EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A STEKLOV SYSTEM INVOLVING THE (p, q)-LAPLACIAN

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Abstract. In this paper, we prove the existence of at least three weak solutions for a quasilinear elliptic system involving a pair of (p, q)-Laplacian operators with Stekov boundary value conditions. Using the variational method; the technical approach is an adaptation of a three critical points theorem due to Ricceri.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, with a smooth boundary $\partial\Omega$ and $N < p < \infty, \ \ N < q < \infty$. We consider the system

$$\begin{cases} -\Delta_p u = \lambda F_u(x, u, v) & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + |u|^{p-2} u = \mu G_u(x, u, v) & \text{on } \partial\Omega, \\ -\Delta_q v = \lambda F_v(x, u, v) & \text{in } \Omega, \\ |\nabla v|^{q-2} \frac{\partial v}{\partial \nu} + |v|^{q-2} v = \mu G_v(x, u, v) & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\lambda, \mu \geq 0$ are real numbers, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian, $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ two functions are fulfilling appropriate conditions that we give later. F_t and G_t denote the partial derivatives of F and G with respect to t.

The existence of multiple solutions for the problems involving p-Laplacian type elliptic operators in divergence form and related eigenvalue problems

$$\begin{cases} -\text{div}(a(x, \nabla u)) = \lambda F(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

was studied in [8, 9, 11, 12, 15], these results are based on some three critical points theorems of Bonanno [5] and Ricceri [20].

The quasilinear elliptic systems involving a general (p, q)-Laplacain operator has been received considerable attention in recent years. This is partly due to their frequent appearance in applications such as; the reaction-diffusion problems, the non-Newtonien fluids, astronomy, etc. (see for example [2]). Also these problems are very interesting from a purely mathematical point of view as well. Many results have been obtained on this kind of problems such as [3, 7, 17]. The authors in [3] studied the existence of solutions for the following problem

$$\begin{cases} -\triangle_p u = F_u(x, u, v) & \text{in } \Omega, \\ -\triangle_q v = F_v(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

2010 *Mathematics Subject Classification.* 35J65, 35J60, 47J30, 58E05.

Key words and phrases. (p,q)-Laplacian operator, critical point, three weak solutions.
Submitted ;;;; XX. Published ;;;;;XX..

where $p, q > 1$.

In the paper [21], Seyyed and al. proved the existence of three weak solutions of the following problem

$$\begin{cases} -\text{div}(a_1(x, \nabla u)) = \lambda g_1(x, u) + \mu F_u(x, u, v) & \text{in } \Omega, \\ -\text{div}(a_2(x, \nabla v)) = \lambda g_2(x, v) + \mu F_v(x, u, v) & \text{in } \Omega, \\ u = 0, \quad v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $1 < p, q \leq N$, their main tool is an adaptation of a three critical points theorem due to Recceri.

Remark 1.1. If $N < r$ for $r \in \{p, q\}$, by Theorem 2.2 in [13] and Remark [1] in [18], we have $W^{1,r}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Defining $\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|$, we find that there exist a positive constant $C > 0$ such that

$$\|u\|_\infty \leq C\|u\|_r \text{ for all } u \in W^{1,r}(\Omega). \tag{1.2}$$

For our work, we make the following assumptions on the functions F and G.

$(\mathbf{H_0})$ $F(., s, t)$ is measurable in Ω for all $(s, t) \in \mathbb{R} \times \mathbb{R}$ and $F(x, ., .)$ is C^1 in $\mathbb{R} \times \mathbb{R}$ for a.e. $x \in \Omega$.

$(\mathbf{H_1})$ There exist $d(x) \in L^\infty(\Omega)$ and $0 < \alpha < p,\ 0 < \beta < q$, such that

$F(x, s, t) \leq d(x)(1 + |s|^\alpha + |t|^\beta)$ for a.e. $x \in \Omega$ and for all $(s, t) \in \mathbb{R} \times \mathbb{R}$.

$(\mathbf{H_2})$ $F(x, 0, 0) = 0$ for a.e. $x \in \Omega$.

$(\mathbf{H_3})$ $F(x, s_1, t_1) > 0$ for any $x \in \Omega$ and $|s_1|, |t_1|$ large enough; and there exist $M, M' > 0$ such that

$$F(x, s_1, t_1) \leq 0, \quad x \in \Omega, \quad |s_1| \leq M, \quad |t_1| \leq M'.$$

$(\mathbf{H_4})$ There exist $s_2, t_2 \in \mathbb{R}$ with $|s_2|, |t_2| \geq 1$ such that

$$|\Omega| \sup_{(x,|s|,|t|) \in \Omega \times [0, C\alpha_p] \times [0, C\beta_q]} F(x, s, t) \leq \frac{(\frac{1}{p} + \frac{1}{q}) \int_\Omega F(x, s_2, t_2) dx}{|\partial\Omega|(\frac{1}{p}|s_2|^p + \frac{1}{q}|t_2|^q)},$$

where $|\partial\Omega||s_2|^p > 1$, $|\partial\Omega||t_2|^q > 1$ and C is the constant given in Remark 1.1.

