q- Continuity Equation

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Abstract: ***For*** $q \in (0,1)$, ***a*** $q -$ ***deformation of the continuity equation is introduced using the q- derivative (or Jackson derivative). By quantum calculus, we solve these equations.***

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1. Introduction

A continuity equation in physics is an equation that describes the transport of some quantity. It is particularly simple and powerful when applied to a conserved quantity, but it can be generalized to apply to any extensive quantity. Since mass, energy, momentum, electric charge and other natural quantities are conserved under their respective appropriate conditions, a variety of physical phenomena may be described using continuity equations. Continuity equations are a stronger, local form of conservation laws. For example, a weak version of the law of conservation of energy states that energy can neither be created nor destroyed i.e., the total amount of energy in the universe is fixed. This statement does not rule out the possibility that a quantity of energy could disappear from one point while simultaneously appearing at another point. A stronger statement is that energy is locally conserved: energy can neither be created nor destroyed, nor can it ”teleport” from one place to another, it can only move by a continuous flow.Acontinuityequationisthemathematicalwaytoexpressthis kindofstatement. For example, the continuity equation for electric charge states that the amount of electric charge in any volume of space can only change by the amount of electric current flowing into or out of that volume through its boundaries. An alternative expression of the continuity equation for a species in the atmosphere can be derived relative to a frame reference moving with the local flow; this is called the Lagrangian approach. Consider a fluid element at location X_0 at time t. We wish to know where this element will be located at a later time t. We define a transition probability density such that the probability that the fluid element will have moved to within a volume (dx, dy, dz)centered at location X at time t. Continuity equations more generally can include “source” and “sink” terms, which allow them to describe quantities that are often but not always conserved, such as the density of a molecular species which can be created or destroyed by chemical reactions. In an everyday example, there is a continuity equation for the number of people alive; it has a ”source term” to account for people being born, and a ”sink term” to account for people dying.

Any continuity equation can be expressed in an ”integral form” in terms of a flux integral, which applies to any finite region, or in a differential form interms of the divergence operator) which applies at a point. Continuity equations underlie more specific transport equations such as the convection diffusion equation, Boltzmann transport equation, and Navier-Stokes equations.

A continuity equation is useful when a flux can be defined. To define flux, first there must be a quantity Q which can flow or move, such as mass, energy, electric charge, momentum, number of molecules, etc. Let ρ be the volume density of this quantity, that is, the amount of Q per unit volume.

By the divergence theorem, a general continuity equation can also be written in a differential form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = \sigma$$

where ∇ is divergence, ρ is the amount of the quantity Q per unit volume, j is the flux of Q, t is time, σ is the generation of Q per unit volume per unit time. Terms that generate Q (i.e. $\sigma > 0$) or remove Q (i.e. $\sigma < 0$) are referred to as a ”sources” and ”sinks” respectively. This general equation may be used to derive any continuity equation, ranging from as simple as the volume continuity equation to as complicated as the Navier-Stokes equations. This equation also generalizes the advection equation. Other equations in physics, such as Gauss’s law of the electric field and Gauss’s law for gravity, have a similar mathematical form to the continuity equation, but are not usually referred to by the term ”continuity equation”, because j in those cases does not represent the flow of a real physical quantity. In the case that Q is a conserved quantity that cannot be created or destroyed (such as energy), $\sigma = 0$ and the equations becomes:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.$$

In electromagnetic theory, the continuity equation is an empirical law expressing (local) charge conservation. Mathematically it is an automatic consequence of Maxwell’s equations, although charge conservation is more fundamental than Maxwell’s equations. It states that the divergence of the current density j (in amperes per square meter) is equal to the negative rate of change of the charge density ρ (in coulombs per cubic metre),

$$\nabla \cdot j = -\frac{\partial \rho}{\partial t}$$

Current is the movement of charge. The continuity equation says that if charge is moving out of a differential volume (i.e. divergence of current density is positive) then the amount of charge within that volume is going to decrease, so the rate of change of charge density is negative. Therefore, the continuity equation amounts to a conservation of charge. If magnetic monopoles exist, there would be a continuity equation for monopole currents as well, see the monopole article for background and the duality between electric and magnetic currents. In fluid dynamics, the continuity equation states that the rate at which mass enters a system is equal to the rate at which mass leaves the system plus the accumulation of mass within the system. Conservation of energy says that energy cannot be created or destroyed. (See below for the nuances associated with general relativity.) Therefore, there is a continuity equation for energy flow:

$$\frac{\partial u}{\partial t} + \nabla \cdot Q = 0$$

Where u is a local energy density (energy per unit volume), Q is energy flux (transfer of energy per unit cross-sectional area per unit time) as a vector. An important practical example is the flow of heat. When heat flows inside a solid, the continuity equation can be combined with Fourier's law (heat flux is proportional to temperature gradient) to arrive at the heat equation. The equation of heat flow mayalso have source terms: although energy cannot becreated or destroyed, heat can be created from other types of energy, for example via friction or joule heating.

