On Essential Numerical Range and Davis-Wielandt Shell of Hilbert Space Operators

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Abstract: ***Let the numerical range of the operator T be denoted by W(T), the essential numerical range by $W_e(T)$ and the Davis-Wielandt shell by DW(T). We discuss the equality of the essential numerical range and the numerical range of the operator T in a complex infinite dimensional Hilbert space H. The results obtained are used to study the instance when the essential numerical range can replace the numerical range in the Davis-Wielandt shell.***

Keywords: Numerical range, Essential numerical range, Davis-Wielandt shell

1. Introduction

Let H be an infinite dimensional Hilbert space and $B(H)$be the algebra of bounded linear operators acting on H. The numerical range of an operator $T \in B(H)$ is defined by;

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\} \quad \text{See [1, 2, 3]}$$

which is useful for studying operators. In particular, the geometrical properties of the numerical range often provide useful information about the algebraic and analytic properties of the operator T. For instance, $W(T) = \mu$ if and only if $T = \mu I$; $W(T)$ is real if and only if $T = T^*$, $W(T)$ has no interior points if and only if there are complex numbers, a and b with $a \neq 0$ such that $aT + bI$ is self-adjoint. Moreover, the closure of $W(T)$, denoted by $\overline{W(T)}$, always contains the spectrum of T denoted by $\sigma(T)$. See[1]

Let $K(H)$ denote the set of compact operators on H and $\pi : B(H) \to B(H)/K(H)$ be the canonical quotient map. The essential numerical range of T, denoted by $We(T)$ is the set;

$$W_e(T) = \cap\, \overline{W(T+K)} \quad \text{See[4]}$$

where the intersection runs over the compact operators $K \in K(H)$. To continue our study, we develop an alternative definition of the essential numerical range and note how the corresponding definition relates to the definition of the numerical range. The alternative definition states that; for an operator $T \in B(H)$, $\lambda \in W_e(T)$ if and only if there exists an orthonormal sequence $\{x_n\}_{n\geq 1}$ such that $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = \lambda$. Note that an operator T is compact if and only if $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = 0$ for every orthonormal set $\{x_n\}_{n\geq 1}$. The essential numerical range is non-empty, closed and convex. The non-emptiness of the essential numerical range follows from the fact that for any orthonormal sequence $\{x_n\}_{n\geq 1}$, $(\langle Tx_n, x_n\rangle)_{n\geq 1}$ is a sequence of complex numbers bounded by $\|T\|$ and thus has a convergent subsequence. Thus $W_e(T)$ is non-empty. Moreover, it is clear that $W_e(T)$ is closed and convex being the intersection of closed, convex sets.

The Davis-Wielandt shell is a generalization of the classical numerical range and it is defined by;

$$DW(T) = \{\langle Tx, x\rangle, \langle Tx, Tx\rangle : \|x\| = 1,\ x \in H\} \quad \text{See [5, 6, 7]}$$

The first co-ordinate is the classical numerical range, $W(T)$, while the second co-ordinate denotes the numerical range of the operator T^*T , i.e. $W(T^*T)$

So the Davis-Wielandt shell captures more information about the operator T than the numerical range of the operator. For instance, in the finite dimensional case, normality of operators can be completely determined by the geometrical shape of their Davis-Wielandt shells. However we restrict our study to an infinite dimensional Hilbert space. This is because the essential numerical range does not make sense in the finite dimensional case.

2. Main Results

We begin by outlining some theorems, corollaries and proofs that are important in achieving the objective of this paper.

Theorem 1. [Anderson-Stampfli]. See [8]
Let $T \in B(H)$. Then T is of the form $T = \lambda I_H + K$ where K is compact if and only if $W_e(T) = \{\lambda\}$
Proof
It is clear that;
$W_e(\lambda I_H + K) = W_e(\lambda I_H)$
$= \lambda W_e(I_H)$
$= \{\lambda\}$
which completes one direction.

