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Methods for Determining Fractal Dimensions

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Abstract: ***In this paper, A is considered to be a non-empty compact subset of a metric space X. For each small positive number δ, the minimum number of closed balls of radius δ needed to cover A is denoted by N(A,δ). The objective is to obtain the fractal dimension of A, denoted by D(A), which is the limit as δ tends to zero of the ratio $\ln N(A,\delta)/\ln(1/\delta)$ if this limit exists. Three main methods are discussed with relevant geometrical and theoretical ideas leading to determination of fractal dimension of the given set. The methods showed that the fractal dimension, D(A), exists for all compact subset of n-dimensional Euclidean space, $\mathbb{R}^n$, where D(A) is equal to both the Hausdorff-Besicovitch dimension, D_H and the Packing dimension, D_P due to self-similarity property of fractals.***

Keywords: Fractals, Fractal Dimension, Hausdoff-Besicovitch Dimension, Packing Dimension and Self-similarity

1. Introduction

Our present knowledge of fractals is due to an increasing interest in their behaviour. Some of the basic properties of objects with anomalous dimensions were noticed and investigated at the beginning of last century mainly by Hausdorff and Besicovitch. The relevance of fractals to Physics, Mathematics and many other fields was pointed out by Mandelbrot, who demonstrated the richness of fractal geometry and presented further important results in his research works on the subject [25]. [6] considered certain fractal subsets. The concept of fractals has caught the imagination of scientists in many fields of study. Some of these scientist include:- Voss [36,37], Oppenheimer [31], Feder [18], Peitgen and Richter [32], Edgar [15]- Vicsek [35], Hutchinson [22], Barnsley [7,8], Rushing [33], Barlow and Taylor [5,6], Barnsley and Demko [9]. Besicovitch [10,11,12], Sergio [34], Khoshnevisan [13]. Lindstrom [24] find the famous scientist (in fractal geometry) Mandelbrot [25,26,27,28,29]. The common features of a fractal are its dimension and self-similarity. The idea of dimension is fundamental in the study of fractals [25]. The fractal dimension, D, has a linear relationship with Hurst parameter, H. [1] reviewed various fractal dimensions while [2] gave further fractal computation of certain boundary sets. [3] presented the equality of the covering and topological dimensions for the linear model of Nigerian map, while the estimation of Hurst parameter, H, as a useful relevant self-similar process in the interpretation of certain financial market dynamics obtained in a long-range dependent phenomenon was given by [4].

2. Definitions

In this section, we give some brief explanations about ideas relevant to the understanding of fractal concepts

2.1 The Hausdorff-Besicovitch Dimension

The Hausdorff-Besicovitch dimension, D_H, of the set S in a metric space, that is, in any space (not necessarily Euclidean), is the critical dimension $d = D_H$ obtained in the limit $\delta \to 0$ for which the measure M_d changes from zero to infinity:

$$M_d(S) = \sum_{i=1}^{N(\delta)} \gamma_i(d)\delta^d = \gamma(d)N(\delta)\delta^d \to \begin{cases} 0, & d > D_H \\ \infty, & d < D_H \end{cases}$$

where $\gamma_i(d)$ are geometrical factors such that $\gamma_1(d) = 1$ for lines, squares and cubes;

$\gamma_2(d) = \dfrac{\pi}{4}$ for disks and $\gamma_3(d) = \dfrac{\pi}{6}$ for spheres [21].

2.2 Packing Dimension

Let F be a Borel set in a metric space. There exist $s_0 \in [0,\infty)$ and $s > 0$ such that

$$P^s(A) = \begin{cases} 0, & \forall\ s > s_0 \\ \infty, & \forall\ s < s_0 \end{cases}$$

where $P^s(A)$ is a measure called the s-dimensional packing measure of A. This value s_0 is called the packing dimension of the set A, denoted by $D_P(A)$ i.e. $s_0 = D_P(A)$. For self-similar sets, $D_H = D_P$ [15].

