Obtaining Fractional Fringe Shift in LIGO Type Interferometers using Newtonian Laws of Vector Addition

Siddhant Pinjarkar

Knowledge Centre, Priyadarshani College of Engineering, Nagpur – 440022, India

Abstract: ***The LIGO interferometers which are inspired by a setup of Michelson interferometer which consists of a beam splitter, light source, frequency resonators and mirrors as their basic working tools.LIGO-type interferometerswhich are used to detect gravitational waves are installed in LIGO-Hanford and LIGO-Livingston.Light travelling in the arms of the interferometers should experience a change in time dilation due to uniform rotation of the earth as inertial gravitational forces come into picture. By using Newtonian method of vector addition, we find the path of light with respect to rotation of earth whilst neglecting third-degree order terms of angular velocity of the earth. The change in time of arrival and fractional fringe shift induced in LIGO-type interferometer due to the uniform rotation of the earth is obtained in this paper. A direct relationship between the fractional fringe shift andthe angular velocity of the earth is found, and other parameters dependent are arm's length, latitude of object placed, wavelength of light source and speed of light.***

Keywords: Michelson interferometer, Newtonian vector addition, fractional fringe shift

1. Introduction

To understand the absolute motion and existence of hypothetical aether the Michelson-Morley experiment was performed. Similar to the Michelson interferometer setup, LIGO uses kilometre-scaled Michelson interferometers to detect gravitational waves and their source [1][2] [9]. There are two such setups operating at Livingston and Hanford, USA. The setup uses two orthogonal coherent beams of light that travels between two mirrors and the interferometer compares the time taken to travel between two mirrors [2] [6] [9]. Light leaving the source splits perpendicularly at beam-splitter and travels towards the end mirrors. Thislight then reflects back to the beam-splitter to form an interference pattern if there is a change in path of lights [1] [6].We would obtain interference fringes at the detector if the path of light is changed. As the apparatus moves about in space, the experiment should obtain a change in dark fringes [3].

Each LIGO interferometer consists of 4 km long arms andthe laser type usedisNd: YAG ofa wavelength of 1064 nm [4] [6]. For amplification of the signal to the LIGO interferometers use Fabry-Perot cavity to increase the sensitivity and interaction time between mirrors [11]. The angle of entry at the Fabry-Perot cavity at zero degrees.Figure (1) below represents the components and illustrates the working of the Michelson type interferometer used in LIGO.

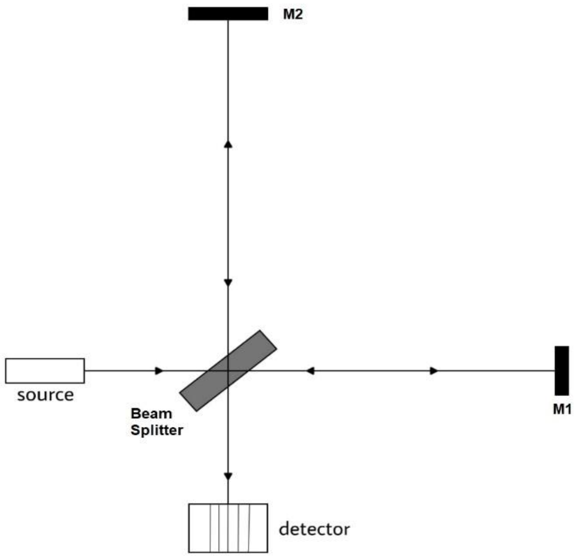

Figure 1: Schematic diagram of LIGO interferometers where M_1 and M_2 are end mirrors.(Source: International journal of Science and Research; Article:ART20201134)

To sense the gravitational waves, LIGO uses kilometre-scaled Michelson type interferometers ofthe sensitivity of 10^{-23} in metric strain. The beam splitter of the detector measures the relative distance between the two orthogonally placed mirrors for an incident gravitational wave [6]. Where the path length or change in the lengths between the mirrors is given by [6]:

$$\Delta L = hL/2 \qquad (a)$$

Here h denotes the amplitude of the gravitational wave incident on the setup and L being the arm length. This induced change in path can be used to measure the fractional fringe shift in the setup.

