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Multiply Fibred Manifolds

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Abstract: ***This paper aimed at investigating the product of longitudinal pseudodifferential operators of C^*- algebras $\prod_{\propto=1}^{r} \Psi_c^{-\infty}(F_\propto)$ contained in $C^* - closure$ $\overline{\Psi_c^{-\infty}}(F)$by fibrations manifolds. Such that $\{F_\propto\}$ is a family of fibred manifolds. A locally homogenous condition of foliationsis required to obtain us a generator of foliation$F$.this condition equivalent to requiring that Lie subalgebras generate Lie algebra. we also show a brief application of noncommutative harmonic analysis for compact Lie groups.***

Keywords: SemisimpleLie groups; C^*- algebras; Pseudodifferential Operators; Flag Varieties

1. Introduction

The purpose of this paper is to study a manifolds of multiple foliations with foliation algebras. This work is depending on an idea in papers ([AS71, CON82]). A main goal throughout index theory is the study of longitudinal pseudo differential operators, which (pseudo) differentiate along the leaves of foliation. In [Yun 11], we explained how the analysis of these operators can be used to index theory. BGG- complex of SL(3,ℂ) is used to construct an explicit model for Kasparove's γ −element as the image of an element of the equivariant K-homology of the flag variety$K_G(M)$. In [Yun 10], the construction γ for SL(3,ℂ) was compactness theorem for products of negative order Pseudodifferential Operators along the foliations of a manifold M. In this article, for any generalized flag manifold, our approach are changed by using noncommutative harmonic analysis in the sense of M, [Tay84].

Let $F = \{F_\alpha\}, \alpha = 1, \dots, r$ be a collection of smooth foliations of a manifold M. and $F = \Psi_c^{-\infty}(F_j)$ is denoted the set of longitudinally smoothing operators along F_j with compact support. These act as bounded operators on L^2M, and their norm- closure $\overline{\Psi_c^{-\infty}}(F_j)$ is a $C^* - algebra$. It contains the order $-d$ longitudinal Pseudodifferential Operators $\Psi_c^d(F_j)$ for any $-\infty \le -d < 0$. We will explain what we mean by this in section 2. If F is the tangent bundle to a smooth fibration $p: M \to N$, then the elements of $\Psi^d(F)$ are families of psudodifferential operators of order d on the fibers. And $\Psi_c^d(F)$ is the subset of closed and bounded in $\times M$.

The basic idea of our construction is to consider the families of psudodifferential operators along the foliations is the holonomygroupoid$\mathcal{G}: = \mathcal{G}(M, F)$; it follows from the definition of differentiable groupoid that these foliations are smooth manifold. We impose a condition of local homogeneity on foliations which include that they generate a foliation F, under the structure of Lie bracket of tangential vector fields. We extend a non-commutative miccrolocal analysis as it belongs to longitudinal Pseudo differential operators on Lie groups. We prove the outcomes of integral operators in nonsingular coordinate systems. Let $F = \{F_1, \dots, F_\propto\}$ with Codimenstion c ,$(0 < c < n)$ be a collection of smooth foliations of a manifold M^n, such that n is the dimension of M . we suppose a condition of local homogeneity on $\{F_1, \dots, F_\propto\}$ which include F under the structure of Lie bracket of tangential vector fields . let g be The Lie algebra of connected Lie group G, and $\eta_1, \dots, \eta_\propto$ is a family of Lie subalgebras and η is the Lie algebra they generate;$\mathcal{H}_1, \dots, \mathcal{H}_\propto$ and $\mathcal{H}$ are the equivalent left-coset foliations . we also use $\eta_1, \dots, \eta_\propto$ of Lie subalgebras of its Lie algebras g, we let H_j denote the foliation of G generated by left translates of η_j, and $h_1, \dots, h_\propto$ be a collection of Lie sub algebras of its Lie algebra g , such that $h_1, \dots, h_\propto$ generate g as a Lie algebra ,for semi simple Lie groups much of the representation theory focus on the generalized principle series representations, which act on section spaces of line bundle over the flag manifold $M: G/B$, of Lie group G.

Definition (1.1). The family $F_1, \dots, F_\propto$,is called locally homogeneous if there exist an atlas of local charts $\emptyset_\beta: U_\beta \to M$ with $U_\beta \subseteq G$ such that $d\emptyset_\beta$ isomorphism maps and $h_\propto \le g$, each $H_j \cong F_j$ on its domain.