2.3 Covering Dimension

Let X be a metric space. Let $n \geq -1$ be an integer. We denote covering dimension by D_C. We say that X has covering dimension $D_C \leq n$ iff every finite open covering of X has an open refinement with order $\leq n$. $D_C = n$ iff $D_C \leq n$ but not $n-1$. If $D_C \leq n$, $\forall n \notin \mathbb{N}$, then $D_C = \infty$, see [15], [21], [13], [16] and [19]. For example, if $D_C = 1$, then the open covering {S} is refined by a covering of order -1, which is necessarily $\{\phi\}$. This implies $S = \phi$. So $D_C = -1$ iff $S = \phi$.

2.4 Fractal

A set $A \subseteq \mathbb{R}^n$ is a fractal (in the sense of Taylor) if and **o**nly if (iff)

$$D_C(A) < D_H(A) = D_P(A)$$

where $D_C(A)$ is the covering dimension of' A, and $D_P(A)$ is the packing dimension or A.

REMARK 1: D_H is of interest for mathematically defined fractals but not for the analysis of experimental data $D_H(A)$ is non-integer in this case.

2.5 Fractal Geometry

Fractal geometry is defined as the study of geometric shapes that seem chaotic when compared with those of standard geometry (lines, spheres, cubes, etc.) but exhibit extreme orderliness because they possess a property of invariance under suitable contractions or dilations. The contractions and dilations can be characterized by numerical quantities, the most important being those that enter as exponents and are called fractal dimensions [30].

2.6 Self-Similar Set

A (non-random) set S in a Euclidean space $\mathbb{R}^n$ is called self-similar if it call be represented in the form $S = \bigcup_{n=1}^{N} S_n$

2.7 Compact Set

A compact set $A \in \mathbb{R}^n$ (or more generally in a complete metric space) is self-similar if there exists contraction maps $\omega_i : \mathbb{R}^n \to \mathbb{R}^n \ (i = 1, 2, \ldots, n)$ such that $A = \bigcup_{n=1}^{n} \omega_i$

The collection of maps $\omega_i, \ldots, \omega_n$ is an iterated function system (IFS) and self-similar set A is the attractor of the IFS [25].

3. Methods for Determining Fractal Dimensions

Three main approaches which are used for the determination of fractal dimension include: (a) Theoretical (b) Experimental (c) Computer. A number of experimental techniques has been used to measure the fractal dimension of scale invariant structures grown in various experiments. The most widely applied methods can be divided into the following categories:- (i) sDigital image processing of two-dimensional pictures; (ii) Scattering experiments; (iii) Covering the structures with monolayers; (iv) Direct measurement.

3.1 Theoretical Method

The following theorem shows how the theoretical approach call he used to simplify the process of calculating fractal dimension.

Theorem 1 [15]

Let A be a non-empty compact subset of a metric space X. Let $\varepsilon_n = cr^n$ for real numbers $r \in (0,1)$ and $c > 0$, and integers $n = 1, 2, 3, \ldots$ If

$$D(A) = \lim_{n\to\infty} \frac{\ln N(A, \varepsilon_n)}{\ln(1/\varepsilon_n)} \qquad (3.1)$$

then A has the fractal dimension $D(A)$.

Proof: Let the real numbers r and C, and the sequence of numbers $S = \{\varepsilon_n : n = 1, 2, 3, \ldots\}$ be defined in the statement of the theorem. Define

$$f(\varepsilon) = \max\{\varepsilon_n \in S : \varepsilon_n \le \varepsilon\}. \qquad (3.2)$$

Assume that $\varepsilon \le r$. Then $f(\varepsilon) \le \varepsilon \le f(\varepsilon)/r$ and $N(A, f(\varepsilon)) \ge N(A, \varepsilon) \ge N(A, f(\varepsilon)/r)$

Since $\ln x$ is an increasing positive function of x for $x > 1$, it follows that

$$\frac{\ln N(A, f(\varepsilon)/r)}{\ln(1/f(\varepsilon))} \le \frac{\ln N(A, \varepsilon)}{\ln(1/\varepsilon)} \le \frac{\ln N(A, f(\varepsilon))}{\ln(r/f(\varepsilon))} \qquad (3.3)$$