$\alpha_p = (1 + \frac{p}{q})^{\frac{1}{p}}$, $\beta_q = (1 + \frac{q}{p})^{\frac{1}{q}}$. We denote by $|\Omega|$, (resp $|\partial\Omega|$) the Lebesgue measure of Ω, (resp $\partial\Omega$).

$(\mathbf{G_0})$ G is a Carathéodory function;

$(\mathbf{G_1})$ $G(x, 0, 0) \in L^1(\partial\Omega)$ for all $x \in \partial\Omega$;

$(\mathbf{G_2})$ $G_u(x, u, v)$ and $G_v(x, u, v)$ are continuous with respect to u and v, for all $x \in \partial\Omega$;

$(\mathbf{G_3})$ there exist $c > 0$ such that $|G_u(x, u, v)| \leq c(1 + |u|^{p-1} + |v|^{\frac{q(p-1)}{p}})$ and $|G_v(x, u, v)| \leq c(1 + |u|^{\frac{p(q-1)}{q}} + |v|^{q-1})$, for a.e. $x \in \partial\Omega$ and for all $(u, v) \in \mathbb{R} \times \mathbb{R}$.

Our main results in this paper is the proof of the following theorem which is based on the Recceri Theorem.

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

Theorem 1.2. Assume $(\mathbf{G_0})-(\mathbf{G_3})$,$(\mathbf{H_0})-(\mathbf{H_2})$ and $(\mathbf{H_3})$ or $(\mathbf{H_4})$ hold. Then there exist an open interval $\Lambda \subseteq [0,\infty)$ and a positive real number ρ with the following property: for each $\lambda \in \Lambda$, there exists $\sigma > 0$ such that for each $\mu \in [0,\sigma]$, problem (1.1) has at least three weak solutions whose norms are less than ρ.

This paper is organized as follows, section 1 contains an introduction and the main results. In section 2, which has a preliminary character, we will give some assumptions and facts that will be needed in the paper, in section 3 we will give the proof of our main result.

2. Preliminaries

Consider the space $\mathrm{W} = \mathrm{W}^{1,\mathrm{p}}(\Omega) \times \mathrm{W}^{1,\mathrm{q}}(\Omega)$ equipped with the norm

$$\|w\| = \|u\|_{1,p} + \|v\|_{1,q}, \text{ for } w = (u,v) \in \mathrm{W},$$

where

$$\|u\|_{1,p} = \Big(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p d\sigma\Big)^{\frac{1}{p}},$$

and

$$\|v\|_{1,q} = \Big(\int_\Omega |\nabla v|^q dx + \int_\Omega |v|^q d\sigma\Big)^{\frac{1}{q}}.$$

We introduce a new norm, which will be used later in this work that

$$\|w\|_{p,q} = \|u\|_p + \|v\|_q,$$

where

$$\|u\|_p = \Big(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} |u|^p d\sigma\Big)^{\frac{1}{p}},$$

and

$$\|v\|_q = \Big(\int_\Omega |\nabla v|^q dx + \int_{\partial\Omega} |v|^q d\sigma\Big)^{\frac{1}{q}}.$$

$\|.\|_r$ is also a norm on $\mathrm{W}^{1,\mathrm{r}}(\Omega)$ which is equivalent to $\|u\|_{1,r}$ for $r \in \{p,q\}$. Then $\|.\|_{p,q}$ is a norm on W which is equivalent to $\|.\|$(see [Theorem 2.1] [10]).

Definition 2.1. We say that $(u,v) \in \mathrm{W}$ is a weak solution of (1.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \lambda \int_\Omega F_u(x,u,v)\varphi dx + \mu \int_{\partial\Omega} G_u(x,u,v)\varphi d\sigma - \int_{\partial\Omega} |u|^{p-2}u\varphi d\sigma,$$
$$\int_\Omega |\nabla v|^{q-2}\nabla v \nabla\psi dx = \lambda \int_\Omega F_v(x,u,v)\psi dx + \mu \int_{\partial\Omega} G_v(x,u,v)\psi d\sigma - \int_{\partial\Omega} |v|^{q-2}v\psi d\sigma,$$

for all $(\varphi,\psi) \in \mathrm{W}$.

3. Proof of main result

To prove our Theorem 1.2 we shall give a variant of Ricceri's three critical points theorem [19]. On the basis of [4], we state an equivalent formulation of the three critical points theorem in [19] as follows.

