Extension of Mercer Theorem for Reproducing Kernel Hilbert Space on Noncompact Sequence of Sets

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Abstract: In this paper we extend the Mercer theorem to noncompact sequence of Sets, and to establish a functional analysis characterization of the reproducing square kernel Hilbert spaces on general domains.

Keywords: Mercer kernel; Reproducing kernel Hilbert spaces; Nondegenerate Borel measure; Positive Semidefiniteness

1. Introduction

Let $(X,d)$ be a metric space and $K : X \times X \to \mathbb{R}$ be continuous and symmetric. Wesay that $K$ is a Mercer kernel if it is positive semidefinite, i.e., for any finite sequence set of points $(x_n)^m_{i=1} \subseteq X$ and $(c_n)^m_{i=1} \subseteq \mathbb{R}$, there holds $\sum_{i,j=1}^m (c_i) (c_j) K(x_i,x_j) \geq 0.$

The reproducing kernel Hilbert space $\mathcal{H}_K$ associated with the square Mercer kernel $K$ is defined [1] to be the closure of $\{K(x_{\cdot},x_{\cdot}) : x_{\cdot} \in X\}$ with the inner product given by:

$$
\langle f,g \rangle_K = \sum_{i=1}^n \sum_{j=1}^m c_i d_j K(x_i,x_j)
$$

for

$$
f = \sum_{i=1}^n c_i (K(\cdot|x_i) , g = \sum_{j=1}^m d_j (K(\cdot|x_j)

The reproducing kernel property takes the form:

$$
f(x) = \langle f,K(\cdot|x) \rangle_K, \forall f \in \mathcal{H}_K, (x_{\cdot}) \in X \times X.
$$

As $\mathcal{H}_K$ is a Hilbert space, the orthogonal projection of an arbitrary function $f \in \mathcal{H}_K$ onto the finite-dimensional space, span$\{K(x_{\cdot},x_i)\}_{i=1}^m$, denoted as $P(f)$, satisfies $(f - P(f)K(x_{\cdot},x_i))_{i=0}^m = 0$ for each $1 \leq i \leq m$. Then the reproducing kernel property (1.1) implies:

$$
P(f)(x_{\cdot}) = P(f) (K_{\cdot}(x_{\cdot}))_{i=1}^m = \frac{1}{m} \sum_{i=1}^m \langle f,K(\cdot|x_i) \rangle_K = f(x_{\cdot})
$$

Therefore if $f$ minimizes (1.2), then $P(f)$ also does, hence $f$ must be equal to $P(f), i.e., \sum_{i=1}^m c_i (K(\cdot|x_i)) \in \text{span}(\{K(\cdot|x_i)\}_{i=1}^m)$ and the minimization problem (1.2) can be solved by solving a linear system

$$
\left[K(x_{\cdot},x_{\cdot})\right]_{i,j=1}^m m + \lambda I \left[c_j\right]_{i=1}^m = \left((x_{\cdot+1})\right)_{i=1}^m.
$$

See [9,10].When the domain $X$ is compact, the Hilbert space structure of the reproducing kernel Hilbert space $\mathcal{H}_K$ is wellunderstood from a functional analysis point of view, by means of the Mercer theorem. To see this, let $\mu$ be a nondegenerate Borel measure on $(X,d)$. Then the integral operator $L_K$ on $L^2(X,\mu)$ defined by

$$
L_Kf(x_{\cdot}) = \int_X K(x_{\cdot},x_{\cdot+1}) f(x_{\cdot+1}) d\mu(x_{\cdot+1})
$$

is compact, positive and symmetric. It has at most countably many positive eigen values $\{\lambda_i\}_{i=1}^\infty$ and corresponding orthonormal eigenfunctions $\{\phi_i\}_{i=1}^\infty$. The Mercer theorem [7] asserts that:

$$
K(x_{\cdot},x_{\cdot+1}) = \sum_{i=1}^\infty \lambda_i \phi_i(x_{\cdot}) \phi_i(x_{\cdot+1})
$$