Suppose that $T \in B(H)$ is such that $W_e(T) = \lambda$ for some complex number λ. Then, $W_e(T - \lambda_H) = \{0\}$. Thus $\|\pi(T - \lambda I H = 0$. Therefore $T - \lambda IH$ is compact and thus there exists a compact operator $K \in K(H)$ such that $T = \lambda I_H + K$ as desired.

Theorem 2
Let $T \in B(H)$ and let $\lambda \in \mathbb{C}$. The numerical range of T, $W(T) = \{\lambda\}$ if and only if $T = \lambda I_H$

Proof

It is clear that $W(\lambda I_H) = \{\lambda\}$ for any $\lambda \in \mathbb{C}$
Suppose that $W(T) = \{\lambda\}$ for some $T \in B(H)$. Then for all $x \in H$ with $\|x\| = 1$, we have

$\langle(\lambda I_H - T)x, x\rangle = \lambda - \langle Tx, x\rangle$
$= \lambda - \lambda$
$= 0$

Hence $\langle(\lambda I_H - T)x, x\rangle = 0$ for all $x \in H$ and thus $\lambda I_H - T = 0$ See [8]

Lemma 1. [Characterization of the Essential numerical range] See [9]
Let $T \in B(H)$. Each of the following conditions is necessary and sufficient in order that $\lambda \in W_e(T)$

(1) $\langle Tx_n, x_n\rangle \to \lambda$ for some sequence x_n of unit vectors such that $x_n \rightharpoonup 0$ weakly
(2) $\langle Te_n, e_n\rangle \to \lambda$ for some orthonormal sequence (e_n)

The relationship between the numerical range and the essential numerical range is given in the result by John Lancaster.

Theorem 3 [John Lancaster theorem]

For $T \in B(H)$ we have $\overline{W(T)} = conv\{W(T) \cap W_e(T)\}$ (see, [10])

Proof
Clearly, $W(\alpha T + \beta) = \alpha W(T) + \beta$ for all complex α and β. Therefore by rotation and translation, we can assume that $\overline{W(T)}$ is contained in the closed right half plane and $0 \in Ext(\overline{(W(T)}) - W(T)$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors of H such that $\langle Tx_n, x_n\rangle \to 0$. By weak sequential compactness of the unit ball of H, we can assume that $\{x_n\}_{n=1}^{\infty}$ converges weakly to $x \in H$ with $\|x\| \leq 1$. We prove that x is the 0 vector, and hence $0 \in W_e(T)$

If $\|x\| = 1$, then $x_n \to x$ strongly. But;

$$\begin{aligned}|\langle Tx, x\rangle| &\leq |\langle T(x - x_n), x\rangle| + |\langle Tx_n, x - x_n\rangle| + |\langle Tx_n, x_n\rangle| \\ &\leq |\langle x - x_n, T^*x\rangle| + \|T\|\,\|x - x_n\| + |\langle Tx_n, x_n\rangle| \\ &\to 0\end{aligned}$$

Hence $\langle Tx, x\rangle = 0$ and $0 \in W(T)$.

So assume $0 < \|x\| < 1$. Clearly, the operator ReT is positive since $W(T)$ is contained in the closed right half plane. Then;

$$\begin{aligned}\left\|(ReT)^{\frac{1}{2}}x_n\right\|^2 &= \langle (ReT)x_n, x_n\rangle \\ &= Re\langle Tx_n, x_n\rangle \to 0\end{aligned}$$

So $\|(ReT)x_n\| \to 0$. This clearly yields $Re\langle Tx, x\rangle = 0$ so $\langle Tx, x\rangle$ is purely imaginary. On the other hand;

$$\begin{aligned}\langle T(x - x_n), x - x_n\rangle &= \langle Tx, x - x_n\rangle - \langle Tx_n, x\rangle + \langle Tx_n, x_n\rangle \\ &\to -\langle Tx, x\rangle\end{aligned}$$