Theorem (1.2). Let $F_{1,\dots,}F_\propto$ be locally homogenous family of foliations. with F ,sequent smoothing along the directions of $F_1, \dots, F_\propto$ yields an operator which is smoothing in all directions generated from them via Lie brackets.
$\overline{\psi_c^{-\infty}}(F_1) \dots \overline{\psi_c^{-\infty}}(F_\propto) \subseteq \overline{\psi_c^{-\infty}}(F)$.

Definition (1.3). we say that the family of foliations $F = \{F_\alpha\}$ satisfy Hörmander's condition (the vector fields with all their Lie bracket and coefficients in the space of smooth foliations span the whole tangent space at each point), if the Lie algebra of all smooth vector fields on M is generated by $C^\infty(TF_1), \dots, C^\infty(TF_\propto)$.

Corollary (1.4). Let $F_1, \dots, F_{\propto-1}, F_\propto, F_{\propto+1}$ be a locally homogeneous family of foliations which satisfy Hörmander's condition. for each $(j+1)$, if $A_{j+1} \in \Psi_c^{-1}(F_{j+1})$ then the product $A_1, \dots, A_{\propto-1}, A_\propto, A_{\propto+1}$ is a compact operator.

Assume that$K$ is a compact Lie group, contains a finitely generated subgroup. And K_1 and K_2 are closed subgroups which generate $K$and let U be a unitary group action of Kon a vector space which is Hilbert space H for which all irredducible K −types have finite multiplicity.

If π_1, and π_2 are orthogonal groups have an irreducible representation on K_1 and K_2respectively and they have arbitrarily small inner products on the other hand they have arbitrarily small inner products on the component of some finite- dimensional subspace.(see theorem 6.4).

Let$H_{1,...,}H_{\propto}$ be closed subgroups of connected Lie group G. and Lie algebra g of connected Lie group G; $\eta_1,\dots,\eta_{\propto}$ is a family of Lie subalgebras and η is the Lie algebra they generate;$\mathcal{H}$ and $\mathcal{H}_{1,...,}\mathcal{H}_{\propto-1,}\mathcal{H}_{\propto}$ are corresponding left-coset foliations.

2. Smooth Family of Pseudodifferential Operators

If F is the tangent bundle to a smooth fibration $\ell : M \to N$. with a smooth foliation F then elements of $\Psi^d(F)$ can be defined as smooth families of pseudodifferential operators of order d on the fibers[201].Then $\Psi_c^d \subseteq M \times M$ with compact support. We have already known a groupoid is a small grouping $\mathcal{G}$, in which every morphism is invertible.

If G is differentiable groupoid, then $C_c^{\infty}(G)$ will have the structure of convolution algebra on the L^2-space of the leaves, and the out coming c^*- algebra is Conne's foliation algebra $C^*(M,F)$.

We are looking for the different structure which is be convolution algebras to be work on Hilbert space.
Let $C_c^{\infty}(\mathcal{G})$ acting on L^2(M) and $k \in C_c^{\infty}(\mathcal{G})$, $f \in L^2$(M)

$$k.f(x) := \int_{\mathcal{G}_x} k(\gamma)\ f(r(\gamma))\quad d\gamma$$

Such that $\mathcal{G}_x := s^{-1}(x)$, and r, s are the range and source maps of $\mathcal{G}$ and $\mathcal{G}_x$.it is the image of this statement in $B(L^2M)$ as $\Psi_c^{-\infty}(F)$ with norm- closure $\overline{\Psi_c^{-\infty}}(F)$ in $B(L^2M)$, there is a homeomorphism mapmakes $\overline{\Psi_c^{-\infty}}(F) \cong C_r^*(M,F)$ if F obtains from a fibration. for any $d \in [-\infty, 0)$, $\Psi_c^{-d}(F)$ is dense in $\overline{\Psi_c^{-\infty}}(F)$.

Let H be a connected Lie subgroup of a connected smooth Lie group G. And let H denote the foliation of G by left-cosets of H. And $K \in \psi_c^{-\infty}(H)$ is a longitudinally differentiable operator is given by

$$K\ u\,(x) := \int_H k(x,h)\ \ u\,(h^{-1}x)\,dh \qquad (2.1)$$

Where $k \in C_c^{\infty}(G \times H)$. Define $L^2(G)$ with respect to left-invariant Haar measure.

$$k^*(x,h) = \overline{k(h^{-1}x, h^{-1})}\Delta_H(h). \qquad (2.2)$$

We call such an operator by $O_{PH}\ k$. such that $O_{PH}\ (k^*)$ is the adjoint of $O_{PH}\ k$.