where the series converges absolutely and uniformly on $X \times X$. Here one needs to assumethat $\mu$ nondegenerate in the sense that $\mu(S) > 0$ for any nonempty open set $S \subseteq X$, i.e., the complement of any set of measure zero is dense in $X$. For a simple proof of the Mercer theorem, when $X = [0,1]$ and $d\mu = dx$, see [6]. The same proof works for general nondegenerate measures $\mu$, as pointed out by Cucker and Smale [2,3].
An interesting consequence of the Mercer theorem is that \( \{\sqrt{\lambda_i \phi_i}\}_{i=1}^{\infty} \) forms an orthonormal basis of \( \mathcal{H}_{k^2} \). This was proved in [2, 4].

2. Noncompact sequence of Sets with Mercer theorem

We show how to check the assumptions above in the Mercer theorem on a general domain, and discuss the Hilbert space structure of the reproducing kernel Hilbert space \( \mathcal{H}_{k^2} \).

Let \((X, d)\) be a metric space, and \( \mu \) be a nondegenerate Borel measure on \( X \), that means for every open set \( U \subset X, \mu(U) > 0 \). Assume a (sequence) compactness structure for \( X: X = \bigcup_{n=1}^{\infty} X_n \), where \( X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \), and each \( X_n \) is compact with finite measure: \( \mu(X_n) < \infty \). Moreover, any compact subset of \( X \) is contained in \( X_i \) for some \( i \).

Let \( K: X \times X \to \mathbb{R} \) be a Mercer kernel. Define the integral operator \( L_K \) on \( L^2(X, \mu) \) as

\[
L_K(f)(x_n) = \int_X K_j(x_n, x_{n+1})f(x_{n+1}) d\mu(x_{n+1}).
\]

Concerning the kernel \( K \) and the measure \( \mu \) we assume the following:

Assumption 1. \( (K_j)_{x_n} \in L^2(X, \mu) \) for every \( x_n \in X \).

Assumption 2. \( L_k \) is a bounded and positive operator on \( L^2(X, \mu) \), and for every \( g \in L^2(X, \mu), L_k(g) \in C(X) \).

Assumption 3. \( L_{k,j} \) has at most countably many positive eigenvalue \( \{\lambda_j\}_{j=1}^{\infty} \), and corresponding orthonormal eigen functions \( \{\phi_j\}_{j=1}^{\infty} \).

The above assumptions in connection with the reproducing property of the reproducing kernel Hilbert space yield the following.

Lemma 2.1. If \( f \in C(X) \) is supported on \( X_n \) for some \( n \in \mathbb{N} \), then \( L_{k_j}(f) \in \mathcal{H}_{k^2} \) and for \( h \in \mathcal{H}_{k^2} \), holds

\[
(L_{k_j}(f), h)_{k_j} = \int f(x_n)h(x_n) d\mu(x_n) \tag{2.1}
\]

Proof. Since \( f \) is supported on \( X_n \), we have

\[
L_{k_j}(f)(x_n) = \int_X K_j(x_n, x_{n+1})f(x_{n+1}) d\mu(x_{n+1})
\]

Take a sequence \( \{\delta_k > 0\}_{k \in \mathbb{N}} \) such that \( \lim_{k \to \infty} \delta_k = 0 \). For each \( k \), the compactness of \( X_n \) enables us to partition \( X_n \) into subsets \( \{X_{k,i}\}_{i=1}^{m_k} \) such that \( X_{k,i} \cap X_{k,j} = \emptyset \). Then each \( X_{k,i} \) has diameter less than \( \delta_k \). This can be obtained by taking a finite subcovering of the open balls with radius \( \delta_k \) centered at points in \( X_n \).