Theorem 3.1. Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ a C^1 functional with compact Gâteaux derivative. Assume that

(1) $\lim_{||u||\to\infty}(\Phi(u)+\lambda\Psi(u))=\infty$ for all $\lambda>0$, there exist $r\in\mathbb{R}$ and $u_0,u_1\in X$ such that
(2) $\Phi(u_0)<r<\Phi(u_1)$,
(3) $\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)\geq\frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exists a non-empty open set $\wedge\subseteq[0,\infty)$ and a positive real number ρ with the following property: for each $\lambda\in\wedge$ and every C^1 functional $J:X\to\mathbb{R}$ with compact Gâteaux derivative, there exists $\sigma>0$ such that for each $\mu\in[0,\sigma]$, the equation $\Phi'(u)+\lambda\Psi'(u)+\mu J'(u)=0$ has at least three solutions in X whose norms are less than ρ.

In order to apply Ricceri's result we define $\Phi,\Psi,J:\mathrm{W}\to\mathbb{R}$ by:

$$\Phi(w)=\frac{1}{p}\int_\Omega|\nabla u|^p dx+\frac{1}{p}\int_{\partial\Omega}|u|^p d\sigma+\frac{1}{q}\int_\Omega|\nabla v|^q dx+\frac{1}{q}\int_{\partial\Omega}|v|^q d\sigma, \tag{3.1}$$

$$\Psi(w)=-\int_\Omega F(x,u,v)dx, \tag{3.2}$$

$$J(w)=-\int_{\partial\Omega}G(x,u,v)d\sigma, \tag{3.3}$$

where $w=(u,v)\in\mathrm{W}$. It is clear that the weak solution of (1.1) is a solution of

$$\Phi'(w)+\lambda\Psi'(w)+\mu J'(w)=0. \tag{3.4}$$

It follows that we can seek for weak solutions of problem (1.1) by applying Theorem 3.1.
We start by proving some properties of the operator Φ, we first give the following result.

Lemma 3.2. Let Φ be defined as above in (3.1), then Φ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous C^1 functional and $(\Phi')^{-1}:\mathrm{W}^*\to\mathrm{W}$ exists and it is continuous.

Proof. It is clear that the functional Φ is Gâteaux differentiable at every $(u,v)\in\mathrm{W}$ and

$$\begin{aligned}(\Phi'(u,v),(\varphi,\psi))=&\int_\Omega|\nabla u|^{p-2}\nabla u\nabla\varphi dx+\int_{\partial\Omega}|u|^{p-2}u\varphi d\sigma\\&+\int_\Omega|\nabla v|^{q-2}\nabla v\nabla\psi dx+\int_{\partial\Omega}|v|^{q-2}v\psi d\sigma,\end{aligned}$$

for all $(\varphi,\psi)\in\mathrm{W}$.
Φ is sequentially weakly lower semicontinuous by Lemma 3.6 [21].
Moreover Φ' is of (S_+) type. Indeed, let $(w_n)=(u_n,v_n)$ be a sequence of W such that $w_n\rightharpoonup w=(u,v)$ weakly in W as $n\to+\infty$ and $\limsup_{n\to+\infty}(\Phi'(w_n),w_n-w)\leq 0$,

$$\begin{aligned}(\Phi'(w_n),w_n-w)=&\textstyle\int_\Omega|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n-\nabla u)dx+\int_{\partial\Omega}|u_n|^{p-2}u_n(u_n-u)d\sigma\\&+\textstyle\int_\Omega|\nabla v_n|^{q-2}\nabla v_n\nabla(v_n-v)dx+\int_{\partial\Omega}|v_n|^{q-2}v_n(v_n-v)d\sigma.\end{aligned}$$

Using the compact embedding $W^{1,p}(\Omega)\hookrightarrow L^p(\partial\Omega)$ and $W^{1,q}(\Omega)\hookrightarrow L^q(\partial\Omega)$, we obtain $\lim_{n\to+\infty}\int_{\partial\Omega}|u_n|^{p-2}u_n(u_n-u)d\sigma=0$, $\lim_{n\to+\infty}\int_{\partial\Omega}|v_n|^{q-2}v_n(v_n-v)d\sigma=0$.

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

Thus

$$\limsup_{n\to+\infty}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n-\nabla u)dx+\int_\Omega |\nabla v_n|^{q-2}\nabla v_n(\nabla v_n-\nabla v)dx\le 0,$$

by [[16]theorem 4.1], we have $(u_n,v_n)\to(u,v)$ strongly in W as $n\to+\infty$.