One of the sources that can be responsible for obtaining fringe patterns in LIGO like interferometers is the uniform rotation of the earth. The light in the interferometer propagatingperpendicular in the arms should experience difference in time dilation as inertial gravitational forces come into picture [5]. As a result, the lightreturns to the beam splitter at different times, causing change in path length experienced by the light rays in both arms.This paperattempts to evaluate the difference in time taken for the light beams to reach the beam splitter andderive the effect of rotation on fringe shift for kilometre scaled Michelson type interferometers such as LIGO using Newtonian method of vector addition. Next section explains, in brief, the method used to calculate the fractional fringe shift for kilometre scaled Michelson type interferometer.

2. Evaluating the effect of the earth's rotation on the interferometer using Newtonian vector Addition

2.1 Equation of path of light

Let us consider the centre of earth as the non-rotating frame of reference, where the origin is at the centre of the earth. Let another frame of reference be at a latitude λ whose origin is at that latitude and is rotating, this is somewhere on the earth's surface. Let x', y' and z' be the coordinate axes of the non-rotating frame and x,y,z are the coordinates axes of the rotating frame, where x is pointed northward, y is pointed westward and z is perpendicular to earth's surface pointing vertically upwards. Therefore, the acceleration *a'* of body moving relative to non-rotating frame is related to its acceleration *a*, body moving also relative to rotating frame, as:

$$\vec{a} = \vec{a'} - \vec{A} - \vec{\dot{\Omega}} \times \vec{r} - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) - 2\vec{\Omega} \times \vec{v} \quad (1)$$

Which can also be written as:

$$\vec{a} = \vec{a'} - \vec{A} - \vec{\dot{\Omega}} \times \vec{r} + \Omega^2 \vec{r} - (\vec{\Omega} \cdot \vec{r})\vec{\Omega} - 2\vec{\Omega} \times \vec{v} \quad (2)$$

Where, $\vec{A}$ is the acceleration of the body relative to the non-rotating frame and is given by:

$$\vec{A} = (-\Omega^2 R_E \cos\lambda \sin\lambda\, ,0, \Omega^2 R_E \cos^2\lambda) \quad (3)$$

Over head dot denotes time derivative, $R_E = 6.371 \times 10^6\ m$ is the radius of the earth, $\Omega = 7.27\ \times 10^{-5}\ rad\ per\ sec$ is angular velocity of the earth.

The body under consideration here is light which has a uniform velocity of *c*, so the acceleration $\vec{a'}$ relative to the non-rotating frame of reference tends to zero.

Therefore, from equation (2), we obtain,

$$\ddot{x} = +\Omega^2 R_E \sin\lambda \cos\lambda + \dot{\Omega} y \sin\lambda + \Omega^2 x \sin^2\lambda - \Omega^2 z \sin\lambda \cos\lambda + 2\Omega \sin\lambda\, \dot{y} \quad (4)$$

$$\ddot{y} = +\dot{\Omega}(z\cos\lambda - x\sin\lambda) + \Omega^2 y - 2\Omega \sin\lambda\, \dot{x} + 2\Omega\cos\lambda\, \dot{z} \quad (5)$$

$$\ddot{z} = +\Omega^2 R_E \cos^2\lambda - \dot{\Omega} y \cos\lambda + \Omega^2 z \cos^2\lambda - \Omega^2 x \sin\lambda\cos\lambda - 2\Omega\cos\lambda\, \dot{y} \quad (6)$$