Assume that $N(A, \varepsilon) \to \infty$ as $\varepsilon \to 0$ if not, then the theorem is true. The right-hand-side of inequality (3.3) satisfies Equation (3.4) while the left-hand-side of inequality (3.3) satisfies Equation (3.5) as follows:

$$\lim_{\varepsilon\to 0} \frac{\ln N(A, f(\varepsilon)/r)}{\ln(1/f(\varepsilon))} = \lim_{n\to\infty} \frac{\ln N(A, \varepsilon_{n-1})}{\ln(1/\varepsilon_n)} = \lim_{n\to\infty} \frac{\ln N(A, \varepsilon_{n-1})}{\ln(1/r) + \ln(1/\varepsilon_{n-1})} = \lim_{n\to\infty} \frac{\ln N(A, \varepsilon_n)}{\ln(1/\varepsilon_n)} \qquad (3.4)$$

$$\lim_{\varepsilon\to 0} \frac{\ln N(A, f(\varepsilon)/r)}{\ln(1/f(\varepsilon))} = \lim_{n\to\infty} \frac{\ln N(A, \varepsilon_{n-1})}{\ln(1/\varepsilon_n)} = \lim_{n\to\infty} \frac{\ln N(A, \varepsilon_{n-1})}{\ln(1/r) + \ln(1/\varepsilon_{n-1})} = \lim_{n\to\infty} \frac{\ln N(A, \varepsilon_n)}{\ln(1/\varepsilon_n)} \qquad (3.5)$$

So as $\varepsilon \to 0$, both the left-hand-side and right-hand-side of inequality (3.3) approach the same value, claimed in the theorem. Thus, the limit as $\varepsilon \to 0$ of the quantity in the middle term of inequality (3.3) also exists, and is equal to the same value. This completes the proof of the theorem.

3.2 Experimental Method

Theorem 1 of section 3.1 of the fractal dimension can be used in practice. The object can under consideration can be covered with a set of squares with edge length δ, with the unit of length taken to be equal to the edge of the frame. Counting the number of squares needed to cover the required area gives the number $N(\delta)$.

Now, we may proceed as implied by Section 2.1 and calculate $M_d(\delta)$ or we may simply find $N(\delta)$, for smaller values of δ. Since asymptotically in the limit of small δ, $N(\delta) \sim \frac{1}{D}$, we may determine the fractal dimension of the set A by finding the slope, D, as a function of $\ln N(A,\delta)$ plotted against $\ln \delta$. The dimension, D, determined by counting the number of boxes needed to cover the set A as a function of the box size is now called the Box-counting or Box dimension where the symbol D is replaced by D_B.

The lower and upper box-counting dimension of a set are defined as in [33] and [16] by

$$\underline{\dim}_B(A) := \lim_{\varepsilon \to 0^+} \inf \frac{\log N(A,\varepsilon)}{|\log \varepsilon|} \qquad (3.6)$$

and $$\overline{\dim}_B(A) := \lim_{\varepsilon \to 0^+} \sup \frac{\log N(A,\varepsilon)}{|\log \varepsilon|} \qquad (3.7)$$

where $N(A,\varepsilon)$ is the least number of sets of diameter at most ε which are needed to cover A. By Equations (3.4) & (3.5), it easily follows that

$$\underline{\dim}_B(A) = \inf\{s \mid \lim_{\varepsilon > 0} \inf \varepsilon^S N(A,\varepsilon) < +\infty\} \qquad (3.8)$$

and $$\overline{\dim}_B(A) = \inf\{s \mid \lim_{\varepsilon > 0} \sup \varepsilon^S N(A,\varepsilon) < +\infty\} \qquad (3.9)$$