3. Singular Coordinate Systems with Integral Operators

We have to generalize the operators of (2.1) by reparametrizing the variable $h \in H$in singular coordinate systems. With respect to $\phi: M \to N$ is smooth function.

For smooth manifold M with smooth measure m , $a \in C_c^{\infty}(G \times M)$.we consider operators A is

$$A\ u\,(x) := \int_M a(x,m)\,u(\emptyset(m)^{-1}x)\,dm \qquad (3.1)$$

Define the critical set of ϕ: $M \to H$ as C(ϕ) := { m∈M | D ϕ (m) is not onto}. The M-support of a ∈ C∞ c (G× M), denoted M-supp (a), is the projection of the support of a onto M.

Lemma (3.1): Aisaboundedoperator on $L^2(G)$ with norm $\|A\| \leq \|a\|_{\infty}$ VOI $(M - supp(a))$.

Proof. We can write $A = \int_M A_m\ \ dm$, where A_m is the operator $u \to a(.,m)L_{\phi(m)}\,u$
$A_m = 0$ for $m \notin M$- supp(a).

Lemma (3.2). Suppose that M-supp(a) ∩ C(ϕ) = ∅. Then the operator $A \in \Psi_c^{-\infty}(\mathcal{H})$.

Proof. We shall prove that $A \in \Psi_c^{-\infty}(\mathcal{H})$.
Let $d = \dim(H)$, and $n = \dim(M)$, with $n \geq d$, and $\Phi: M^n \to H^d$.

Let $= \begin{bmatrix} n \\ d \end{bmatrix}$. by using a partition of unity subordinate to local charts on M, we may reduce to the case where M is a bounded open subset of $\mathbb{R}^n$.

Let $E_{\binom{n_1}{n_2}}, \dots, E_{\binom{n}{d}}$ be the coordinate d-planes of $\mathbb{R}^n$.
We will define a Jacobian ∅ of these coordinates of spirit of the implicit function theorem.

$\Phi_i: M \to H \times E_i^{\perp}$; $m \to (\emptyset(m), p_i(m)).\Phi_i$ is a local diffeomorphisim at m on M-supp(a_i) if and only if $D\emptyset(m)|_{E_i}$ is onto.

Define J_i as the Radon-Nikodym derivative of Φ_i ; where de' is Lebesgue measure on $E_i^{\perp}$:

$$\Phi_i(M) = \int_M J_i \quad dm$$

$$J_i(m) = \frac{\Phi_{i*}(dm)}{d_H x de'} \qquad (3.2)$$

Let $J: M \to \mathbb{R}^n$, and $J = (J_1, \dots, J_N)$. (3.3)

There is some $\mathcal{E} > 0$ such that the open sets $U_i = J_i^{-1}(\mathcal{E}, \infty)$ cover M- supp(a). Put $a_i(x,m) = \Psi_i(m)a(x,m)$, so that $A = \sum_i A_i$ where

$$A_i u(x) = \int_{U_i} a_i(x,m)u(\emptyset(m)^{-1}x)\,dm \quad (3.4)$$

We choose $\Psi_i \in C_c^{\infty}(U_i)$, and $\sum_i \Psi_i \equiv 1$ on M-supp(a). Fix $i \in \{1, \dots, N\}$.

Choose a finite cover $\{V_j\}$ of M-supp(a_i) by relatively compact open sets on which Φ_i is a diffeomorphism to its range. $\Phi_{ij} = \Phi_{ij}|V_j$ for diffeomorphisms.

Let $X_j \in C^{\infty}(V_j)$ be a partition of unity subordinate to $\{V_j\}$ and $a_{ij}(x,m) = X_j(m)a_i(x,m)$.then

$$A_i u(x) \quad = \sum_j \int_{V_j} a_{ij}(x,m)u(\phi(m)^{-1}x)\,dm.$$

$$= \sum_j \int_{(h,e')\in\Phi_{ij}(V_j)} a_{ij}\left(x, \Phi_{ij}{}^{-1}(h,e')\right) u(\phi(\Phi_{ij}{}^{-1}((h,e'))^{-1}x)\, d\Phi_{ij}{}^{-1}((h,e')$$

$$= \sum_j \int_{(h,e')\in\Phi_i(V_j)} a_{ij}\left(x, \Phi_{ij}{}^{-1}(h,e')\right) u(h^{-1}x) J_i(\Phi_{ij}{}^{-1}(h,e'))^{-1} dh\, e'$$

$$= \sum_j \int_{(h,e')\in\Phi_i(V_j)} k_{ij}(x,h,e') u(h^{-1}x)\, dh de' \quad (3.5)$$