Now we show that $(\Phi')^{-1}:\mathrm{W}^*\to\mathrm{W}$ exists and it is continuous. First, we show that (Φ') is uniformly monotone. In fact, for any $\zeta,\eta\in\mathbb{R}^{\mathbb{N}}$, we have the following inequality (see [14]):

$$\begin{aligned}&(|\zeta|^{p-2}\zeta-|\eta|^{p-2}\eta)(\zeta-\eta)\ge \tfrac{1}{2^p}|\zeta-\eta|^p,\quad p\ge 2.\\ &(|\zeta|+|\eta|)^{2-p}(|\zeta|^{p-2}\zeta-|\eta|^{p-2}\eta)(|\zeta|-\eta)\ge(p-1)|\zeta-\eta|^2,\quad 1<p<2.\end{aligned}\tag{3.5}$$

For $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$, we have

$$\begin{aligned}(\Phi'(w_1)-\Phi'(w_2),w_1-w_2)=&\int_\Omega(|\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1-u_2)dx\\ &+\int_{\partial\Omega}(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2)(u_1-u_2)d\sigma\\ &+\int_\Omega(|\nabla v_1|^{p-2}\nabla v_1-|\nabla v_2|^{p-2}\nabla v_2)\nabla(v_1-v_2)dx\\ &+\int_{\partial\Omega}(|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_1-v_2)d\sigma.\end{aligned}$$

Applying (3.5), we deduce that

$$\begin{aligned}(\Phi'(w_1)-\Phi'(w_2),w_1-w_2)\ge&\int_\Omega\left(\frac{1}{2^p}|\nabla u_1-\nabla u_2|^p+\frac{1}{2^q}|\nabla v_1-\nabla v_2|^q\right)dx\\ &+\int_{\partial\Omega}\left(\frac{1}{2^p}|u_1-u_2|^p+\frac{1}{2^q}|v_1-v_2|^q\right)d\sigma\\ &\ge\min\{\frac{1}{2^p},\frac{1}{2^q}\}\left(\|u_1-u_2\|_p^p+\|v_1-v_2\|_q^q\right)\end{aligned}$$

for any $w_1=(u_1,v_1),w_2=(u_2,v_2)\in\mathrm{W}$, i.e.,$\Phi'$ is uniformly monotone.
We can see that for any $w\in\mathrm{W}$, we have that

$$\frac{(\Phi'(w),w)}{\|w\|_{p,q}}\ge\frac{\|u\|_p^p+\|v\|_q^q}{\|w\|_{p,q}},$$

that's meaning Φ' is coercive on W.
The strict monotonicity of Φ' implies its injectivity and the coercivity implies the surjectively, consequently the operator Φ' admits an inverse mapping. Therefore, the conclusion follows by applying Theorem 26.A [22].
It suffices to show the continuity of Φ'^{-1}. Let $(f_n)_n=(f_{1_n},f_{2_n})$ be a sequence of W^* such that $f_n\to f=(f_1,f_2)$ in W as $n\to+\infty$. Let $w_n=(u_n,v_n)$ and $w=(u,v)$ in W such that

$$\Phi'^{-1}(f_n)=w_n\ \ and\ \ \Phi'^{-1}(f)=w.$$

By the coercivity of Φ', the sequence (w_n) is bounded in the reflexive space W. For a subsequence $\hat w=(\hat u,\hat v)$, we have $w_n\rightharpoonup\hat w$ weakly in W as $n\to+\infty$, which implies

$$\lim_{n\to+\infty}(\Phi'(w_n)-\Phi'(w),w_n-\hat w)=\lim_{n\to+\infty}(f_n-f,w_n-\hat w)=0.$$

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

By the property (S_+) and the continuity of Φ' it follows that $w_n \to \hat{w}$ strongly in W and $\Phi'(w_n) \to \Phi'(\hat{w}) = \Phi'(u)$ in W* as $n \to +\infty$, since Φ' is an injection, we conclude $w = \hat{w}$ □

Lemma 3.3. Let $J : \mathrm{W} \to \mathbb{R}$ be defined as above. If $(\mathbf{G_0}) - (\mathbf{G_3})$ hold, then $J \in C^1(\mathrm{W}, \mathbb{R})$. In particular $J' : \mathrm{W} \to \mathrm{W}^*$ is continuous and compact.