Choose a set of points \( \{(x_{n+1})_{k,i}^{m_k}\}_{i=1}^{m_k} \) such that \( (x_{n+1})_{k,i} \in X_{k,i} \). Then for each function \( g \in C(X_n) \), there holds

\[
\lim_{k \to \infty} \sum_{i=1}^{m_k} g(x_{n+1})_{k,i} \mu(X_{k,i}) = \int g(x_{n+1}) d\mu(x_{n+1}).
\]

In fact, for any \( \varepsilon > 0 \), there exists some \( \delta > 0 \) such that \( |g(x_{n+1}) - g(x_{n+1})_{k,i}| < \varepsilon \) whenever \( (x_{n+1})_{k,i} \leq \delta \). When \( \delta_k \leq \delta \), we have

\[
\left| \sum_{i=1}^{m_k} g(x_{n+1})_{k,i} \mu(X_{k,i}) - \int g(x_{n+1}) d\mu(x_{n+1}) \right| \leq \sum_{i=1}^{m_k} \varepsilon \mu(X_{k,i}) = \varepsilon \mu(X_n).
\]

It follows that

\[
\lim_{k \to \infty} \sum_{i=1}^{m_k} f((x_{n+1})_{k,i}) \mu(X_{k,i}) = \int f(x_{n+1}) d\mu(x_{n+1}) \tag{2.2}
\]

In the same way, we have

\[
= \int f(x_{n+1}) K_j(x_{n+1}, x_{n+1}) d\mu(x_{n+1})
\]

Let \( \Phi_k(x_n) = \sum_{j=1}^{m_k} f((x_{n+1})_{k,i}) \mu(X_{k,i}) K_j(x_n, (x_{n+1})_{k,i}) \).

Then \( \Phi_k \in \mathcal{H}_{k^2} \). We have

\[
||\Phi_{(t+\varepsilon)} - \Phi_t||_{k_j}^2 = 2\Phi_t \Phi_{(t+\varepsilon)} K_j + \Phi_{t+\varepsilon} \Phi_t_{X_j}. \tag{2.3}
\]

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\[ \left( \Phi_{(i+1)} \cdot \Phi_{(i)} \right)_{K_j} \]

as \( t \to +\infty \). So \( (\Phi_k) \) is a Cauchy sequence in \( \mathcal{H}_K \) and has a limit \( \Phi \in \mathcal{H}_K^2 \). By (2.2), for each \( x_n \in X, \lim_{k \to \infty} \Phi_k(x_n) = L_K(f)(x_n) \). Therefore \( L_K(f) = \Phi \in \mathcal{H}_K^2 \).

The function \( h \in \mathcal{H}_K \) is continuous on \( X_n \) for each \( n \in \mathcal{N} \). Since \( \lim_{k \to \infty} \Phi_k = L_K(f) \) in \( \mathcal{H}_K^2 \), we have

\[ \lim_{k \to \infty} \left( L_K(f)(x_n) \cdot h(x_n) \right) = L_f(x_n)h(x_n) \]

which equals

\[ \lim_{k \to \infty} \int_{x_n} f(x_n)h(x_n)d\mu(x_n) = \int_{x_n} f(x_n)h(x_n)d\mu(x_n). \]

This proves Lemma 2.1.

Define

\[ \mathcal{C}_0(X) = \{ f \in C(X) : f \text{ is supported on } X_n \text{ for some } n \}. \]

It is easy to see that \( \mathcal{C}_0(X) \subset L^2(X, \mu) \) and \( \mathcal{C}_0(X) \) is dense in \( L^2(X, \mu) \).

**Lemma 2.2.** Under Assumptions 1 and 2, for any \( g \in L^2(X, \mu) \), we have \( L_K^2(g) \in \mathcal{H}_K^2 \), and

\[ \| L_K^2(g) \|_{\mathcal{H}_K^2} = \left( L_K^2(g) \cdot g \right)_{L^2(X, \mu)} \]

Also, for any \( h \in \mathcal{H}_K \cap L^2(X, \mu) \), there holds

\[ \left( L_K^2(g), h \right)_{\mathcal{H}_K^2} = \left( g, h \right)_{L^2(X, \mu)}. \]

**Proof.** Since \( g \in L^2(X, \mu) \), there is a sequence \( \{ g_n \} \subset \mathcal{C}_0(X) \) such that \( g_n \to g \) in \( L^2(X, \mu) \). By Lemma 1, \( L_K^2(g_n) \in \mathcal{H}_K^2 \). Moreover,

\[ \| L_K^2(g_n) - g_m \|_{\mathcal{H}_K^2} = \left( \int_{x_n} (g_n(t) - g_m(t))K^2(x, t)d\mu(t) \right)^{\frac{1}{2}} \]

and

\[ L_K^2(g_n) - g_m \leq 0 \quad \text{as } n, m \to \infty \quad (2.6) \]