In recent years the q-deformation of the Heisemburg commutation relation has drawn attention. In the paper [10], the purpose was to understand the probability distribution of a non-commutative random variable $a + a^*$, where a is a bounded operator on some Hilbert space satisfying

$$aa^* + qa^*a = 1, \tag{1}$$

for some $q \in [-1,1)$. The calculation is inspired by the case, $q = 0$, where a and a^*, turn out to be the left and right shift on $l^2(\mathbb{N})$, In this case a and a^*, can be quite nicely represented as operators on the Hardy class $\mathcal{H}^2$of all analytic functions on the unit disk with L^2 limits toward the boundary.

Subsequently, they find a measure μ_q, $q \in [-1,1)$, on the complex plane that replaces the Lebesgue measure on the unit circle in the above: μq is concentrated on a family of concentric circle, the largest of which has the radius$\frac{1}{\sqrt{1-q}}$. Their representation space (see[10]) will be $\mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$ the completion of the analytic functions on $\mathfrak{I}_q = \{z \in \mathbb{C} |z|^2 <$ $11-q$ (1−q)}with respect to the inner product defined byμq. In this space annihilation operator a is represented by a q difference operator D_q. As q tends to 1, $\mu_q$will tend to the Gauss measure on $\mathbb{C}$ and D_q becomes differentiation. So, it is natural to ask what is the q - deformation of the continuity equation. This paper organized as follows. In Section 2, we briefly recall well-known results on q - calculus, Jackson derivative (or q -derivative) and useful representations. In Section 3, we introduce q -continuity equation and we deduced some theorems.

2. Preliminaries

We recall some basic notations of the language of q-calculus (see [1, 2, 7, 9, 10]). The natural number n has the following q deformation:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, \text{ with } [0]_q = 0.$$

Occasionally we shall write $[\infty]_q$ for the limit of these numbers: $\frac{1}{(1-q)}$. The q factorials and q binomial coefficients are defined naturally as

$$[n]_q! := [1]_q \cdot [2]_q \cdots [n]_q \text{ with } [0]_q := 1$$

Recall that from [10], for $q \in (-1,1)$. relation (1) admits, up to unitary equivalence, a unique non-trivial bounded irreducible representation given on the canonical basis $\{e_n | n \in \mathbb{N}\}$ of $l^2(\mathbb{N})$by:

i. $a^* e_n = e_{n+1}$
ii. $a e_n = [n]_q e_{n-1}$
iii. $\langle e_n, e_m \rangle = \delta_{n,m} [n]_q!$.

For $q \in (0,1)$ and analytic $f: \mathbb{C} \longrightarrow \mathbb{C}$define operators Z and D_q as follows (see [7, 9, 10])

$$(Zf)(z) := zf(z)$$

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, & z \neq 0 \\ f'(0) & \end{cases}$$

The operatorD_q has the following properties:

i. $lim_{q\uparrow 1}(D_q f)(z) = f'(z)$
ii. $D_q(z^n) = [n]_q z^{n-1}$
iii. $D_q(f(z)g(z)) = (D_q f)(z)g(z) + f(qz)(D_q g)(z)$,
iv. $D_q\left(\frac{f(z)}{g(z)}\right) = \frac{(D_q f)(z)g(z) - f(z)(D_q g)(z)}{g(z)g(qz)}$.

It is well known ([10]) that the operators D_q and Z give a bounded representation of (1), i.e., D_q and Z satisfy

$$D_q Z - qZD_q = 1$$

With respect to the measure (see [10])

$$\mu_q(dz) = (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \lambda_{r_k}(dz),$$

$$0 < q < 1 \text{ and } r_k = \frac{q^{\frac{k}{2}}}{\sqrt{1-q}}$$

where λ_{r_k} is the normalized Lebesgue measure on the circle with radiusr_k, they define the inner product

$$\langle f, g \rangle_{\mu_q} := \int_{\mathbb{C}} \overline{f(z)}\, g(z) \mu_q(dz)$$

for all $f, g \in \mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$. Note that $\mu_q \longrightarrow \mu_0$ when $q \longrightarrow 0$, where μ_0 is the normalized Lebsgue measure on the unit circle and that, in the limit $q \uparrow 1$,μ_q tends to the Gauss measure on the complex plane. The identification $a = D_q$and $a^* = Z$ determine a representation of (1) on $\mathfrak{H}^2(\mathfrak{D}_q, \mu_q)$. In particular, with $e_n := z^n$, (i),(ii) and (iii) are satisfied, and therefore $D_q^* = Z$. For more details see Ref. [10].