Proof. Since $G(x, u, v)$ is C^1 with respect to u, v, then for every $x \in \partial\Omega$ there exist $\alpha(x), \beta(x)$ in $(0, 1)$ such that

$$\begin{aligned}|G(x,u,v) - G(x,0,0)| &\leq |G(x,u,v) - G(x,u,0)| + |G(x,u,0) - G(x,0,0)|,\\ &\leq |G_u(x,\alpha(x)u,0||u| + |G_v(x,u,\beta(x)v||v|,\\ &\leq c(1+|u|^{p-1})|u| + c(1+|u|^{\frac{p(q-1)}{q}} + |v|^{q-1})|v|\\ &\leq K(p,q,c)(1+|u|^p + |v|^q).\end{aligned}$$

Let $(u, v) \in \mathrm{W}$ for every $(\varphi, \psi) \in \mathrm{W}$ and $0 < |t| < 1$, by applying the Mean Value Theorem we obtain

$$\begin{aligned}&(J'(u,v),(\varphi,\psi)) = \lim_{t\to 0} \frac{J(u+t\varphi, v+t\psi) - J(u,v)}{t}\\ &= \lim_{t\to 0} -\frac{1}{t}\Big(\int_{\partial\Omega} G(x,u+t\varphi,v+t\psi) - G(x,u,v)d\sigma\Big)\\ &= -\lim_{t\to 0}\Big(\int_{\partial\Omega} G_u(x,u+t\alpha\varphi,v+t\beta\psi)\varphi d\sigma + \int_{\partial\Omega} G_v(x,u+t\alpha\varphi,v+t\beta\psi)\psi d\sigma\Big),\end{aligned}$$

with $0 < \alpha = \alpha(x), \beta = \beta(x) < 1$, for every $x \in \partial\Omega$, G_u is continuous and $\lim_{t\to 0} G_u(x, u+t\alpha\varphi, v+t\beta\psi) = G_u(x, u, v)$. On the other hand for $|t| < 1$ we have

$$\begin{aligned}|G_u(x,u+t\alpha\varphi,v+t\beta\psi)\varphi| &\leq c(1+|u+t\alpha\varphi|^{p-1} + |v+t\beta\psi|^{\frac{q(p-1)}{p}})|\varphi|,\\ &\leq c(1+(|u|+|\varphi|)^{p-1} + \ (|v|+|\psi|)^{\frac{q(p-1)}{p}})|\varphi|.\end{aligned}$$

Notice that the right hand side of the above inequality is independent of t and integrable on $\partial\Omega$, then the dominated convergence Theorem implies

$$\lim_{t\to 0}\int_{\partial\Omega} G_u(x,u+t\alpha\varphi,v+t\beta\psi)\varphi d\sigma = \int_{\partial\Omega} G_u(x,u,v)\varphi d\sigma.$$

Similarly we have

$$\lim_{t\to 0}\int_{\partial\Omega} G_v(x,u+t\alpha\varphi,v+t\beta\psi)\psi d\sigma = \int_{\partial\Omega} G_v(x,u,v)\psi d\sigma.$$

Therefore

$$\begin{aligned}(J'(u,v),(\varphi,\psi)) &= \lim_{t\to 0} \tfrac{J(u+t\varphi,v+t\psi)-J(u,v)}{t}\\ &= -\textstyle\int_{\partial\Omega} G_u(x,u,v)\varphi d\sigma - \int_{\partial\Omega} G_v(x,u,v)\psi d\sigma,\end{aligned}$$

and J is Gâteaux differentiable at any $(u, v) \in W$ and for every $(\varphi, \psi) \in W$.
It's clear that $(J'(u, v), (\varphi, \psi))$ is a linear operator. Moreover, the Nemytskii operator $N_u(u, v) :\to G_u(x, u, v)$ (resp. $N_v(u, v) :\to G_v(x, u, v)$) is continuous bounded operator from $L^p(\partial\Omega)$ into $L^{p'}(\partial\Omega)$ (resp.$L^q(\partial\Omega)$ into $L^{q'}(\partial\Omega)$), where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$.

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

Now we prove that $J' : \mathrm{W} \longrightarrow \mathrm{W}^*$ is continuous, suppose that $(u_n, v_n) \to (u, v)$ in W by the Hölder inequality and the compact embedding $W \hookrightarrow L^p(\partial\Omega) \times L^q(\partial\Omega)$ then for every $(\varphi, \psi) \in \mathrm{W}$ we have

$$\begin{aligned}
&|(J'(u_n, v_n) - J'(u, v), (\varphi, \psi))| \\
&\leq \int_{\partial\Omega} |(G_u(x, u_n, v_n) - G_u(x, u, v))\varphi| + |(G_v(x, u_n, v_n) - G_v(x, u, v))\psi| d\sigma, \\
&\leq \|G_u(x, u_n, v_n) - G_u(x, u, v)\|_{L^{p'}(\partial\Omega)} \|\varphi\|_p \\
&+ \|G_v(x, u_n, v_n) - G_v(x, u, v)\|_{L^{q'}(\partial\Omega)} \|\psi\|_q, \\
&\leq \max\{\|G_u(x, u_n, v_n) - G_u(x, u, v)\|_{L^{p'}(\partial\Omega)}, \|G_v(x, u_n, v_n) - G_v(x, u, v)\|_{L^{q'}(\partial\Omega)}\} \\
&\times \|(\varphi, \psi)\|_{p,q}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|(J'(u_n, v_n) - J'(u, v)\|_{W^*} \\
&\leq \max\{\|G_u(x, u_n, v_n) - G_u(x, u, v)\|_{L^{p'}(\partial\Omega)}, \|G_v(x, u_n, v_n) - G_v(x, u, v)\|_{L^{q'}(\partial\Omega)}\}.
\end{aligned}$$