And
$\|x - x_n\|^2 = 1 - 2Re\langle x - x_n, x\rangle - \|x\|^2$, so $\langle Ty_n, y_n\rangle \to -\langle Tx, x\rangle \,/\, (1 - \|x\|^2)$ where $y_n = \frac{x - x_n}{\|x - x_n\|}$

Thus we have produced a non-zero purely imaginary points in $\overline{W(T)}$ which lie in the upper and lower half planes. However this implies that 0 is a non-extreme point of $\overline{W(T)}$ thus completing the proof of the inclusion. The equality follows from the inclusion by the Krein-Milman theorem

This theorem by John Lancaster [10] is reinforced by the theorem by J. Christophe (see, [11])

Theorem 4 [J. Christophe theorem] **See [11]**
Let T be an operator, then:
(1) If $W_e(T) \subset W(T)$ then $W(T)$ is closed.
(2) There exist normal finite rank operators R of arbitrarily small norm such that $W(T + R)$ is closed.

Theorem 5
For a compact operator T on an infinite dimensional Hilbert space,
(1) If $0 \in W(T)$ then $W(T)$ is closed
(2) If 0 is not in $W(T)$ then 0 is an extreme point of $\overline{W(T)}$, and $\overline{W(T)} \setminus W(T)$ consists at most of line segments in $\delta W(T)$ which contain 0 but no other extreme point of $\overline{W(T)}$ See [12]

Proof
If λ is a cluster point of $W(T)$, then there exists a sequence $\{\langle Tx_n, x_n\rangle\}$, where $\|x_n\| = 1$ for all n, converging to λ. Since the unit ball in a Hilbert space is weakly sequentially compact, there exists a subsequence $\{x_{nk}\}$ which is weakly convergent to an x where $\|x\| \leq 1$. Since T is a compact operator, $\{Tx_{nk}\}$ is strongly convergent to Tx. However,

$$\begin{aligned}|\langle Tx_{nk}, x_{nk}\rangle - \langle Tx, x\rangle| &= |\langle Tx_{nk}, x_{nk}\rangle - \langle Tx, x_{nk}\rangle| \\ &\quad + |\langle Tx, x_{nk}\rangle - \langle Tx, x\rangle| \\ &\leq \|x_{nk}\|\,\|Tx_{nk} - Tx\| \\ &\quad + |\langle x_{nk}, Tx\rangle - \langle Tx, x\rangle|\end{aligned}$$

Therefore $\{\langle Tx_{nk}, x_{nk}\rangle - \langle Tx, x\rangle\}$ converges to $\langle Tx, x\rangle$ and so $\langle Tx, x\rangle = \lambda$. If $\lambda \neq 0$, clearly $x \neq 0$. So,

$$\frac{\lambda}{\|x\|^2} = \left(T\frac{x}{\|x\|}, \frac{x}{\|x\|}\right) \in W(T)$$

Since $\|x\| \leq 1$ it follows that \$\lambda\$ belongs to the interval $\left(0, \frac{\lambda}{\|x\|^2}\right]$, using an obvious notation for line segments in the complex plane.

(1) If $0 \in W(T)$ we have from convexity of $W(T)$ that $\lambda \in W(T)$ and so $W(T)$ is closed.
(2) If λ, $(\lambda \neq 0)$ is an extreme point of the intersection of a ray from 0 with $\overline{W(T)}$, then, since $0 \in \overline{W(T)}$, we have that $\lambda = \frac{\lambda}{\|x\|^2}$ and so $\|x\| = 1$ and $\lambda \in W(T)$.

If 0 is in the interior of a line segment in $\delta W(T)$ then the intersection of the line with $\overline{W(T)}$ has two extreme points and these are in $W(T)$; but these are on either side of 0, so from the convexity of $W(T)$, we have that $0 \in W(T)$; we conclude that if 0 is not in $W(T)$, then 0 is an extreme point of $\overline{W(T)}$.