This means that \( \{ L_K^2(g_n) \} \) is a Cauchy sequence in \( \mathcal{H}_K^2 \) and has a limit \( f \in \mathcal{H}_K^2 \). This in connection with the reproducing kernel property (1.1) implies that for each \( m \in \mathcal{N} \),

\[ \sup_{x_n \in X_n} |L_K^2(g_n)(x_n) - f(x)| \leq \| L_K^2(g_n) - f \|_{\mathcal{H}_K^2} \sup_{x_n \in X_n} K^2(x, x_n) \to 0 \quad \text{as } n \to \infty. \]

Hence \( \{ L_K^2(g_n) \} \) converges to \( f \) uniformly on \( X_m \). By Assumptions 2, \( L_K^2(g_n) \) are all continuous on \( X \) and \( \lim_{n \to \infty} L_K^2(g_n) = L_K^2(g) \) in \( L^2(X, \mu) \). Since \( \mu \) is nondegenerate, \( L_K^2(g_n) \to L_K^2(g) \) almost everywhere on \( X_m \) for each \( m \in \mathcal{N} \). Thus, \( g_n(x) = f \) almost everywhere on \( X_m \). But \( L_K^2(g) \) and \( f \) are both continuous on \( X_m \), we have \( L_K^2(g) = f \) on \( X_m \) and hence on \( X \). Therefore \( L_K^2(g) \in \mathcal{H}_K^2 \). By (2.1)

\[ \left( L_K^2(g), h \right)_{\mathcal{H}_K^2} = \lim_{n \to \infty} \left( L_K^2(g_n), h \right)_{\mathcal{H}_K^2} \]

and

\[ \| L_K^2(g) \|^2_{\mathcal{H}_K^2} = \left( L_K^2(g), L_K^2(g) \right)_{L^2(X, \mu)} = \left( L_K^2(g), g \right)_{L^2(X, \mu)} \]

Thus, both (2.4) and (2.5) hold. We first claim that \( \{ \sqrt{\lambda_i} \phi_i \}_{i=1}^{\infty} \) is an orthonormal system.

**Theorem 2.3:** Under Assumptions 1–3, \( \{ \sqrt{\lambda_i} \phi_i \}_{i=1}^{\infty} \) is an orthonormal system in \( \mathcal{H}_K^2 \).

**Proof.** Since \( \phi_i^2 = - \int L_K^2(\phi_i) \), by Lemma 2.2, \( \phi_i^2 \in \mathcal{H}_K^2 \cap L^2(X, \mu) \). Then (2.5) yields

\[ \left( \sqrt{\lambda_i} \phi_i \right)^2 \quad \text{in } L^2(X, \mu), \]

This proves our statement.

**Theorem 2.4.** Suppose Assumptions 1–3 hold. Then

\[ K^2(x_n, x_{n+1}) = \sum_{i=1}^{\infty} \lambda_i \phi_i^2(x_n) \phi_i^2(x_{n+1}) \]

where the series converges absolutely and uniformly on \( Y_1 \times Y_2 \) with \( Y_1 \) and \( Y_2 \) being any compact subsets of \( X \).

**Proof.** For an arbitrarily fixed point \( x_n \in X, K_{x_n} \in \mathcal{H}_K^2 \cap L^2(X, \mu) \). By Theorem 2.3, the orthogonal projection of \( K_{x_n} \) onto \( \text{span}(\sqrt{\lambda_i} \phi_i)_{i=1}^{\infty} \) equals

\[ \sum_{i=1}^{\infty} \left( K_{x_n}, \sqrt{\lambda_i} \phi_i \right)_{\mathcal{H}_K^2} \sqrt{\lambda_i} \phi_i(x_{n+1}) \]

and

\[ \sum_{i=1}^{\infty} \lambda_i \phi_i^2(x_n) \phi_i^2(x_{n+1}) \]