Therefore the operator $T : L^{p'}(\partial\Omega) \times L^{q'}(\partial\Omega) \longrightarrow \mathrm{W}^*$ defined by

$$T(G_u(x, u, v), G_v(x, u, v)) = J'(u, v)$$

is continuous, then the composite operator $J' = ToN_G oI : (u, v) \to J'(u, v)$ from W into W^* is continuous, where $N_G : W \to L^{p'}(\partial\Omega) \times L^{q'}(\partial\Omega)$ is the composite operator Nemytskii defined by $N_G(u, v) = (N_u(u, v), N_v(u, v))$. This implies that $J \in C^1(\mathrm{W}, \mathbb{R})$, and

$$(J'(w), (\varphi, \psi)) = -\int_{\partial\Omega} G_u(x, u, v)\varphi d\sigma - \int_{\partial\Omega} G_v(x, u, v)\psi d\sigma$$

Therefore $J' : \mathrm{W} \longrightarrow \mathrm{W}^*$ is compact. □

Lemma 3.4. Let Ψ defined as above in (3.2), then Ψ is C^1, in particular Ψ' is continuous and compact.

Proof. It can be show easily that Ψ is a C^1 functional ([1] Theorem 2.9), and

$$(\Psi'(u, v), (\varphi, \psi)) = -\int_{\Omega} F_u(x, u, v)\varphi dx - \int_{\Omega} F_v(x, u, v)\psi dx.$$

The continuity of Ψ' can be proved like the continuity of J'.
Using the compactly of the embedding $W^{1,r}(\Omega) \hookrightarrow L^r(\Omega)$ where $r \in \{p, q\}$, we deduce that Ψ' is compact. □

Proof of Theorem 1.2. From Lemma 3.2, 3.3, 3.4, the functional Φ is continuously Gâteaux differentiable and sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of W, whose Gâteaux derivative admits a continuous inverse on W^*; $\Psi : \mathrm{W} \to \mathbb{R}$ a C^1 functional with compact Gâteaux derivative, and J is well defined and continuously Gâteaux differentiable on W, with compact derivative.

Now we show that the hypotheses (1), (2) and (3) of Theorem 3.1 are fulfilled. For $w \in \mathrm{W}$, we have

$$\begin{aligned}\Phi(w) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{p}\int_{\partial\Omega} |u|^p d\sigma + \frac{1}{q}\int_\Omega |\nabla v|^q dx + \frac{1}{q}\int_{\partial\Omega} |v|^q d\sigma,\\ &= \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q\\ &\geq \min\{\frac{1}{p},\frac{1}{q}\}(\|u\|_p^p + \|v\|_q^q)\\ &\geq c_1(\|u\|_p^p + \|v\|_q^q),\end{aligned}$$

where $c_1 = \min\{\frac{1}{p},\frac{1}{q}\}$.
From $(\mathbf{H_1})$ we have $\Psi(w) \geq -\int_\Omega d(x)(1+|u|^\alpha+|v|^\beta)dx$, thus

$$\Psi(w) \geq -\|d(x)\|_{L^\infty(\Omega)}(|\Omega| + \|u\|^\alpha_{L^\alpha(\Omega)} + \|v\|^\beta_{L^\beta(\Omega)}),$$

so

$$\Psi(w) \geq -c_2(1 + \|u\|^\alpha_{L^\alpha(\Omega)} + \|v\|^\beta_{L^\beta(\Omega)}),$$

consequently we obtain

$$\Psi(w) \geq -c_2'(1 + \|u\|_p^\alpha + \|v\|_q^\beta),$$

for any $w=(u,v) \in$ W, where c_2 and c_2' are positives constant.
Combining two inequalities above, we have

$$\Phi(w) + \lambda\Psi(w) \geq c_1(\|u\|_p^p + \|v\|_q^q) - \lambda c_2'(1 + \|u\|_p^\alpha + \|v\|_q^\beta)$$