If 0 does not belong to $W(T)$ and 0 does not belong to a line segment $\delta W(T)$ then every point in $\delta W(T) \setminus \{0\}$ is an extreme point of the intersection of a ray from 0 with $\overline{W(T)}$, and so $\overline{W(T)} \setminus W(T) = \{0\}$.

Corollary
For a compact operator T on a non-separable Hilbert space, $W(T)$ is closed. See [12]

Proof
Since the range of a compact operator is separable, 0 is an eigen-value of T, so $0 \in W(T)$.

From the $\boldsymbol{Theorem}$ 1 and $\boldsymbol{Theorem\ 2}$ discussed above, it can be noted that the numerical range and the essential numerical range strike an equality if and only if the operator $T = \lambda I_H$ for any complex λ. Thus it is interesting to discuss how the Davis-Wielandt shell will look like if the operator T is replaced by λI_H. As outlined in the introduction part of this paper, the Davis-Wielandt shell is defined by;

$$DW(T) = \{\langle Tx, x\rangle, \langle Tx, Tx\rangle \ : \ \|x\| = 1\,,\ x \in H\}$$

Thus, if $T = \lambda I_H$, we have the Davis-Wielandt shell as;

$$DW(\lambda I_H) = \{\langle(\lambda I_H)x, x\rangle, \langle(\lambda I_H)x, (\lambda I_H)x\rangle \ : \ \|x\| = 1\,,\ x \in H\}$$

If we let $I = I_H$ then we obtain $\lambda I = \lambda I_H$. Substituting λI for λI_H in the above definition of the $Davis - Wielandt\ shell$, we get;

$$DW(\lambda I) = \{\langle(\lambda I)x, x\rangle, \langle(\lambda I)x, (\lambda I)x\rangle \ : \ \|x\| = 1\,,\ x \in H\}$$

We first note that the first co-ordinate of $DW(\lambda I)$ is $W(\lambda I)$ while the second co-ordinate is $W(\lambda^* \lambda)$

If $T = \lambda I$ then the $essential\ numerical\ range$ behaves in the same way as the first co-ordinate of the $Davis - Wielandt\ shell$. That is, for the essential numerical range, we have;

$$\begin{aligned} W_e(\lambda I) &= \lambda W_e(I) \\ &= \lambda \end{aligned}$$

This is as a result of **Theorem 1** by Anderson and Stampfli.

The first co-ordinate (numerical range) of the $Davis - Wielandt\ shell$ which is equal to the essential numerical range when $T = \lambda I$ is given as;

$$\begin{aligned} W(\lambda I) &= \langle(\lambda I)x, x\rangle \\ &= \lambda\langle Ix, x\rangle \\ &= \lambda \end{aligned}$$

This follows from **Theorem 2** above

As for the second co-ordinate of the $Davis - Wielandt\ shel$, we have;

$$\begin{aligned} W(\lambda^* \lambda) &= \langle(\lambda I)x, (\lambda I)x\rangle \\ &= \langle\big(\lambda\bar{\lambda} I\big)x, x\rangle \\ &= \langle(|\lambda|^2)Ix, x\rangle \\ &= |\lambda|^2\langle Ix, x\rangle \\ &= |\lambda|^2 \end{aligned}$$

Following the computation of the first and second co-ordinates of the $Davis - Wielandt\ shell$. Thus we obtain;

$$DW(\lambda I) = \{\lambda, |\lambda|^2\}$$

3. Conclusion

The numerical range and the essential numerical range of an operator on a complex, infinite dimensional Hilbert space are equal when the operator is λI where $\lambda \in \mathbb{C}$ and I is the identity operator. Thus this equality has helped in investigating what the $Davis - Wielandt\ shell$ looks like when the first co-ordinate is substituted with the $essential\ numerical\ range$ of the operator. Thus our conclusion that $DW(\lambda I) = \{\lambda, |\lambda|^2\}$

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