Moreover,

\[ \sum_{i=1}^{\infty} \lambda_i \phi_i^2(x_n) \phi_i^2(x_{n+1}) = 0. \]

Notice that as functions of the variable \( y \), series (2.8) converges in \( \mathcal{H}_K^2 \) and in \( L^2(X, \mu) \). Set \( K_1 \) as

\[ \left( K_1 \right)_{x_n} = \sum_{i=1}^{\infty} \lambda_i \phi_i^2(x_n) \phi_i^2(x_{n+1}) - K_{x_n} \]

Then \( \left( K_1 \right)_{x_n} \in \mathcal{H}_K^2 \cap L^2(X, \mu) \) as a function of the variable \( y \). By (2.9),
Proof.

Theorem 2.5. An orthonormal system $\{\phi_i\}_{i=1}^\infty$ is complete.

Under Assumptions 1–3, $\mathcal{H}_{K^2}$ is the range of $L^2_{K^2+1}$, where $L^2_{K^2+1}: \mathcal{H}_{K^2+1} \to \mathcal{H}_{K^2+1}$ is an isometric isomorphism, with $\mathcal{D}_{K^2+1}$ being the closure of $D_{K^2+1} = \text{span}(\{K^2(x, \cdot) : x \in X\})$ in $L^2(X, \mu)$.

Proof. By the proof of Theorem 2.4, $D_{K^2+1} := \mathcal{D}_{K^2+1}$ is complete. If $f$ is orthogonal to $D_{K^2+1}$, then $\langle f, (K^2 + 1)\phi_i \rangle = 0$ for each $\phi_i \in D_{K^2+1}$. This implies $L^2_{K^2+1}(f) = 0$. Hence $\mathcal{D}_{K^2+1}$ is complete.

2. The integral operator and $\mathcal{H}_{K^2}$

We show how to fulfill the conditions concerning the operator $L_K$, assumed. It is well known that if $L_K$ is compact and positive, then $L_K$ has at most countably many positive eigenvalues $\{\lambda_i\}_{i=1}^\infty$, and corresponding orthonormal eigenfunctions $\{\phi_i\}_{i=1}^\infty$. Hence Assumptions 2 and 3 are satisfied. So we now investigate when $L_K$ is compact and positive.

For the purpose of Theorems 2.4 and 2.5, we also want to know when $L_K(L^2(X, \mu)) \subset C(X)$. Let $(X, d)$ be a metric space, $\mu$ be a Borel measure on $X$, and $K^2: X \times X \to \mathbb{R}$ be a Mercer kernel satisfying

$$\|K_x\| := \int \int (K_x(y, y'))^2 d\mu(x) d\mu(y) < \infty \quad (3.1)$$

Proposition 3.1. If Assumption 1 and (3.1) hold, then $L_K$ is bounded, compact and positive.

Proof. The boundedness of $L_K$ with $\|L_K\| \leq \|K\|$ follows from (3.1) and the Schwartz inequality:

$$\|g\|_{L^2(K^2+1)}^2 \leq \int \int \left( \sum_{x=1}^X |K_x(x, y)|^2 d\mu(x) d\mu(y) \right)^{1/2}$$

The positivity of $L_K$ is a consequence of the positive semidefiniteness of the kernel $K$. Let us now prove that...
Let $\{\phi_i\}_{i=1}^{\infty}$ be an orthonormal basis of $L^2(\mathcal{X}, \mu)$. Fixed a point $x_0 \in \mathcal{X}$. Then we have

$$
\left\| (L_{K_j})_{x_0} \right\|_{L^2(\mathcal{X}, \mu)}^2 \leq \sum_{i=1}^{\infty} \left| (K_j)_{x_0,i} \phi_i \right|^2 
$$

and the series expansion in $L^2(\mathcal{X}, \mu)$:

$$
K_j(x_0, x_{n+1}) = \sum_{i=1}^{\infty} \left( (K_j)_{x_0,i} \phi_i \right)^2 \phi_i(x_{n+1}).
$$