3. q-Continuity Equation

Let $q \in (0,1)$. Recall that

$$D_{q,x} f(t,x) = \frac{f(t,x) - f(t,qx)}{x(1-q)} \tag{2}$$

And

$$D_{q,t} f(t,x) = \frac{f(t,x) - f(qt,x)}{t(1-q)} \tag{3}$$

As a q-deformation of the continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot j=0$$

we will study the following equation

$$D_{q,t}\rho=-D_{q,x}j \qquad (4)$$

Theorem3.1. For $q\in(0,1)$, the q-continuity equation (4) gives

$$\rho(t,x)=\rho_0(x)+\frac{t}{x}\sum_{i=0}^{\infty}q^i\left(j(q^it,qt)-j(q^it,x)\right)$$

Where $j(t,.)\in\mathfrak{H}^2(\mathfrak{D}_q,\mu_q)$ and $lim_{i\to 0}\rho_0(t,.)=\rho_0\in\mathfrak{H}^2(\mathfrak{D}_q,\mu_q)$ are given.

Proof: From equation (2) we get

$$\frac{\rho(t,x)-\rho(qt,x)}{t(1-q)}=\frac{j(t,qx)-j(t,x)}{x(1-q)}$$
$$\frac{\rho(qt,x)-\rho(q^2t,x)}{qt}=\frac{j(qt,qx)-j(qt,x)}{x}$$

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$$\frac{\rho(q^{k-1}t,x)-\rho(q^k,x)}{q^{k-1}t}=\frac{j(q^{k-1}t,qx)-j(q^{k-1}t,x)}{x}$$

Then, we obtain

$$\rho(t,x)-\rho(qt,x)=\frac{t}{x}\left(j(t,qt)-j(t,x)\right)$$
$$\rho(qt,x)-\rho(q^2t,x)=\frac{qt}{x}\left(j(qt,qx)-j(qt,x)\right)$$

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$$\rho(q^{k-1}t,x)\text{-}\rho(q^kt,x)\text{=}q^{k-1}\frac{t}{x}\left(j(q^{k-1}t,qt)\text{-}j(q^{k-1}t,x)\right)$$

Therefore, we deduce that

$$\rho(t,x)-\rho(q^k,x)=\frac{t}{x}\sum_{i=0}^{k-1}q^i\left(j(q^it,qx)-j(q^it,x)\right)$$

Ask $\to\infty$ we get

$$\rho(t,x)=\rho_0(x)+\frac{t}{x}\sum_{i=0}^{\infty}q^i\left(j(q^it,qt)-j(q^it,x)\right)$$

which completes the proof.

Theorem3.2. For $q\in(0,1)$, the solution of the q-continuity equation (4) is given by

$$j(t,x)=j(t,0)-\frac{x}{t}\sum_{i=0}^{\infty}q^i\left(\rho(t,q^ix)-\rho(qt,q^ix)\right)$$

Where $\rho\in\mathfrak{H}^2(\mathfrak{D}_q,\mu_q)$ and $lim_{x\to\infty}j(t,.)=j(t,.)\in\mathfrak{H}^2(\mathfrak{D}_q,\mu_q)$ are given.

Proof: From equation (2) we get

$$\frac{\rho(t,x)-\rho(qt,x)}{t(1-q)}=\frac{j(t,qt)-j(t,x)}{x(1-q)}$$

Which gives

$$\frac{\rho(t,qt)-\rho(qt,qx)}{t(1-q)}=\frac{j(t,q^2x)-j(t,qx)}{qx(1-q)}$$

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$$\frac{\rho(t,q^{k-1}x)-\rho(qt,q^{k-1}x)}{t(1-q)}=\frac{j(t,q^kx)-j(t,q^{k-1}x)}{q^{k-1}x(1-q)}$$

Then, we get

$$\frac{x}{t}\left(\rho(t,x)-\rho(qx,x)\right)=j(t,qx)-j(t,x)$$
$$q\frac{x}{t}\left(\rho(t,qx)-\rho(qt,qx)\right)=j(t,q^2x)-j(t,qx)$$

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$$q^{k-1}\frac{x}{t}\left(\rho(t,q^{k-1}x)\text{-}\rho(qt,q^{k-1}x)\right)\text{=}j(t,q^kx)\text{-}j(t,q^{k-1}x)$$

Then, we obtain

$$\frac{x}{t}\sum_{i=0}^{k-1}q^i\left(\rho(t,q^it)\text{-}\rho(qt,q^ix)\right)\text{=}j(t,q^kx)\text{-}\,j(t,x)$$

Ask $\to\infty$ we get

$$j(t,x)=j(t,0)-\frac{x}{t}\sum_{i=0}^{\infty}q^i\left(\rho(t,q^ix)-\rho(qt,q^ix)\right)$$

as desired.

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