The rotation of the earth is constant over period of time therefore $\dot{\Omega}$ can be neglected.Also from here on we can neglect Ω^3 order or greater terms as they are too less to be taken into account. Differentiating $\ddot{y}$ with respect to time gives us equation that consists $\ddot{x}$ and $\ddot{z}$. Thus, substituting equation (4) and (6) into $\dot{\ddot{v_y}}$ we get the equation of light as observed by topo-centric (rotating) observer in *y* direction. We also obtain the velocity of light in *y* direction relative to rotating observer. Substituting the value of *y* and v_y into equation (4) and (6) and setting the arbitrary constants to zero. We obtain a second order differential equation for *x* and *z*. Solving the equations we obtain final equations of path of light with respect to the rotating observer. The equations are given by:

$$x = x_0 + \dot{x_0} t + \dot{y_0} t^2 \Omega \sin\lambda + \left(x_0 + R_E \sin\lambda\cos\lambda + \frac{\dot{x_0}t}{3}\right)\frac{\Omega^2 t^2}{2} \quad (7)$$

$$y = \dot{y_0} t - \frac{\dot{y_0}}{2}\Omega^2 t^3 + y_0 \quad (8)$$

$$z = z_0 + \dot{z_0} t - \dot{y_0} t^2 \Omega \cos\lambda + \left(z_0 + R_E \cos^2\lambda + \frac{\dot{z_0}t}{3}\right)\frac{\Omega^2 t^2}{2} \quad (9)$$

The equation for the velocity of light for the rotating observer is:

$$\dot{x} = \dot{x_0} + 2\dot{y_0} t\Omega \sin\lambda + \left(x_0 + R_E \sin\lambda\cos\lambda + \frac{\dot{x_0}t}{2}\right)\Omega^2 t \quad (10)$$

$$\dot{y} = \dot{y_0}\left[1 - \frac{3\Omega^2 t^2}{2}\right] \quad (11)$$

$$\dot{z} = \dot{z_0} - 2\dot{y_0} t\Omega\cos\lambda + \left(z_0 + R_E\cos^2\lambda + \frac{\dot{z_0}t}{2}\right)\Omega^2 t \quad (12)$$

Using the equations above we can find the distance travelled by light in time t, which is given by:

$$d(t) \equiv [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{\frac{1}{2}}$$
$$= v_0 t + \frac{1}{2}\alpha t^2 + \frac{1}{2}\beta t^2 \quad (13.0)$$

Where,

$$\alpha = [\dot{x_0}\sin\lambda - \dot{z_0}\cos\lambda]\frac{\dot{y_0}}{v_0}(2\Omega)$$

Where v_0 is the square root of sum of squares of initial velocity of light components.

And

$$\beta = 2\dot{x_0}\left(x_0 + R_E\cos\lambda\sin\lambda + \frac{\dot{x_0}t}{3}\right)\frac{\Omega^2}{2}$$

Here, v_0 is the effective velocity of the light. Also, it evident from the geometry that the velocity of light in z-direction will always be zero. Hence, $\dot{z_0} = 0$. Therefore, the velocity component in the z direction vanishes from the values of α and β.

Furthermore, we can also obtain the expression for velocity by taking the square root of the sum of velocity components of light. Then by using $T \equiv d(t)/v(t)$, we can obtain the expression for the time of travel of light when covering distance $d(t)$. Which is given by:

$$T \equiv t - \frac{1}{2}\frac{\alpha}{v_0}t^2 - \frac{1}{2}\frac{\beta}{v_0}t^2 \quad (13.1)$$

For our consideration of light, we can ignore the β term from the distance and total time equations as it consists of Ω^2 terms which then multiplies to initial velocity component of x axes and goes to order of cube. Also, after substituting the values of the constants it leads very low value of the β term.

2.2 Bent path of light

The two arms of the LIGO interferometers act as small tangents to the surface of the earth, as the surface area of the earth is too large as compared to the arms of the interferometer. The equation of path of light described above includes terms that are dependent on the angular velocity of the earth which is assumed to be uniform. This means when we relate the normal of the mirrors to the velocity of light in different components, we ought to obtain the deflection in the path of light due to rotation of the earth.

Before we dwell into the mathematics of obtaining the bent path of light it is important to understand the construction of the interferometer and the changes in the setup due to uniform rotation of the earth.Point ‘A’ (in figure 2)is the point where the light splits in the beam splitter, the light rays travel orthogonally to each other towards M_1 and M_2. Both mirrors are aligned perpendicular to the propagation of light. For LIGO type detectors, the arm lengths are equal as discussed above.