Set \((L_{K_j})_{x_0} \phi_i \in L^2(\mathcal{X}, \mu)\) is a finite rank operator. For each $x_n \in \mathcal{X}$,

$$
\left\| (L_{K_j} - L_{(K_j)_{x_0}}) \phi_i \right\|_{L^2(\mathcal{X}, \mu)}^2 = \int_{\mathcal{X}} \left| (K_j(x, x_{n+1}) - (K_j)_{x_0,i}(x_{n+1}) \right|^2 d\mu(x_{n+1})
$$

From the dominated convergence theorem, we have

$$
\lim_{n \to \infty} \int_{\mathcal{X}} \left| (K_j(x, x_{n+1}) - (K_j)_{x_0,i}(x_{n+1}) \right|^2 d\mu(x_{n+1}) = 0.
$$

Thus $\left\| L_{K_j} - L_{(K_j)_{x_0}} \right\|_{L^2(\mathcal{X}, \mu)} \to 0$, and $L_{K_j}$ is compact. □

Proposition 3.2. Suppose $K_j$ satisfies (3.1). Then $L_{K_j}$ is positive if and only if $K_j$ is positive semidefinite. The proof of Proposition 3.2 is trivial, but it is necessary that $\mu$ is nondegenerate.

Proposition 3.3. If Assumption 1 holds and $k(x_0, z) := \int_{\mathcal{X}} |K_j(x_0, x_{n+1})|^2 d\mu(x_{n+1})$ is bounded on each $\mathcal{X}_i$, then for every $g \in L^2(\mathcal{X}, \mu)$, $L_{K_j}(g) \in C(\mathcal{X})$.

Proof. Let $g \in L^2(\mathcal{X}, \mu)$. By the dominated convergence theorem,

$$
\lim_{n \to \infty} \int_{\mathcal{X}} \left| g(x_{n+1}) \right|^2 d\mu(x_{n+1}) = 0.
$$

Let $(x_0, z) \in \mathcal{X}$ We show that $L_{K_j}(g)$ is continuous at $(x_0, z)$. To this end, let $U((x_0, z))$ be a bounded neighborhood of $(x_0, z)$ and $(x_0, z) \in U((x_0, z))$ be a sequence tending to $(x_0, z)$. Then $U((x_0, z)) \subseteq \mathcal{X}_i$ for some $i_0$. Denote $M := \sup_{x_0, z \in \mathcal{X}_i} k(x_0, z)^2 < \infty$. Then

$$
\left| L_{K_j}(g)(x_0, z) - L_{K_j}(g)(x_0, z) \right|^2 \leq \int_{\mathcal{X}_i} \left| K_j(x_0, z) - K_j(x_0, z) \right|^2 d\mu(x_{n+1}).
$$

Consider the sequence of functions in the integrand. For any $n \in \mathcal{N}$,

$$
\int_{\mathcal{X}_i} \left| (K_j(x_0, z) - K_j(x_0, z))^2 d\mu(x_{n+1}) \right| \leq \int_{\mathcal{X}_i} \left| (K_j(x_0, z) - K_j(x_0, z))^2 d\mu(x_{n+1}) \right| \leq \left\| (K_j(x_0, z))^2 \right\|_{L^2(\mathcal{X}, \mu)}^2.
$$

That means the sequence of functions of the variable $x_0$ is dominated by an integrable function:

$$
\int_{\mathcal{X}_i} \left| (K_j(x_0, z))^2 d\mu(x_{n+1}) \right| \leq \left\| (K_j(x_0, z))^2 \right\|_{L^2(\mathcal{X}, \mu)}^2.
$$

Also, for each fixed $x_0 \in \mathcal{X}$,

$$
\lim_{n \to \infty} \int_{\mathcal{X}_i} \left| \left( (K_j(x_0, z)) x_{n+1})^2 \right| \right|_2 d\mu(x_{n+1}) = 0.
$$

Therefore, by the dominated convergence theorem, we have

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As $K_2$ is uniformly continuous on the compact set $X_0 \times X_0$, we know that
\[
\lim_{y \to x} \int_{X_0} |K_2(x,y) - K_2(x,x)|^2 d\mu(y) = 0.
\]
Therefore,
\[
\lim_{y \to x} L_{K_2}(g)(x) = L_{K_2}(g)(x).
\]
This proves the continuity of $L_{K_2}(g)$. □

**Proposition 3.4.** If Assumption 1 and (3.1) hold, then $\mathcal{H} \subset L^2(X,\mu)$.

**Proof.** Since $D_K \subset L^2(X,\mu) \cap K$ and $D_K$ is dense in $K$, we only need to compare the norm of $L^2(X,\mu)$ and the norm of $K$.

