International Journal of Science and Research (IJSR)
ISSN: 2319-7064
Index Copernicus Value (2016): 79.57 | Impact Factor (2017): 7.296

Zeroforcing and Power Domination for a Graph of Cartesian Products of Two Cycles m ≥ n ≥ 3

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Abstract: ***The power domination number arose from the monitoring of electrical networks, and methods for its determination have the associated application. The zero forcing number arose in the study of maximum nullity among symmetric matrices described by a graph (and also in control of quantum systems and in graph search algorithms). There has been considerable effort devoted to the determination of the power domination number, the zero forcing number, and maximum nullity for specific families of graphs. In this paper we exploit the natural relationship between power domination and zero forcing to obtain results for the power domination number of Cartesian products and the zero forcing number of lexicographic products of graphs. We also establish results for the zero forcing number and maximum nullity of Cartesian products graphs.***

Keywords: Minimum rank, Matrices, Placement, Sets, Zero Forcing, Power domination

1. Introduction

Electric power companies need to monitor the state of their networks continuously to prevent system failure; a standard method is to place Phase Measurement Units (PMUs) at selected locations in the system, called electrical nodes or buses, where transmission lines, loads, and generators are connected. A PMU placed at an electrical node measures the voltage at the node and all current phasors at the node [1]; it also provides these measurements at other vertices or edges according to certain propagation rules. Due to the cost of a PMU, it is important to minimize the number of PMUs used while maintaining the ability to observe the entire system. This problem was first modeled using graphs by Haynes et al. in [2], where the vertices represent the electric nodes and the edges are associated with the transmission lines joining two electrical nodes (see Section 1.3 for the details and formal definitions). In this graph model, the power domination problem consists of finding a minimum set of vertices from where the entire graph can be observed according to certain rules; these vertices provide the locations where the PMUs should be placed in order to monitor the entire electrical system at minimum cost. Since its introduction in [2], the power domination number and its variations have generated considerable interest.

As was pointed out in [3], a careful examination of the definition of power domination leads naturally to the study of zero forcing. The zero forcing number was introduced in [4] as an upper bound for the maximum nullity of real symmetric matrices whose nonzero pattern of off-diagonal entries is described by a given graph, and independently by mathematical physicists studying control of quantum systems [5], and later by computer scientists studying graph search algorithms [6]. The study of maximum nullity, or equivalently, maximum multiplicity of an eigenvalue, was motivated by the inverse eigenvalue problem of a graph (see [7] and [8] for surveys of results on maximum nullity and zero forcing containing more than a hundred references). Since its introduction, zero forcing has attracted the attention of a large number of researchers who find the concept useful to model processes in a broad range of disciplines. There has been extensive work on determining the values of the power domination number and the zero forcing number for families of graphs. It is worth noting that the problem of deciding whether a graph admits a power dominating set of a given size is NP-complete [2], as is the analogous problem for zero forcing [9].

In Section 2 we establish results for the zero forcing number and maximum nullity of some families of Cartesian products. A zero forcing lower bound has not previously been applied to graphs other than the hypercube in [3]. Note that in [10] the author claimed to have obtained the first general lower bound for the power domination number, but a family of counterexamples to his claim was given in [11]. The remainder of this introduction contains formal definitions of power domination and zero forcing, graph terminology, and matrix terminology.

1.1 Power domination and zero forcing definitions

A graph G = (V, E) is an ordered pair formed by a finite nonempty set of *vertices* V = V (G) and a set of *edges* E = E(G) containing unordered pairs of distinct vertices (that is, all graphs are simple and undirected). The order of G is denoted by |G| := |V (G)|. We say the vertices *u* and *v* are *adjacent* or are *neighbors*, and write *u* ∼*v*, if {*u, v*} ∈ E. For any vertex *v*∈V , the *neighborhood* of *v* is the set N(*v*) = {*u*∈ V : *u* ∼ *v*} (or $N_G(v)$ if G is not clear from context), and the closed *neighborhood* of *v* is the set N[*v*] = N(*v*) ∪ {*v*}. Similarly, for any set of vertices S, N(S) = $U_{v\in S}$N(*v*) and N[S] =$U_{v\in S}$ N[v].

A vertex *v* in a graph G is said to *dominate* itself and all of its neighbors in G. A set of vertices S is a *dominating* set of G if every vertex of G is dominated by a vertex in S. The minimum cardinality of a dominating set is the *domination number* of G and is denoted by γ(G).

In [2] the authors introduced the related concept of power domination by presenting propagation rules in terms of vertices and edges in a graph. In this paper we use a simplified version of the propagation rules that is equivalent to the original, as shown in [12]. For a set S of vertices in a graph G, Define P D(S) ⊆ V (G) recursively:

1. P D(S):= N[S] = S ∪ N(S).
2. While there exists v∈ P D(S) such that |N(v) ∩ (V (G) \ P D(S))| = 1: P D(S) := P D(S) ∪ N(v).

We say that a set S ⊆ V (G) is a *power dominating* set of a graph G if at the end of the process above P D(S) = V (G). A *minimum power dominating set* is a power dominating set of minimum cardinality, and the power domination number, γ_P(G) of G is the cardinality of a minimum power dominating set.