Since $0<\alpha<p,\ \ 0<\beta<q$, it follows that

$$\lim_{\|w\|\to+\infty}(\Phi(w) + \lambda\Psi(w)) = +\infty.$$

Then condition (1) of Theorem 3.1 is satisfied.
Next, we will prove the condition (2) and (3), for that we consider two cases:
case (I): The assumption $(\mathbf{H_3})$ holds, i.e., there exist $|s_1|>1,\ \ |t_1|>1$ such that $F(x,s_1,t_1)>0$ for any $x\in\Omega$, and there exist $M>0,\ \ M'>0$ such that $F(x,s_1,t_1)\leq 0$ for any $x\in\Omega$ and $|s_1|\leq M,\ \ |t_1|\leq M'$, set $a=\min\{C,M\},\ \ b=\min\{C,M'\}$, where C is defined in remark 1.1, then we have

$$\int_\Omega \sup_{(|s|,|t|)\in[0,a]\times[0,b]} F(x,s,t)dx \leq 0 < \int_\Omega F(x,s_1,t_1)dx. \tag{3.6}$$

Now we set $w_0=(0,0)$ and $w_1=(s_1,t_1)$ and $r=\min\{\frac{1}{p}(\frac{a}{C})^p,\frac{1}{q}(\frac{b}{C})^q\}>0$, it is clear that

$$\Phi(w_0) = 0 = \Psi(w_0) \text{ and } \Phi(w_0) < r < \Phi(w_1),$$

so (2) of Theorem 3.1 is satisfied.
When $\Phi(w)\leq r$ it's means that $\frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q \leq r$, we deduce that $C\|u\|_p\leq a$ and $C\|v\|_q\leq b$, from (1.2) we obtain $\|u\|_\infty\leq a$ and $\|v\|_\infty\leq b$. On the other hand, we have

$$\frac{(\Phi(w_1)-r)\Psi(w_0) + (r-\Phi(w_0))\Psi(w_1)}{\Phi(w_1)-\Phi(w_0)} = r\frac{\Psi(w_1)}{\Phi(w_1)} = -r\frac{\int_\Omega F(x,s_1,t_1)dx}{|\partial\Omega|(\frac{1}{p}|s_1|^p + \frac{1}{q}|t_1|^q)} < 0. \tag{3.7}$$

From (3.2) and (3.6), we deduce

$$-\inf_{w\in\Phi^{-1}((-\infty,r])}\Psi(w) = \sup_{w\in\Phi^{-1}((-\infty,r])} -\Psi(w) \leq \int_\Omega \sup_{(|u|,|v|)\in[0,a]\times[0,b]} F(x,u,v)dx \leq 0. \tag{3.8}$$

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

From (3.7) and (3.8) we obtain

$$\inf_{w\in\Phi^{-1}((-\infty,r])}\Psi(w)\geq\frac{(\Phi(w_1)-r)\Psi(w_0)+(r-\Phi(w_0))\Psi(w_1)}{\Phi(w_1)-\Phi(w_0)},$$

thus (3) of Theorem 3.1 is hold.

case (II) $(\mathbf{H_4})$ holds, then there exist $s_2,t_2\in\mathbb{R}$ with $|s_2|,|t_2|\geq 1$ such that

$$|\Omega|\sup_{(x,|s|,|t|)\in\Omega\times[0,C\alpha_p]\times[0,C\beta_q]}F(x,s,t)\leq\frac{(\frac{1}{p}+\frac{1}{q})\int_\Omega F(x,s_2,t_2)dx}{|\partial\Omega|(\frac{1}{p}|s_2|^p+\frac{1}{q}|t_2|^q)},\tag{3.9}$$

where $|\partial\Omega||s_2|^p>1$, $|\partial\Omega||t_2|^q>1$, C is the constant given in Remark 1.1, $\alpha_p=(1+\frac{p}{q})^{\frac{1}{p}}$ and $\beta_q=(1+\frac{q}{p})^{\frac{1}{q}}$.

We set $w_2=(s_2,t_2)$ and denote $r=\frac{1}{p}+\frac{1}{q}>0$, then it is easy to see that $\Phi(w_0)=0<\frac{1}{p}+\frac{1}{q}$ and $\Phi(w_2)=|\partial\Omega|(\frac{1}{p}|s_2|^p+\frac{1}{q}|t_2|^q)>\frac{1}{p}+\frac{1}{q}$ we see that

$$\Phi(w_0)<r<\Phi(w_2),$$

so the assumption (2) is satisfied. On the other hand we have

$$\frac{(\Phi(w_2)-r)\Psi(w_0)+(r-\Phi(w_0))\Psi(w_2)}{\Phi(w_2)-\Phi(w_0)}=r\frac{\Psi(w_2)}{\Phi(w_2)}=-r\frac{\int_\Omega F(x,s_2,t_2)dx}{|\partial\Omega|(\frac{1}{p}|s_2|^p+\frac{1}{q}|t_2|^q)}.\tag{3.10}$$