For fixed $f = \sum_{i,j=1}^m a_{ij} K_1(x_{ij}) \in K$, there hold
\[
\|f\|_K^2 = \sum_{i,j=1}^m a_{ij} K_1(x_{ij}),
\]
and
\[
\|f\|_2^2 = \int_X \sum_{i,j=1}^m a_{ij} K_1(x_{ij}) d\mu(x).
\]

Let $b = \frac{1}{2} \left\| L_{K_1}(\mathbb{I}_K) \right\|_2^{-1}$, and $K_1(x_{ij}) = K_1(x_{ij}) - b \int_X K_1(t) d\mu(t).

Now we want to prove that $L_{K_1}$ is a positive operator. Notice that
\[
L_{K_1}(g)(x_{ij}) = L_{K_1}(g)(x_{ij}) - b L_{K_1}(L_{K_1}(g))(x_{ij}).
\]
Hence
\[
\begin{align*}
\langle L_{K_1}(g), g \rangle &= \langle L_{K_1}(g), g \rangle - b \langle L_{K_1}(g), L_{K_1}(g) \rangle, \\
&= b \left\| L_{K_1}(g) \right\|_2^2.
\end{align*}
\]

By Proposition 3.2, $K_2$ is positive semidefinite. This implies
\[
\sum_{i,j=1}^m a_{ij} K_1(x_{ij}), \quad (3.7)
\]
Thus we have $\mathcal{H} \subset L^2(X,\mu)$.

**Example 3.5.** Let $Y = \mathbb{R}^n$, $K(x_{ij}, x_{ij+1}) = e^{-\frac{(x_{ij}-x_{ij+1})^2}{c^2}}$ with $c > 0$. If $r \in L^2(\mathbb{R}^n)$ is positive almost everywhere and $d\mu = r(x) dx$, then Assumption 1 and (3.1) hold. Hence Theorems 1–3 are valid.

**Proof:** Let $K_{x_n}(t) = K(x_n, t) = e^{-\frac{(x_n-x_n+1)^2}{c^2}}$. Then
\[
\int_{\mathbb{R}^n} K^2_x(t) d\mu(t) = \int_{\mathbb{R}^n} K^2_x(t) r(t) dt \leq \int_{\mathbb{R}^n} K_{x_n}(t) r(t) dt.
\]
Therefore $K_{x_n} \in L^2(\mathbb{R}^n)$ for each $x_n \in \mathbb{R}^n$ and Assumption 1 holds.

Set $A = \int_{\mathbb{R}^n} e^{-\frac{(x_n-x_n+1)^2}{c^2}} dx_n$. Then $0 < A < +\infty$ and
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K^2(x_n, x_{n+1}) d\mu(x_{n+1}) d\mu(x_n) \leq \int_{\mathbb{R}^n} r(x_n) \int_{\mathbb{R}^n} e^{-\frac{(x_n-x_{n+1})^2}{c^2}} r(x_{n+1}) dx_{n+1} dx_n
\]
\[
= \int_{\mathbb{R}^n} r(x_n) \int_{\mathbb{R}^n} e^{-\frac{(x_n-x_t)^2}{c^2}} r(x_{n-t}) dx_{n} dt
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{(x_n-x_t)^2}{c^2}} r(x_{n-t}) dx_{n} dt 
\]
\[
\leq \int_{\mathbb{R}^n} \left\| L_{K_1}(g) \right\|_2^2 dt \leq \left\| L_{K_1}(g) \right\|_2^2 A < \infty.
\]
This verifies (3.1). Hence our statements hold true.

**References**