$m = \Phi_{ij}{}^{-1}(h,e')$, and $k_{ij}(x,h,e')$

$$= a_{ij}\left(x, \Phi_{ij}{}^{-1}(h,e')\right) J_i(\Phi_{ij}{}^{-1}(h,e'))^{-1}$$

Since $J_i(m) \geq \mathcal{E}$ on M-supp(a_{ij}),$k_{ij}(x,h,e')$ extends to a smooth compactly supported function on $G \times H \times E_i^{\perp}$.

The summation $\left(\sum_j \int_{E_i^{\perp}} k_{ij}(x,h,e') d e'\right)$)is a smooth compactly supported function of $(x,h) \in G \times H$, SO

$$A_i u(x) = \int_G \left(\sum_j \int_{E_i^{\perp}} k_{ij}(x,h,e') d e'\right) u(h^{-1}x) dh$$

Then $A_i \in \Psi_c^{-\infty}(\mathcal{H})$. The proof is completed.

Corollary (3.3). For any A of the form (3.1) is s multiplier of the C^*- algebras $\overline{\Psi_c^{-\infty}}(\mathcal{H})$.

Proof. Let $K \in \Psi_c^{-\infty}(\mathcal{H})$ be given in the form (2.1), for some $k \in C_c^{\infty}(G \times H)$. Then

$$AKu(x) = \int_{M\times H} a(x,m) k(\phi(m)^{-1}x, h) u((\phi(m)h)^{-1}x)\, dm\, dh$$

The submersion map $M \times H \to H$; takes $(m,h) \to \phi(m)h$. Similar the previous lemma $AK \in \Psi^{-\infty}$, and $KA \in \Psi^{-\infty}(\mathcal{H})$.

Corollary (3.4). Any operators $A_i \in \Psi_c^{-\infty}(\mathcal{H})$, is a multiplier of C^*- algebras $\overline{\Psi_c^{-\infty}}(\mathcal{H})$.

Proof. Let $K_i \in \Psi_c^{-\infty}(\mathcal{H})$ and $A_i \in \Psi_c^{-\infty}(\mathcal{H})$, for some $k_{ij} \in C_c^{\infty}(G \times H)$ and $a_{ij} \in C_c^{\infty}(G \times H)$, then

$$A_i K_i u(x) = \int_{M\times H} a_{ij}(x,m) k_{ij}(\phi(m)^{-1}x, h) u((\phi(m)h)^{-1}x)\, dm\, dh$$

So $A_i K_i \in \Psi^{-\infty}$. Similarly $K_i A_i \in \Psi^{-\infty}(\mathcal{H})$.

Theorem (3.5). Let $a \in C_c^{\infty}(G \times M)$ and $\phi: M \to G$ be smooth. If $C(\phi)$ has measure zero, then the operator A of the equation (6.80) is in $\overline{\Psi_c^{-\infty}}(\mathcal{H})$.

Proof. Fix $\epsilon > 0$. Choose an open neighbourhood U of $C(\phi)$ with measure less than ϵ. Let X_1 and X_2 be a smooth partition of unity on M with supp(X_1) $\subset U$ and supp(X_2) $\subset M \setminus C(\phi)$. Then $A = A_1 + A_2$ with

$$A_i u(x) = \int_M Xi(m) a(x,m) u(\phi(m)^{-1}x)\, dm.$$

Therefore $A_2 \in \Psi_c^{-\infty}(\mathcal{H})$ by the lemma (3.2) and by lemma (3.1)$\|A_1\| < \epsilon$.

Corollary (3.6): Let $a \in C_c^{\infty}(G \times M)$ and $\phi_{\propto}: M \to G$ be smooth. If $C(\phi_{\propto})$ has measure zero, then the operator $A_{\propto-1}$ of equation (6.80) is in $\overline{\Psi_c^{-\infty}}(\mathcal{H})$.

Proof: Fix $\epsilon > 0$. Choose an open neighbourhood U of $C(\phi_{\propto})$ with measure less than ϵ.

