimierz Kuratowski Polish mathematician. The k_5 and $k_{3,3}$ graphs are non-planar (Buzz & Zofran, 2009).

Theorem: The Γ graph is planar if and only if it does not have k_5, $k_{3,3}$ as subgraph.

Proof: Refer to (Aleve's, 2011).

Definition 12: Suppose that R is a ring and $R^*=R-\{0\}$. The zero-divisor graph of R which we denotes with $\Gamma(R)$ is a graph that its vertices are elements of R^*. The two different elements of $a,b\in R^*$ are adjacent, if $ab=ba=0$. Whereas a and b both are adjacent vertices of $\Gamma(R)$ edge.

Definition 13: Assume that $R=P(X)$ is a power ring on non-empty set of X and $R^*=R-\{\varnothing\}$. The zero-divisor graph of R which we denote with $\Gamma(R)$ is a graph that its vertices are elements of R^*. If $AB=A\cap B=\varnothing$. The two different elements of $A,B\in R^*$ are adjacent, whereas A and B both are adjacent vertex of edge (Anderson, Levy, & Shapiro, 2003).

Theorem 2: If the elements of $X\neq\varnothing$ be minor than 4, in this case the zero-divisor graph of $P(X)$ ring is planar.

Proof:

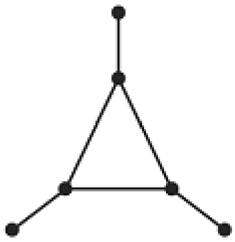
a) If $X=\{a\}$, thus $R=P(X)=\{X,\varnothing\}\Rightarrow R^*=R-\{\varnothing\}=\{X\}\Rightarrow\Gamma(R)=K_1$ ‘ Therefore $\Gamma(R)$ is a planar graph.

b) If $X=\{a,b\}$, thus $R=P(X)=\{\{a\},\{b\},X,\varnothing\}\Rightarrow R^*=R-\{\varnothing\}=\{\{a\},\{b\},X\}\Rightarrow\Gamma(R)=P_2\cup K_1$ ‘ Therefore $\Gamma(R)$ is a planar graph.

c) If $X=\{a,b,c\}$, thus

$$R=P(X)=\{\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X,\varnothing\}$$

$$\Rightarrow R^*=R-\{\varnothing\}=\{\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}$$



Shape 1

Note: The zero-divisors graph of $P(X)$ ring is a triangular which an edge is connected to every vertex of it (Aleve's, 2011).

Theorem 3: If $X=\{a,b,c,d\}$, thus the zero-divisor graph of $P(X)$ is not planar.

Proof:

$$R=P(X)=\left\{\begin{matrix}\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\\ \{c,d\},\{a,b,c\},\{a,b,d\}\{a,c,d\},\{b,c,d\},X,\varnothing\end{matrix}\right\}$$

$$\Rightarrow R^*=R-\{\varnothing\}=\left\{\begin{matrix}\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\\ \{c,d\},\{a,b,c\},\{a,b,d\}\{a,c,d\},\{b,c,d\},X\end{matrix}\right\}$$

Assume that $A=\{\{b\},\{d\},\{b,d\}\}$, $B=\{\{a\},\{c\},\{a,c\}\}$ and H are subgraph of $\Gamma(P(X))$ which $V(H)=A\cup B$. We can see easily that H with $K_{3,3}$ is co-strength . therefore, according to theorem (1) the zero-divisor graph of $P(X)$ ring is not planar. But the complete graph of K_4 with vertices set of $V(K_4)=\{\{a\},\{b\},\{c\},\{d\}\}$ has as subgraph in its own (Anderson, Levy, & Shapiro, 2003).

Result 1: If the elements numbers of X be bigger or equal to 4, thus the zero-divisors graph of $P(X)$ is not planar.

Definition 14: Suppose that S is ring with a single element, $R=M_2(S)$, matrices ring of $2x2$ with elements in S and $R^*=R-\{0_{2\times2}\}$. The zero-divisor graph in R ring that we denote with $\Gamma(R)$ is a graph which its vertices are elements of R^*. The two different elements of $A,B\in R^*$ are adjacent, if $AB=BA=0$. Whereas A and B both show adjacent vertex of $\Gamma(R)$ edge (Tongue, 2005).

Theorem 4: Assume that $R=M_2(F)$ be the matrices ring of $2x2$ with elements in field of F , in this case:

a) If $U=\left\{\begin{pmatrix}a & \circ\\ \circ & b\end{pmatrix}:a,b\in F\right\}$, thus the inductive subgraph on U of zero-divisors in R ring is complete bipartite.

b) If $V=\left\{\begin{pmatrix}\circ & a\\ \circ & \circ\end{pmatrix}:a\in F\right\}$ or $V=\left\{\begin{pmatrix}\circ & \circ\\ a & \circ\end{pmatrix}:a\in F\right\}$, thus inductive subgraph on V of zero-divisors graph in R ring is complete.

c) The zero-divisor graph in $R=M_2(F)$ ring is planar if and only if $|F|\le 3$.

Proof:

a) Suppose that T is matrices set which only a_{11} element of its be nonzero and K is the matrices set which only a_{22} element of its be nonzero, in this case for each $A\in T$, $B\in K$ and $AB=BA=0$, so the graph is a complete bipartite graph.

b) If for every $A,B\in V$, $A\neq B$ and $AB=BA=0$, in this case the graph is a complete graph.

c) With consideration of (a) the theorem is proved (Tongue, 2005).

Theorem 5: Assume that A and B are two endpoints of all $R=A\times B$ and $R^*=A\times B-\{(0,0)\}$, in this case the zero-divisor graph in R ring is complete bipartite.