The concept of zero forcing can be explained via a coloring game on the vertices of G. The *color change rule* is: If u is a blue vertex and exactly one neighbor w of u is white, then change the color of w to blue. We say u forces w and denote this by $u \longrightarrow w$. A *zero forcing set* for G is a subset of vertices B such that when the vertices in B are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of G blue. A *minimum zero forcing set* is a zero forcing set of minimum cardinality, and the *zero forcing number*Z(G), of G is the cardinality of a minimum zero forcing set. The next observation is the key relationship between the two concepts.

1.2 Graph definitions and notation

Let n be a positive integer. The *path* of order n is the graph $P_n$with V (P_n) = {x_i : 1 ≤ i ≤ n} and E(P_n) = {{x_i , x_i+1} : 1 ≤ i ≤ n − 1}. If n ≥ 3, the *cycle* of order *n* is the graph $C_n$with V(C_n) = {x_i : 1 ≤ i ≤ n} and E(C_n) = {{x_i , x_i+1} : 1 ≤ i ≤ n − 1} ∪ {{x_n, x_1}}. The *complete graph* of order *n* is the graph $K_n$with V (K_n) = {x_i: 1 ≤ i ≤ n} and E(K_n) = {{x_i,x_j} : 1 ≤ i <j ≤ n}.

Let G = (V (G), E(G)) and H = (V (H), E(H)) be disjoint graphs. All of the following products of G and H have vertex set V (G) × V (H). The *Cartesian product* of G and H is denoted by G□H; two vertices (g, h) and (g' ,h') are adjacent in G □H if either (1) g =g' and {h,h'} ∈ E(H), or (2) h = $h'$and {g,g'} ∈ E(G).

The *lexicographic product* of G and H is denoted by G ∗ H; two vertices (g,h) and (g',h') are adjacent in G∗H if either (1) {g,g'}∈ E(G), or (2) g = $g'$and {h,h'} ∈ E(H).

Note that H ×G ≅ G×H and H□G ≅ G□H, whereas H ∗ G need not be isomorphic to G ∗ H.

1.3 Matrix definitions and notation

Let S_n(R) denote the set of all n× n real symmetric matrices. For A = [a_{ij}] ∈S_n(R), the graph of A, denoted by G(A), is the graph with vertices {1, . . . , n} and edges {{i, j} :a_{ij}= 0, 1 ≤ i < j ≤ n}. More generally, the graph of A is defined for any matrix that is combinatorially symmetric, i.e.,a_{ij}= 0 if and only if a_{ji} = 0. Note that the diagonal of A is ignored in determining G(A). The set of symmetric matrices described by a graph G of order n is defined as S(G) = {A ∈S_n(R): G(A) = G}. The maximum nullity of G is M(G) = max{null A : A ∈ S(G)}, and the minimum rank of G is mr (G) = min{rank A : A ∈ S(G)}; clearly M(G) +mr(G) = |G|. The term 'zero forcing' comes from using the forcing process to force zeros in a null vector of a matrix A ∈ S(G), implying the following key relationship:

Proposition 1.4 For a graph G, M(G) ≤ Z(G).
Although the relationship M(G) ≤ Z(G) was originally viewed as an upper bound for the maximum nullity of a graph, we will repeatedly use this inequality to provide a lower bound for the zero forcing number. A standard way to construct matrices of maximum nullity for a Cartesian product or of graphs is to use the Kronecker or tensor product of matrices. Let A be an n × n real matrix and B be an m × m real matrix. Then A ⊗ B is the n × n block matrix whose ijth block is the m × m matrix a_{ij}B. It is known that (A ⊗ B) T = A^T ⊗B^T BT and rank (A ⊗ B) = (rank A)(rank B). If A ∈S(G), B ∈ S(H), |G| = n, and |H| = m, then A ⊗ I_m − I_n ⊗ B ∈S(G□H). If x is an eigenvector of A for eigenvalue λ and y is an eigenvector of B for eigenvalue μ, then x ⊗ y is an eigenvector of A ⊗I_m − I_n ⊗ B for eigenvalue λ − μ. Since a real symmetric matrix has an orthonormal basis of eigenvectors, the multiplicity of λ−μ is at least mult A^λ mult B^μ. If A ∈ S(G) and B ∈ S(H) and the diagonal entries of A and B are all zero ,then A⊗B∈ S(G × H). Define M_0(G) = {A ∈ R n×n : G(A) = G and a_{ii} = 0 for i = 1, . . . , n}; in contrast to a matrix in S(G), a matrix in M_0(G) need not be symmetric but must have a zero diagonal and be combinatorially symmetric. If A ∈ M_0(G) and B ∈ M_0(H) then A ⊗ B ∈M_0(G × H).

Proposition 1.5 Let G be a graph that has an edge. Then $\left\lceil \frac{Z(G)}{\Delta(G)} \right\rceil$ ≤ γ_P (G) and this bound is tight.