Let $X_{\propto}$ and $X_{\propto+1}$ be a smooth partition of unity on M with supp($X_{\propto}$) $\subset U$ and supp($X_{\propto+1}$) $\subset M \setminus C(\phi_{\propto})$. Then $A_{\propto-1} = A_{\propto} + A_{\propto+1}$ with

$$(A_{\propto-1})_i u(x) = \int_M (X_{\propto-1}) i(m) a(x,m) u(\phi_{\propto}(m)^{-1}x)\, dm.$$

Therefore $A_{\propto+1} \in \Psi_c^{-\infty}(\mathcal{H})$ by the lemma (3.2) and $\|A_1\| < \epsilon$ by lemma (3.1).

Corollary (3.7): Let M be a connected real-analytic manifold, and $\phi: M \longrightarrow H$ be a real-analytic map with image of nonzero measure, then from(3.1) the operator $A \in \overline{\Psi_c^{-\infty}}(\mathcal{H})$.

Proof: Let $= \binom{n}{d}$, $n = \dim(M)$ and $d = \dim(H)$ as the pervious.

In any analytic chart U of M, $C(\phi)$ is the zero set of the real-analytic function $J: U \to \mathbb{R}^N$ of equation (3.3). by Sard's theorem this function is not everywhere zero. Real-analyticity implies $J^{-1}(0)$ has measure zero.

4. The Product of Longitudinal Pseudodifferential Operators

We shall continue with the notations: g is the Lie algebra of a connected Lie group G; $\eta_{1,\dots,}\eta_{\propto}$ is a family of Lie subalgebras and η is the Lie algebra they generate; $\mathcal{H}_{1,\dots,}\mathcal{H}_{\propto}$ and $\mathcal{H}$ are the corresponding left- coset foliations of G.

$H_1, \dots, H_{\propto}$ are connected Lie subgroups of the connected Lie group G. Let H denote the subgroup they generate: $H = \{x_1 x_2 \dots x_k | Each\ x_i \in H_j\}$. This is a connected Lie algebra generated by $\eta_{1,\dots,}\eta_{\propto}$.

Theorem (4.1)

$$\prod_{i=1}^{\propto} \overline{\Psi_c^{-\infty}}(\mathcal{H}_i) \subseteq \overline{\Psi_c^{-\infty}}(\mathcal{H}).$$

Proof. Let $\prod_{i=1}^{\propto} H_i$ denote the set of products $\{\prod_{i=1}^{\propto} h_i | h_j \in H_j for\ all\ j\}$. We first prove the theorem under the assumption that $\prod_{i=1}^{\propto} H_i$ has nonzero measure in H.

Let $K_j = O_{PH}(k_j) \in \overline{\Psi_c^{-\infty}}(\mathcal{H}_j)$, with $k_j \in C_c^{\infty}(G \times H)$. By equation (2.1), then

$$\prod_{i=1}^{\propto} K_i u(x) = \int_{\prod_{i=1}^{\propto} H_i} \prod_{j=1}^{\propto} k_j\left((h_1 \dots h_{j-1})^{-1}x, h_j\right) u((\prod_{i=1}^{\propto} h_i)^{-1}x) \prod_{i=1}^{\propto} dh_i$$

.

Here, $\prod_{j=1}^{\propto} k_j\left((h_1 \dots h_{j-1})^{-1}x, h_j\right) \in (G \times (\prod_{i=1}^{\propto} H_i))$ and the map $(h_1, \dots, h_{\propto}) \to \prod_{i=1}^{\propto} h_i$ is real analytic, so (3.4) implies that $\prod_{i=1}^{\propto} K_i \in \overline{\Psi_c^{-\infty}}(\mathcal{H})$.

Now we drop the assumption on the product $\prod_{i=1}^{\propto} H_i$. Note that $\prod_{i=1}^{\propto} K_i$ is in the multiplier algebra of $\overline{\Psi_c^{-\infty}}(\mathcal{H})$. By (3.3) since $H_1, \dots, H_{\propto}$ generate G, there is some $n \in \mathbb{N}$ for which the set

$$(H_1H_2 \dots H_\propto H_\propto \dots H_2H_1)(H_1H_2 \dots H_\propto H_\propto \dots H_2H_1) \dots (H_1H_2 \dots H_\propto H_\propto \dots H_2H_1)$$

Has positive measure in G. From (2.2) for the adjoint of K_j, we see that $(K_1 \dots K_\propto K^*{}_\propto \dots K_1{}^*)^n \in \overline{\Psi_c^{-\infty}}(\mathcal{H})$. But from the theory of $C^* - algebra \prod_{i=}^{\propto} K_i$ in $\overline{\Psi_c^{-\infty}}(\mathcal{H})$. (see [Dav96,1.5.3]). This complete the proof.