Similarly when $\Phi(w)\leq r$ where $r=\frac{1}{p}+\frac{1}{q}$, we have $\|u\|_p\leq\alpha_p$, and $\|v\|_q\leq\beta_q$. By (1.2) we obtain $\|u\|_\infty\leq C\alpha_p$, and $\|v\|_\infty\leq C\beta_q$. From (3.2) we have

$$\begin{aligned}-\inf_{w\in\Phi^{-1}((-\infty,r])}\Psi(w)&=\sup_{w\in\Phi^{-1}((-\infty,r])}-\Psi(w)\\&\leq\int_\Omega\sup_{(|u|,|v|)\in[0,C\alpha_p]\times[0,C\beta_q]}F(x,u,v)dx\\&\leq|\Omega|\sup_{(x,|u|,|v|)\in\Omega\times[0,C\alpha_p]\times[0,C\beta_q]}F(x,u,v).\end{aligned}\tag{3.11}$$

From (3.9), (3.10) and (3.11), we can see (3) of Theorem 3.1 is hold.

Then all conditions of Theorem 3.1 are fulfilled. We conclude that there exist a non-empty open set $\wedge\subseteq[0,\infty)$ and a positive real number ρ with the following property: for each $\lambda\in\wedge$ and there exists $\sigma>0$ such that for each $\mu\in[0,\sigma]$, the problem (1.1) has at least three solutions in W whose norms are less than ρ. □

At last, we give two examples

Example 1. Let $\Omega=B(0,1)$ be the unit ball of $\mathbb{R}^N$ with $N\geq 2$, set $p=q=N+1$, $G(x,u,v)=x^2(u^2+v^2)$, $F(x,u,v)=(1+2x^2)(u^4v^2+v^4u^2-2u^2v^2)$ $x\in\Omega$, $u,v\in\mathbb{R}$, in this case the problem (1.1) becomes:

$$\begin{cases}-\Delta_p u=\lambda(1+2x^2)(4u^3v^2+2v^4u-4uv^2) & \text{in }\Omega\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}+|u|^{p-2}u=\mu 2x^2u & \text{on }\partial\Omega\\ -\Delta_q v=\lambda(1+2x^2)(2u^4v+4v^3u^2-4u^2v) & \text{in }\Omega\\ |\nabla v|^{q-2}\frac{\partial v}{\partial\nu}+|v|^{q-2}v=\mu 2x^2v & \text{on }\partial\Omega.\end{cases}\tag{3.12}$$

Obviously $G(x,u,v),F(x,u,v)$ satisfy $(\mathbf{G_0})-(\mathbf{G_3})$ and $(\mathbf{H_1})-(\mathbf{H_2})$ respectively. We can see that

$$F(x,u,v)>0,\ \text{when } |u|>\sqrt{2}\ \ or\ \ |v|>\sqrt{2},$$

and

$$F(x,u,v)<0,\ \ when\ \ |u|<1 \text{ and } |v|<1.$$

It's mean that $(\mathbf{H_3})$ holds. By Theorem 1.2 there exist an open interval $\wedge \in\subseteq [0,\infty)$ and a positive constant ρ such that for any $\lambda \in \wedge$ there exist $\sigma > 0$ and for $\mu \in [0,\sigma]$, the problem (3.12) has at least three weak solutions whose norms are less than ρ.

Example 2. Set $\Omega, p, q, G(x,u,v)$ are the same as in example 1 and $F(x,u,v) = -e^x(e^u + uv - 1)$ In this case we have the problem:

$$\begin{cases} -\Delta_p u = -\lambda e^x(e^u + v) & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + |u|^{p-2}u = \mu 2x^2 u & \text{on } \partial\Omega \\ -\Delta_q v = -\lambda e^x u & \text{in } \Omega \\ |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} + |v|^{q-2}v = \mu 2x^2 v & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

For $|s_2|, |t_2| \geq 1$, we can see that

$$-\frac{1}{2}e(e^{C\alpha_p} + C^2\alpha_p\beta_q - 1) \leq \frac{(e^{s_2} + s_2t_2 - 1)(1-e)}{s_2^{N+1} + t_2^{N+1}},$$

where C is given in Remark 1.1 and $\alpha_p = \beta_q = 2^{\frac{1}{N+1}}$ are positive constants. Then $F(x,u,v)$ satisfies $(\mathbf{H_0}) - (\mathbf{H_2})$ and $(\mathbf{H_4})$, then by Theorem 1.2 there exist an open interval $\wedge \in\subseteq [0,\infty)$ and a positive constant ρ such that for any $\lambda \in \wedge$ there exist $\sigma > 0$ and for $\mu \in [0,\sigma]$, the problem (3.13) has at least three weak solutions whose norms are less than ρ.

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International Journal of Science and Research (IJSR)
ISSN: 2319-7064
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