Proof: Suppose that $A^*=A-\{0\}$, $B^*=B-\{0\}$, $V_1=\{(a,0)\mid a\in A^*\}$ and $V_2=\{(0,b)\mid b\in B^*\}$. Thus A and B are two endpoints, the vertices of $(i=1,2)$ V_i are not connected. On the other hand $(a,0)(0,b)=(0,0)$, so the V_1 vertex connected to V_2 vertex. Therefore $\Gamma(R)=K_{m,n}$ in which $m=|A^*|$, $n=|B^*|$, if $A=\mathbf{Z}_2$, in this case zero-divisor graph of $\Gamma(R)$ with consideration of $|\Gamma(R)|=|B|$ is a star graph (Cretkovic & Droop, 1995).

For more information about concept of zero-divisors graph of commutative ring refer to (Buzz & Zofran, 2009).

As you can see elements of a ring which are not zero the vertices without edges in zero-divisor graph is that ring. In order to not have a single vertex for dependent ring graph. We define zero-divisor graph of a ring as follow:

Definition 15: Assume R as a ring, the $Z\left(R^*\right)$ is the zero-divisor set of R ring except zero. The two different elements of $a,b \in Z\left(R^*\right)$ are adjacent, if $ab = ba = 0$. Whereas a and b both are adjacent vertex of $\Gamma\left(R\right)$ edge (Lucas, 2006).

Theorem 6: Suppose that R is a commutative ring with $Z\left(R^*\right) \neq \varnothing$, in this case the zero-divisor graph of $\Gamma\left(R\right)$ is connected and $diam\left(\Gamma\left(R\right)\right) \leq 3$.

Proof: Assume the two different elements are $x, y \in Z\left(R^*\right)$. If $xy = 0$, thus $d(x,y) = 1$. Then suppose $xy \neq 0$. If $x^2 = y^2 = 0$. According to definition (8), $x - xy - y$ is a 2 length circuit. Therefore $d(x,y) = 2$ if $x^2 = 0$ and $y^2 \neq 0$, so $b \in Z\left(R^*\right) - \{x, y\}$ with $by = 0$ exists. So we conclude that $d(x,y) = 2$. Now suppose that x^2, xy, y^2 are all nonzero. In this case

$a,b \in Z\left(R^*\right) - \{x,y\}$ ب $ax = by = 0$

Exists. If $a = b$, thus $x - a - y$ is 2 length circuit. Then assume $a \neq b$. If $ab = 0$, so $x - a - b - y$ is 3 length circuit. Therefore, it could be written as:

$d(x,y) \leq 3$

If $ab \neq 0$, and $x - ab - y$ is 2 length circuit. Then $d(x,y) = 2$, so $d(x,y) \leq 3$. In conclusion $diam\left(\Gamma\left(R\right)\right) \leq 3$ (Vadis, 2009).

Example 1: If $R = \mathbf{Z}_2 \times \mathbf{Z}_4$, then $diam\left(\Gamma\left(R\right)\right) = 3$.

Solution:

$\left(\bar{0},\bar{1}\right)\left(\bar{0},\bar{2}\right) \neq \left(\bar{0},\bar{0}\right)$, $\left(\bar{0},\bar{1}\right)\left(\bar{1},\bar{2}\right) = \left(\bar{0},\bar{2}\right) \neq \left(\bar{0},\bar{0}\right)$, $\left(\bar{1},\bar{0}\right)\left(\bar{1},\bar{2}\right) \neq \left(\bar{0},\bar{0}\right)$

On the other hand $\left(\bar{0},\bar{1}\right)\left(\bar{1},\bar{0}\right) = \left(\bar{0},\bar{0}\right)$, $\left(\bar{1},\bar{0}\right)\left(\bar{0},\bar{2}\right) = \left(\bar{0},\bar{0}\right)$, $\left(\bar{0},\bar{2}\right)\left(\bar{1},\bar{2}\right) = \left(\bar{0},\bar{0}\right)$, so we have $\left(\bar{0},\bar{1}\right) - \left(\bar{1},\bar{0}\right) - \left(\bar{0},\bar{2}\right) - \left(\bar{1},\bar{2}\right)$ circuit in zero-divisor graph of $R = \mathbf{Z}_2 \times \mathbf{Z}_4$ ring. Therefore, $diam\left(\Gamma\left(R\right)\right) = 3$ (Vadis, 2009).

Result 2: Suppose F_p and F_q are two field of p (initial p) and q (initial q) elements, in this case $\Gamma\left(F_p \times F_q\right) = K_{p-1,q-1}$ (Li, 2011).

2. Conclusion

We conclude from the study of zero-divisor graph in power set ring and matrices ring of 2×2 as:

1) If the $X \neq \varnothing$ number of set elements be minor than 4.
2) If the number of X set elements be bigger or equal to 4, in this case the zero-divisor graph of $P\left(X\right)$ is not planar.
3) Suppose that $R = M_2\left(F\right)$ matrices ring of $2x2$ is in field F with its elements.
4) If $U = \left\{ \begin{pmatrix} a & \circ \\ \circ & b \end{pmatrix} : a, b \in F \right\}$, the inductive subgraph on U is a complete bipartite zero-divisor graph in R ring and if $V = \left\{ \begin{pmatrix} \circ & a \\ \circ & \circ \end{pmatrix} : a \in F \right\}$ or $V = \left\{ \begin{pmatrix} \circ & \circ \\ a & \circ \end{pmatrix} : a \in F \right\}$, in this case the inductive graph on V is a complete zero-divisor graph in R ring.
5) The zero-divisor graph $R = M_2\left(F\right)$ ring is planar if and only if $\left|F\right| \leq 3$.

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