6. Homogeneity and Generalization of Flag Varieties

From the definition (1.1) remember that a family of foliations $F_1, \dots, F_\propto$ of a manifold M is called loacally homogeneous if M admits an atlas of local diffeomorphisims from G, under fibrations map to $\mathcal{H}_1, \dots, \mathcal{H}_\propto$. A family $F_1, \dots, F_\propto$ generates a foliation $F \subseteq TM$, which in each chart is $\mathcal{H}$.

Example (5.1). Let us take the complex orthogonal group $O(n)$ and let $SO(n)$ be a subgroup of $O(n)$. Fix a system of positive roots $\sum^+$, with simple roots $\prod$. Fix a cartan subalgebra η. Let $m = \oplus_{\beta \in \sum^+} SO(n)_\beta$ and $\breve{m} = \oplus_{\beta \in \sum^+} SO(n)_{-\beta}$, and let $\breve{b} = \eta \oplus \breve{m}$ be the lower Borel subalgebra.

Let $M = O(n)/\overline{B}$ be the flag variety of (n) . for any subsimple roots of $\prod$, let $\langle S \rangle = \{ \beta \in \sum^+$ such that $\beta = \sum m_\gamma \gamma$, and $m_\beta \in \mathbb{N}\}$ be a set of positive roots spanned by S. and here $m_S = \oplus_{\beta \in \langle S \rangle} SO(n)_\beta$.

Let $Y_S = O(n)/\overline{P}s$ be the corresponding partial fag variety. and $\overline{P}s$ is the parabolic subalgebra $\breve{b} \oplus m_S$.

The expression

$$\prod_{i=1}^{r} \overline{\Psi_c^{-\infty}}(F_{S_i}) \subseteq \overline{\Psi_c^{-\infty}}(F_S)$$

Is obtained by theorem (1.2) with respect to $S_1, \dots, S_r \subseteq \prod$ and $S = \cup_{i=1}^{r} S_i$.

The fibration of M is denoted by F_S by fibres of the quotient map $\tau_S : M \to Y_S$.

A diffeomorphism map onto its range is defined by $\psi : N \to O(n) \longrightarrow O(n)/\overline{B} = M$. N is clearly equivariant. M is covered by such charts. By taken this chart, the fibres of τ_S pull back to the left cosets of $N_S = N \cap \overline{P}s$. Therefore $(F_S)_{S \subseteq \prod}$ is a family of locally homogeneous fibrations with structural data $(m_S)_{S \subseteq \prod} \leq m$.

7. Essential orthotypical

Let K_t and K_{t+1} be closed subgroups of a compact Lie group K_{t-1}. And $U: K_{t-1} \to B(H)$ be a unitary representation of K_{t-1} with finite multiplicities.

$$P\sigma = \sum_{\pi_{t-1} \in \sigma} P\pi_{t-1}$$

If π_{t-1} has no nontrivial invariant subspaces then π_{t-1} is an irreducible representation of a subgroup.

$K'_{t-1} of\ K_{t-1}$. $P\pi_{t-1}$ is the orthogonal projection onto $H_{\pi_{t-1}}$, the π_{t-1}- isotypical subspace of $U|_{K'_{t-1}}$.

$$Ps = \sum_{\pi_{t-1} \in S} P\pi_{t-1}$$

If $S \subseteq K'^{\wedge}_{t-1}$ is a set of K'_{t-1}- types, and $\pi_{t-1} \in K'^{\wedge}_{t-1}$ if $\partial \in K^{\wedge}_{t-1}$ then we can replace P_σ and P_π , $Ps = \sum_{\sigma \in S} P\sigma$ such that $\sigma \in S \subseteq K'^{\wedge}_{t-1}$.

The inner product of subspaces $H_t , H_{t+1} \leq H_{t-1}$ are defined by:
$< H_t, H_{t+1} > = \sup\{ < \xi_t, \xi_{t+1} > such\ that\ \xi_j \in H_j , \|\xi_j\| \leq 1 \}$

In case the representation H_{t-1} has finite K_{t-1} −multiplicities, $K_{t-1} = K_t \times K_{t+1}$,

A finite dimension intersection $H_{\pi_t \otimes \pi_{t+1}}$ will be gained by the isotypical subspaces H_{π_t} and $H_{\pi_{t+1}}$, with respect to the orthogonality of $H_{\pi_t} \cap (H_{\pi_{t+1}})^\perp$ and $H_{\pi_{t+1}} \cap (H_{\pi_t})^\perp$.

