Relation between Boolean Metric Spaces and Boolean Valued Rings

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Abstract: ***In this paper the terms Boolean metric space, Boolean valued rings, Strong Boolean valued rings, weak Boolean valued rings are defined. Example of a Boolean valued ring which is not a weak Boolean valued ring is given. A few theorems to illustrate the relationship between the various kinds of Boolean valued rings and the Boolean metric spaces are established. Examples to illustrate these theorems are given.***

Keywords: Boolean metric space, Boolean valued rings, Strong Boolean valued rings, weak Boolean valued rings, Translations, motions, Triangle inequality, Relationships between the various kinds of Boolean valued rings and the Boolean metric spaces, Different examples.

1. Introduction

The purpose of this paper is to present the essential features of two terms Boolean valued rings and Boolean metric spaces. These has been a continuing interest in abstract metric spaces in which the distances are elements of a set bearing an algebraic structure less restricted than that of the real numbers. The geometric properties and Boolean valuations are discussed in this paper.

Definition 2.1: An ordered triple (S,B,d) is a Boolean metric space where S is an abstract set, B is a Boolean algebra, and d is a mapping from SxS into B satisfying the following conditions:

a) $d(x,y)$=0 if and only if $x=y$,
b) $d(x,y)$= $d(y,x)$
c) $d(x,z) \subseteq d(x,y) \cup d(y,z)$ for x , y, z in S

Here B is called the distance algebra, the set S is called the ground set of the space, and the mappind d is called the distance function or metric. The subset of B consisting of the function values taken on by d is called the distance set.

Definition 2.2: An ordered triple (R,B,$\emptyset$) is a Boolean valued ring, where R is a ring, B is a Boolean algebra, and $\emptyset$ is a mapping from R into B satisfying the following conditions:

a) $\emptyset(x)=0$ iff $x=0$
b) $\emptyset(x+y) \subseteq \emptyset(x) \cup \emptyset(y)$
c) $\emptyset(xy) \subseteq \emptyset(x) \cap \emptyset(y)$

Definition 2.3: A strong Boolean valued ring is an ordered triple(R,B,$\emptyset$) ,where R is a ring, B is a Boolean algebra, and $\emptyset$ is a mapping from R into B satisfying the following conditions:

a) $\emptyset(x)=0$ iff $x=0$
b) $\emptyset(x+y) \subseteq \emptyset(x) \cup \emptyset(y)$
c) $\emptyset(xy) = \emptyset(x) \cap \emptyset(y)$

Definition 2.4: A weak Boolean valued ring is an ordered triple(R,B,$\emptyset$) ,where R is a ring,
B is a Boolean algebra, and $\emptyset$ is a mapping from R into B satisfying the following conditions:

a) $\emptyset(x)=0$ iff $x=0$
b) $\emptyset(x+y) \subseteq \emptyset(x) \cup \emptyset(y)$
c) $\emptyset(x) = \emptyset(-x)$

The ring is called a weak Boolean valued ring. In general a Boolean valued ring need not be a weak Boolean valued ring.

Example 2.5: Let R be an ordered integral domain and B a Boolean algebra with more than two elements. Let R' be the ring obtained from R by defining all products to be zero.
Let b be an element in B such that $0 \neq b$ *and* $1 \neq b$

For $x \in R'$
set $\emptyset(x)=1$ if x is negative in R
and $\emptyset(x)=b$ if x is positive in R
Let $\emptyset(0)=0$.

Then (R', B,$\emptyset$) is a Boolean valued ring which is not a weak Boolean valued ring.

The relationship between the various kinds of Boolean valued rings and the Boolean metric spaces which they generate **3.1**:

Theorem 3.2: Let (R,B,d) be a Boolean metric space in which all translations $x \rightarrow x+a$ are motions.Then there exists a weak Boolean valuation $\emptyset$ for R such that $\emptyset(a-b=d(a,b)$

Proof: Let (R,B,d) be a Boolean metric space in which R is a ring and in which translations are motions.
Let $\emptyset(x)=d(0,x)$
Then $\emptyset(0)=d(0,0)=0$
and if x$\neq$ 0,.then $\emptyset(x)=d(0,x) \neq 0$