Now, (Figure 2) the light travels from point A to mirrors M_1 and M_2, due uniform rotation of the earth the by the time the light reaches the mirrors have changed their orientation by some angle δθ. The next step is to find the expression for the δθ for both the mirrors. This can be obtained by using dot product rule for the normal of both mirrors and the component velocity of light, which is given by:

For arm 1:$\overrightarrow{N_1} = (0,-1,0)$ and $\dot{x_0} = 0, \dot{y_0} = -c, \dot{z_0} = 0$

For arm 2:$\overrightarrow{N_2} = (-1,0,0)$ and $\dot{x_0} = c, \dot{y_0} = 0, \dot{z_0} = 0$

Also,

$$cos(\vec{N}, \vec{v}) = \frac{\vec{N} \bullet \vec{v}}{|\vec{N}||\vec{v}|} \qquad (14)$$

Where (from figure 2) for arm 1 we find that the normal for mirror 1 has moved by $\delta\theta_1$ and for arm 2 the normal has also moved by $\delta\theta_2$. Solving equation (14) for mirrors 1 and 2 we get:

$$tan\,\delta\Theta_1 = \pm\sqrt{3}\Omega t \qquad (15)$$

$$tan\,\delta\Theta_2 = \pm\frac{(z_0+R_E\,cos^2\lambda)\Omega^2 t}{c+\left(x_0+R_E\,sin\,\lambda\,cos\,\lambda+\frac{\dot{x_0}t}{2}\right)\Omega^2 t} \qquad (16)$$

Where for mirror 2 the value of z_0 is zero as the value of the velocity of light in the z component is zero. We can also substitute $R_E = v_E/\Omega$, where v_E is the velocity of earth rotation in m/s. v_E is comparable to the velocity of light by:

$$v_E = \zeta c$$

Where $\zeta = 1.54 \times 10^{-6}$

By, obtaining the values of deflection from equation (15) and (16) it is evident that there is a change in orientation of mirrors with different values. We can assume $tan\,\delta\Theta_1 \approx \delta\Theta_1$ *and* $tan\,\delta\Theta_2 \approx \delta\Theta_2$as the value of deflection angle is too small. Also, we take positive value of the δθ for both mirrors as the rotation of the earth is anticlockwise.

2.3 Change in the path of light

Using the laws of reflection,it can be seen that the angle of incident and angle of reflection for both the mirrors is the same and the total angle made is ‘2δθ’. Also, using the laws of reflection the light rays should lie on the same plane. As it is observed form equation (15) and (16), the light rays are reflected with reference to the normal of the mirror, this means that the light rays do not trace the same path back to the beam splitter. Due to this shift in the path of light we can obtain the value of change in path length for light rays.

For arm 1 and arm 2 the values of initial velocity components for the changed path of light is:

For arm 1: $\dot{x_0} = 2c\delta\Theta_1, \dot{y_0} = c, \dot{z_0} = 0$

For arm 2: $\dot{x_0} = -c, \dot{y_0} = -2c\delta\Theta_2, \dot{z_0} = 0$

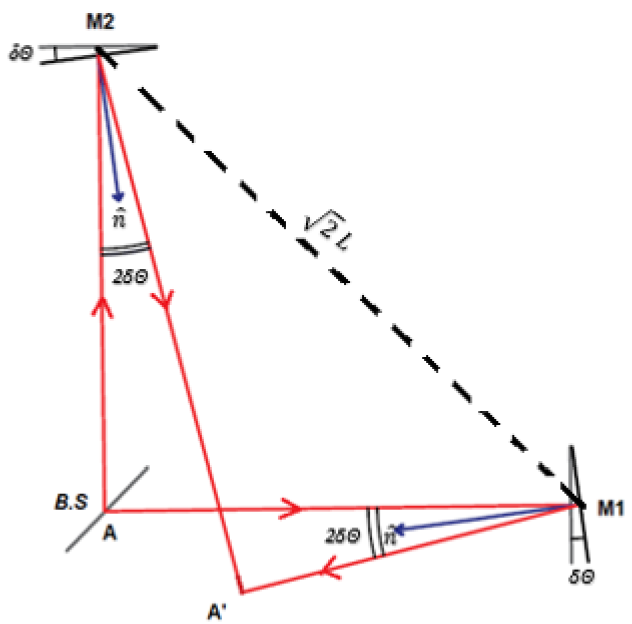

Figure 2: A’ gives the point at which the light meets after reflection.The solid red line gives the path of light followed and blue arrows define the direction of the normal of the mirrors. Black solids give position of mirrors that have changed orientation due to uniform rotation of the earth and grey solids give the initial position of the mirrors. Point A is the position of the beam splitter where the light rays split and the δΘ is different for both the mirrors represented by adding suffixes to each deflection with respect to the mirrors.*(Source: International journal of Science and Research; Article:ART20201134)*

Substituting the above initial velocity values into equation (13.0), we can obtain the value of the distance travelled by light after reflecting from the mirror. Here, the total time taken by the light to reach A’ with reference to the rotating observer (from figure 2) should be equal to $t = L/c$, here we cannot neglect the effect angular velocity of the earth as it plays a major role in finding the difference of effective time of travel. Therefore, the time of travel for arms (1) and (2) can be given by:

$$T_1 = \frac{L}{c} + \frac{L^2}{c^2}2\Omega\sin\lambda\;\delta\Theta_1 \qquad (17)$$

$$T_2 = \frac{L}{c} - \frac{L^2}{c^2}2\Omega\sin\lambda\;\delta\Theta_2 \qquad (18)$$

Here, the terms T_1 and T_2 are the time of travel for light to A’ from M_1 and M_2 respectively. These values of time when substituted into equation (13.0) we obtain the value of A’M_1 and A’M_2 (from figure 2). These values, given by d_1 *and* d_2 when subtracted give the change in path length, which in turn leads to change in time of arrival, which is given by:

$$\Delta T = T_1 - T_2 = 2\Omega\frac{L^2}{c^2}\sin\lambda\,(\delta\Theta_2 + \delta\Theta_1) \qquad (19)$$

Finally, submitting the values of deflection angles, the final expression forthe time change is divided by ‘ƛ/c’, to the fractional fringe shift obtained for the interferometer due to uniform rotation of the earth. Where ‘ƛ’ is the wavelength of the light source. The expression is given by:

$$\Delta N = \frac{2\Omega L^2\sin\lambda}{c ƛ}(\zeta\cos^2\lambda + \sqrt{3})$$

Here, $\zeta \cos^2 \lambda + \sqrt{3}$ term act as constants for the fractional fringe shift value. Where the whole term is close to 2 for any value of λ. Ultimately the fractional fringe shift value is dependent on sin part of the latitude, angular velocity of the earth, arm length, wavelength of light source and speed of light.

3. Conclusion

It is proved using the Newtonian method of vector addition that fractional fringe shift for LIGO like interferometers can be obtained. The dependency of fractional fringe shift onto the speed of light, angular velocity of the earth, arm length and latitude of the object is been proved by above-mentioned expression. It is also evident that if we do not neglect the second order angular velocity term in light path equation, we can observe the effect of angular velocity onto the interferometer. Furthermore, if the latitude is at zero degrees for any Michelson type interferometer the fractional fringe shift value is vanishing, hence placement of any LIGO interferometer for latitude values of degree would mean no observance of fringe shit due to uniform rotation. Whereas, the case opposite when the interferometer is placed at latitude of 90 degrees. LIGO type interferometers also using Fabry-Perot resonators that help enhance the sensitivity of the interferometers by bouncing back the light back and forth in the arms which in turn increases the effective length travelled by the light. Here we have not considered the orientation of arms which could also become multiples into fringe shift values. similar expressionsare obtained for Michelson-Gale experiment [8] and by Silberstein [10] for a different setup of interferometry.

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Author Profile

Siddhant Pinjarkar received his bachelor's degree in Mechanical engineering from Pune University at Sinhgad college of engineering. Further, he has a post graduate degree in Astrophysics from the University of Glasgow. Currently he is working on publications with Astrophysics field and wants pursue PhD in the future to get into research.