The orthogonality is true, whereas in general is not. For example, let $K_{t-1} = SU(N)$ is a group that has rank $(N - 1)$ and K_t and K_{t+1} be the subgroups obtained by embedding $SU(N-1)$ in the upper-left and lower- right corners of K_{t-1} respectively. There exist infinitely many irreducible $SU(N)$-representations which contain a nonzero K_t- fixed vector and a nonzero K_{t+1}- fixed vector.

However, there are limits to how far the idea of asymptotically orthogonal, that makes sense for any positive $\in$, there exist finitely many K- types in which the $K_t -$ and $K_{t+1} -$ fixed subspaces have inner product $< K_t, K_{t+1} > greater\ than\ \epsilon$.

Let V^σ be the vector space depend on an irreducible representation $\sigma \in K^{\wedge}_{t-1}$.

Definition (6.1). The subgroups K_t and K_{t+1} of K_{t-1} are essentially orthotypical if , for any $\pi_t \in K_t{}^{\wedge}, \pi_{t+1} \in K_{t+1}{}^{\wedge}$, and $\epsilon > 0$, there are only finitely many K_{t-1} - types $\sigma \in K_{t-1}{}^{\wedge}$ for which $\langle (V^\sigma)_{\pi_t}, (V^\sigma{}_{\pi_{t+1}}) \rangle \geq \epsilon$.

Lemma (6.2).Let K_1 and K_2 be closed subgroups of a compact Lie group K.
$i)$ K_1 and K_2 are essentially orthotypical
ii) For any $\pi_1 \in K_1{}^{\wedge}$ and $\pi_2 \in K_2{}^{\wedge}$, $P_{\pi_1}P_{\pi_2}$ is a compact operator on every unitary K – representation with finite multiplicities.
(i) and (ii) are equivalent.

Proof: Firstly$(i) \to (ii)$

Let S be the set of $\sigma \in K^{\wedge}$ for which $\langle (V^\sigma)_{\pi_1}, (V^\sigma{}_{\pi_2}) \rangle \geq \epsilon$. Then on V^σ for any $\sigma \notin S$, $\|p_{\pi_1}p_{\pi_2}\| < \epsilon$. Therefore $p_{\pi_1}p_{\pi_2} = ps\, p_{\pi_1}p_{\pi_2} + p\, s^\perp p_{\pi_1}p_{\pi_2}$.

Conversely $(ii) \to (i)$

Let $\epsilon > 0$. Fix any enumeration $\{\sigma_1, \sigma_2, \ldots\}$ of $K^\wedge$, and let $S_n = \{\sigma_1, \ldots, \sigma_n\}$. Put $H = \bigoplus_{n=1}^{\infty} V^{\sigma n}$. The projections ps_n on H converge strongly to 1 as $n \to \infty$, so by the compactness of $p_{\pi_1} p_{\pi_2}$ we have $\|p_{\pi_1} p_{\pi_2}\, p\, s_n{}^{\perp}\| < \epsilon$ for sufficiently large n.

Let σ be any $K - type$ not in S_n and assume that $\xi_j \in (V^\sigma)_{\pi_j}$ for $j = 1,2$, with $\|\xi_j\| \leq 1$. After including V^σ into H, we have

$$|\langle \xi_1, \xi_2 \rangle| = |\langle p\, s_n{}^{\perp} p_{\pi_1} \xi_1 \ , \quad p\, s_n{}^{\perp} p_{\pi_2} \xi_2 \rangle|$$
$$= |\langle \xi_1, p\, s_n{}^{\perp} p_{\pi_1} p_{\pi_2} \xi_2 \rangle| < \epsilon.$$

Thus $\langle (V^\sigma)_{\pi_1}, (V^\sigma)_{\pi_2} \rangle < \epsilon$ for all $\sigma \notin S_n$. The proof is completed.

Remark (6.3). From the previous proof of lemma (6.2) , in the case $(ii) \to (i) H = \bigoplus_{n=1}^{\infty} V^{\sigma n}$, the different representation which contains every $K - type$ could be used.

Corollary (6.4). Let K_t and K_{t+1} be closed subgroups of a compact Lie group K.