Proof. Choose a minimum power dominating set {u_1 ,u_2 ……. u_t}, so t = γ_P (G), and observe that $\sum_{i=1}^{t} \deg u_i$ ≤ tΔ(G). If G has no isolated vertices, Each isolated vertex of G contributes one to both the zero forcing number and the power domination number, hence the result still holds. Since Z(K_n) = Δ(K_n) = n − 1 and γ_P(K_n) = 1, the bound is tight.

Notes:
For a graph G with no edges, Z(G) = γ_P (G) = γ(G) = |G|, so we focus our attention on graphs that have at least one edge.

2. Zero Forcing for Graph Products

In this section we develop a tool for bounding the zero forcing number of Cartesian products of graphs and apply it to compute the zero forcing number and the maximum nullity of the Cartesian product of a complete graph with a path or a cycle.

2.1 Cartesian products

We determine the zero forcing number and maximum nullity of the Cartesian product of two cycles for m ≥ n ≥ 3.

Theorem 2.2. For m ≥ n ≥ 3

$$\gamma_P(C_n \square C_m) = Z(C_n \square C_m) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & if\ n \not\equiv 2\, mod 4 \\ \frac{n}{2}+1, & if\ n \equiv 2\, mod 4 \end{cases}$$

Proof

For m = n ≥ 3,[8 by Theorem 2.18]

$M(C_n \square C_m)$= $Z(C_n \square C_m)$ =n+2[$\frac{n}{2}$], so, $M(C_n \square C_m)$ = $Z(C_n \square C_m)$=$\gamma_P(C_n \square C_m)$=$\frac{n}{2}$ for n is odd

$M(C_n \square C_m)$ = $Z(C_n \square C_m)$ = $\gamma_P(C_n \square C_m)$=$\frac{n}{2}$ +1 for n is even.

We assume that m > n ≥ 3 It is shown in [2,corollary 2.8] that the vertices of two consecutive cycles $C_n$form a zero forcing set,soZ$(C_n \square C_m)$ ≤ 2n.

To complete the proof we construct a matrix in S$(C_n \square C_m)$ with nullity 2n , so 2n≤ M$(C_n \square C_m)$ ≤ Z$(C_n \square C_m)$ ≤2n.

Let k =$\left\lceil \frac{n}{2} \right\rceil$. Let A be the matrix obtained from the adjacency matrix of C_n by changing one pair of symmetrically placed entries from 1 to −1. Then as discussed in the proof of [2, Theorem 3.8], the distinct eigenvalues of A are μ_i= 2cos$\frac{\pi(2i-1)}{n}$, i =1, . . . , k, each with multiplicity 2 except μ_k, which has multiplicity 1 when n is odd. Assuming that there exists a matrix B ∈S(C_m) such thatμ_i is an eigenvalue of B with multiplicity 2 for i = 1, . . . , k, it follows that A $\otimes I_M$−$I_n \otimes$ B has eigenvalue zero with multiplicity at least 2n, because every eigenvalue of A has a corresponding eigenvalue of B with multiplicity 2.

It remains to establish the existence of a matrix B ∈S(C_m) such that μ_i is an eigenvalue of B with multiplicity 2 for i = 1, . . . , k.In [14,Theorem 4.3] Ferguson showed that for any set of $\frac{n}{2}$+ 1 distinct real numbersλ_1> · · · >$\lambda_{\frac{n}{2}+1}$, there is a matrixB ∈ S($C_{\frac{n}{2}+1}$ +1)having eigenvalues λ_1, . . . , $\lambda_{\frac{n}{2}+1}$with multiplicities 1, 2, . . . , 2, respectively.

In 15, Theorem 3.3] Fernandes and da Fonseca extended Ferguson's method to show that for any set of$\frac{n}{2}$distinct real numbers λ_1> · · · >$\lambda_{\frac{n}{2}}$ there is a matrix B ∈ S($C_{\frac{n}{2}}$) having eigenvalues λ_1, . . . ,$\lambda_{\frac{n}{2}}$withmultiplicities 2, . . . , 2. For even m, choose$\lambda_{\frac{n}{2}}$ =$\mu_i$for i = 1, . . . , k.For odd m, since m > n we can choose$\lambda_{\frac{n}{2}+1}$= $\mu_i$for i = 1, . . . , k.

Example 1 Form ≥ n ≥ 3

$$\gamma_P(C_n \square C_m) = Z(C_n \square C_m) = \begin{cases} \frac{n}{2}, & if\ n \not\equiv 2\, mod 4 \\ \frac{n}{2}+1 & if\ n \equiv 2\, mod 4 \end{cases}$$

Solution

For m = n = 3 [17 by Theorem 3.26]

The three vertices on the outer cycle form a zero forcing set, so M(P)≤ Z(P) ≤3. The triangular graph is strongly regular, so by[17 by Theorem 3.25]M(P)≥3 thus we haveM(P)= 3 mr(p)=3.

3. Conclusion

In this paper, we have discussed theZero forcing and power domination for a graph of Cartesian products of two cycles m ≥ n ≥ 3. In future we propose to extend this work with many graphs products of two cyclesm ≥ n ≥ 3

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