Since translations are motions
$d(-x,0)=d(-x+x,0+x)=d(0,x)=d(x,0)$
So that $\emptyset(x)=\emptyset(-x)$

From the triangle inequality
$d(x+y,0) \subseteq d(x+y,x) \cup d(x,0)$
But $d(x+y,x)=d(x+y-x,x-x)=d(y,0)$

Since translations are motions
$d(x+y,0) \subseteq d(x,0) \cup d(y,0)$
and $\emptyset(x+y) \subseteq \emptyset(x) \cup \emptyset(y)$

Hence $d(x,y) = d(x-y, y-y) = d(x-y, 0) = \emptyset(x-y)$

Thus (R,B,d) is the Boolean metric space obtained from (R,B,$\emptyset$) by setting $d(a,b)$=$\emptyset(a-b)$
Where $\emptyset$ is a weak valuation

Theorem 3.3: Let (R,B,$\emptyset$) be a weak Boolean valued ring .Distance defined as $d(a,b)$=$\emptyset(a-b)$, then (R,B,d) is a Boolean metric space in which translations are motions.

Proof: Suppose that (R,B,$\emptyset$) be a weak Boolean valued ring.
Set $d(x,y)$=$\emptyset(x-y)$
Then $d(a,a) = \emptyset(a-a) = \emptyset(0) = 0$
If $a \neq b$, then $a - b \neq 0$ and $d(a,b) = \emptyset(a-b) \neq 0$
Also $d(x,y)$=$\emptyset(x-y) = \emptyset(y-x) = d(y,x)$
And $d(x,y)$=$\emptyset(x-y) = \emptyset(x-z+z-y) \subseteq \emptyset(x-z) \cup \emptyset(z-y) = d(x,z) \cup d(z,y)$
So the triangle inequality is satisfied and (R,B,d) is a Boolean metric space.
To show that translations are motions
$d(x+z, y+z) = \emptyset(x+z-y-z) = \emptyset(x-y) = d(x,y)$

Theorem 3.4: If (R,B,d) is a Boolean metric space, R a ring in which all translations are motions, the following are equivalent

a) Ring multiplication decrease distances from the origin ($d(xy,0) \subseteq d(x,0), d(xy,0) \subseteq d(y,0)$ for all x,y in R)
b) There exists a Boolean valuation $\emptyset$ for R such that $d(a,b)$=$\emptyset(a-b)$
c) Ring multiplication are contraction mappings
   i.e $d(xz,yz) \subseteq d(x,y)$
   $d(zx,zy) \subseteq d(x,y)$ for all x,y,z in R

Proof: (i)$\Longrightarrow (ii)$

Let (R,B,d) be a metric space in which translations are motions and (i) is satisfied.
Let $\emptyset(x) = d(x,0)$.
By the theorem 3.3 (R,B,d) is determined by the weak Boolean valued ring (R,B,$\emptyset$) setting $d(a,b)$=$\emptyset(a-b)$
But since condition (i) is satisfied
$d(xy,0) \subseteq d(x,0)$ and
$d(xy,0) \subseteq d(y,0)$
Hence $d(xy,0) \subseteq d(x,0) \cap d(y,0)$ and
$\emptyset(xy) \subseteq \emptyset(x) \cap \emptyset(y)$
So that (R,B,$\emptyset$) is a Boolean valued ring.
(ii)$\Longrightarrow (iii)$
By definition $d(xz,yz) = \emptyset(xz-yz)$
$\emptyset(xz-yz) = \emptyset((x-y)z) \subseteq \emptyset(x-y) \cap \emptyset(z) \subseteq \emptyset(x-y)$
$= d(x,y)$
Similarly the other inequality can be established.
(iii)$\Longrightarrow (i)$
This can be established, by taking either x or y be 0

Theorem 3.5: If (R,B,d) is a Boolean metric space satisfying the conditions of theorem 3.4 ,where R is a ring with unit , the following are equivalent.

a) Ring multiplications are similarity transformations with respect to the origin (for every x in R ,there is an element $b(x)$ of the distance set such that $d(xz,0) = d(z,0) \cap b(x)$ for all $z \in R$)
b) There exists a strong Boolean valuation $\emptyset$ for R such that $d(a,b)$=$\emptyset(a-b)$
c) Ring multiplications are contraction mappings with fixed constant of contraction (z in R implies the existence of $c(z)$ in the distance set such that $d(xz,yz) \subseteq d(x,y) \cap c(z)$ for all x,y in R)