1) K_t and K_{t+1} are essentially orthotypical
2) For any $\pi_t \in K_t{}^\wedge$ and $\pi_{t+1} \in K_{t+1}{}^\wedge$, $P_{\pi_t} P_{\pi_{t+1}}$ is a compact operator on every unitary K_{t-1} – representation with finite multiplicities.

(i) and (ii) are equivalent.

Proof:

Firstly

$$(i) \quad \to (ii)$$

Let S be the set of $\sigma \in K_{t-1}{}^\wedge$ for which $\langle (V^\sigma)_{\pi_t}, (V^\sigma{}_{\pi_{t+1}}) \rangle \geq \epsilon$. Then on V^σ for any $\sigma \notin S$, $\|p_{\pi_t} p_{\pi_{t+1}}\| < \epsilon$. Therefore $p_{\pi_t} p_{\pi_{t+1}} = ps\, p_{\pi_t} p_{\pi_{t+1}} + p\, s^{\perp} p_{\pi_t} p_{\pi_{t+1}}$

Secondly:

$$(ii) \quad \to (i)$$

Conversely , let $\epsilon > 0$. Fix any enumeration $\{\sigma_1, \sigma_2, \ldots\}$ of $K_{(t-1)}{}^\wedge$, and let $S_n = \{\sigma_1, \ldots, \sigma_n\}$. Put $H = \bigoplus_{n=1}^{\infty} V^{\sigma n}$. The projections ps_n on H converge strongly to 1 as $n \to \infty$, so by the compactness of $p_{\pi_t} p_{\pi_{(t+1)}}$ we have $\|p_{\pi_t} p_{\pi_{(t+1)}}\, p\, s_n{}^{\perp}\| < \epsilon$ for sufficiently large n.

Let σ be any $K_{(t-1)} - type$ not in S_n and assume that $\xi_j \in (V^\sigma)_{\pi_j}$ for $j = 1,2$, with $\|\xi_j\| \leq 1$. After including V^σ into H, we have $|\langle \xi_t, \xi_{(t+1)} \rangle| = |\langle p\, s_n{}^{\perp} p_{\pi_t} \xi_t$, $p\, sn\perp p\pi(t+1)\xi(t+1) = \xi t, p\, sn\perp p\pi t p\pi(t+1)\xi(t+1) < \epsilon$.

Thus $\langle (V^\sigma)_{\pi_t}, (V^\sigma)_{\pi_{(t+1)}} \rangle < \epsilon$ for all $\sigma \notin S_n$.

Theorem (6.5): Let K be a compact Lie group and K_1 , K_2 are closed subgroups of K. K_1, K_2 are essentially orthotypical, if they generate K.

Proof: Fix $\pi_1 \in K_1^\wedge$, $\pi_2 \in K_2^\wedge$ and $\epsilon > 0$. Let $X_{\pi_j} \in C^\infty(K_j)$ denote the character of π_j. Let U be the left – regular representation of K on $L^2(K)$. By the orthogonality of characters,

$$P\pi_j\, f(x) = \int_{k_j} \overline{X_{\pi_j}(x)}\ f(k^{-1}x)\, dx$$

For any $f \in L^2(K)$. That is, $P\pi_j$ is a longitudinally smoothing operator for the coset fibration of $K_j \leq K$. Using (6.3), lemma (6.2) gives the result. The proof is completed.

Corollary (6.6): If K_{t-1} is a compact Lie group and K_t , K_{t+1} are closed subgroups of K_{t-1} ,then they are essentially orthotypical if K_t, K_{t+1} generate K_{t-1}.

Proof: Fix $\pi_t \in K_t^\wedge$, $\pi_{t+1} \in K_{t+1}^\wedge$ and $\epsilon > 0$. Let $X_{\pi_{(t-1)_j}} \in C^\infty(K_{(t-1)_j})$ denote the character of $\pi_{(t-1)_j}$. Let U be the left – regular representation of K_{t-1} on $L^2(K_{t-1})$. By the orthogonality of characters,

$$P\pi_{(t-1)_j}\, f(x) = \int_{k_j} \overline{X_{\pi_{(t-1)_j}}(x)}\ f\left(k_{(t-1)}{}^{-1}x\right) dx$$

For any $f \in L^2(K_{t-1})$. That is, $P\pi_{(t-1)_j}$ is a longitudinally smoothing operator for the coset fibration of $K_{(t-1)_j} \leq K_{t-1}$.the result is given by lemma(6.3) and remark (6.2) .

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