Proof: (i)$\Longrightarrow (ii)$
From (i) it follows that for fixed x and all z,
$d(xz,0) = d(z,0) \cap b(x)$
Let $\emptyset(x) = d(x,0)$.
Then by Theorem 3.4 $\emptyset$ is a Boolean valuation
Hence$d(x,0) = d(x.1,0) = d(1,0) \cap b(x) = \emptyset(1) \cap b(x)$
Thus $\emptyset(x) = \emptyset(1) \cap b(x)$
But $b(x) = \emptyset(t)$ for some $t \in R$, so that $\emptyset(t) = \emptyset(t.1) \subseteq \emptyset(t) \cap \emptyset(1)$
And hence $\emptyset(t) \subseteq \emptyset(1)$
implies that $\emptyset(1) \cap b(x) = \emptyset(1) \cap \emptyset(t) = \emptyset(t) = b(x)$
$\therefore\ b(x) = \emptyset(x)$
$\emptyset(\mathrm{xz}) = \mathrm{d}(\mathrm{xz},0) = \mathrm{d}(\mathrm{z},0) \cap \mathrm{b}(\mathrm{x}) = \mathrm{d}(\mathrm{z},0) \cap \emptyset(\mathrm{x})$
$= \emptyset(\mathrm{z}) \cap \emptyset(\mathrm{x})$
And $\emptyset$ is a strog Boolean valuation
(ii)$\Longrightarrow (iii)$
Let z be fixed ,then $d(xz,yz) = \emptyset(xz-yz) = \emptyset(x-y) \cap \emptyset(z) = d(x,y) \cap \emptyset(z)$.
By letting $y = 0$ we can establish that (iii)$\Longrightarrow (i)$

Example 3.6: A Boolean metric space (R,B,d),R a ring,in which translations are not motions
Let (R,B,d) be a Boolean metric space in which R is a Boolean ring with identity, B is the Boolean algebra associated with R, and $d(a,b) = a - b$

Select distinct elements x,y such that $d(x,y) \neq 1$
Let a be a fixed non-zero element of R with $a \neq x + y$.
Define the function d' as follows:
$d'(b.c) = d(b,c)\ if\ b \neq x+a, c \neq x+a$
$d'(x+a,b) = d'(b,x+a) = 1\ if\ b \neq x+a$
$d'(x+a, x+a) = 0$
Then (R,B,d') is a Boolean metric space, but $d'(x+a, y+a=1\neq d'x,y$

Example 3.7: An example of a Boolean valued ring which is not a strong Boolean valued ring.
Consider the Boolean valued ring (R,B,$\emptyset$) where R is a ring with divisors of zero and B is the two element Boolean algebra.
Let $\emptyset(x) = 1$
if $x \neq 0$ and $\emptyset(0) = 0$ Then if $xy = 0$, and $x,y \neq 0$,
$\emptyset(xy) = 0$, but $\emptyset(x) \cap \emptyset(y) = 1 \cap 1 = 1$.

References

[1] L.M BLUMENTHAL,Boolean Geometry T.Rend.circ.Mat.Palermo,II.Ser.I,343-360(1952)
[2] L.M BLUMENTHAL and C.J.PENNING, Boolean Geometry II.Rend.circ.Mat.Palermo, II.Ser.10,175-192(1962).
[3] D.ELLIS,Autometrized Boolean Algebra I,II.canadian J.Math.3,87-93:145-147(1951)
[4] R.F.KELLER,Motions of Matrix Rings. canadian J.Math.16,159-164(1964)

[5] J.VONNEUMANN,continuous Geometry.princeton 1936-37.
[6] E.A NORDHAUS and L.LAPIDUS,Brouwerian Geometry,Canadian J.Math 6,127-229(1954)
[7] C.J.PENNING,Boolean metrics spaces.Thesis,Technische Hogeschool te Delft 1960.
[8] K.Roy,Newmannian Geometry I,II.Bull Calcutta Math.Soc.52,187-194(1960);54,33-40(1962).