Thinking of $\varepsilon^S N(A,\varepsilon)$ as an s-evaluation of the covering of A, we immediately get:

$$\dim_H(A) \le \underline{\dim}_B(A) \le \overline{\dim}_B(A) \qquad (3.10)$$

If $\underline{D}_B(A) \equiv \underline{\dim}_B(A)$, $\overline{D}_B(A) \equiv \overline{\dim}_B(A)$ and $D_H(A) \equiv \dim_H(A)$ is the Hausdorff dimension of the set A, then Equation (3.10) gives

$$D_H(A) \le \underline{D}_B(A) \le \overline{D}_B(A).$$

If the lower and upper Box-counting dimensions of A, coincides, their common value is the box dimension, D_B. According to [34] the Hausdorff dimension of recurrent net fractals is estimated to show that it coincides with the box-counting dimension. A similar definition (of box-counting dimension) is found in [20] where $A \subseteq \mathbb{R}^2$.

Theorem 2: The Box-Counting Theorem [15]

Let $A \in H(\mathbb{R}^m)$, where the Euclidean metric is used. Cover $\mathbb{R}^m$ by closed just-touching square boxes of side length $\left(\frac{1}{2^n}\right)$ as exemplified in Figure 1 for n = 2 and m = 2. Let $N_n(A)$ denote the number of boxes of side length $\left(\frac{1}{2^n}\right)$ which intersect the attractor. If

$$D_B(A) = \lim_{n \to \infty} \frac{\ln N_n(A)}{\ln 2^n},$$

then A has fractal dimension $D_B(A)$.

Proof: We observe that for m = 1,2,3,… , $2^{-m} N_{n-1} \le N(A, \frac{1}{2^n}) \le N_{k(n)}$ for all n=1,2,3,…, where k(n) is the smallest integer k satisfying $k \ge n - 1 + \frac{1}{2}\log 2^m$. The first inequality holds because a ball of radius $\frac{1}{2^n}$ can intersect at most 2^m "on-grid" boxes of side $\frac{1}{2^{n-1}}$. The second follows from the fact that a box of side s can fit inside a ball of radius r provided

$$r^2 \ge (s/2)^2 + (s/2)^2 + \ldots + (s/2)^2 = m(s/2)^2$$

by the Pythagoras theorem. Now,

$$\lim_{n \to \infty} \frac{\ln N_{k(n)}}{\ln 2^n} = \lim_{n \to \infty} \frac{\ln 2^{k(n)}}{\ln 2^n} \cdot \frac{\ln N_{k(n)}}{\ln 2^{k(n)}} = D_B$$

Since $k(n) \to 1$.

Also, $$\lim_{n \to \infty} \frac{\ln 2^{-m} N_{n-1}}{\ln 2^n} = \lim_{n \to \infty} \frac{\ln N_{n-1}}{\ln 2^{n-1}} = D_B$$

Theorem 2 with $r = \frac{1}{2}$, $\varepsilon \le r$ gives

$$D_B(A) = \lim_{n \to \infty} \frac{\ln N_n(A, \frac{1}{2^n})}{\ln 2^n} = \lim_{n \to \infty} \frac{\ln N_n(A)}{\ln 2^n}$$

This completes the proof.

3.3 Computer Approach

[26] is one of the basic references on fractals, together with a discussion of the application of fractal sets in many branches of science. But it is not as mathematical as mathematicians would like, while it is too mathematical for many others. However, it contains many computer generated pictures [15]. [37] has recently described in some details the idea he uses in the generation of his spectacular pictures. Some of the computer graphics involved in the creation of beautiful fractal objects are described in [31].

The computer approach captures both the theoretical and experimental approaches based on the accuracy of the Researcher's information. Also, the computer method deals with certain other procedures of calculating fractal dimensions, different from the box counting method such as Computer graphics, Times series and wavelet analysis.

4. Conclusion

The three main methods stated in this paper can be interwoven in certain cases in the process of calculating the fractal dimensions of certain compact subsets of a metric space. The experimental method is usually linked with the box counting method while most complicated cases involve computer graphics. Thus computer literacy is relevant in building interest in the geometry of fractal dimensions because there is much to talk about fractals everywhere.

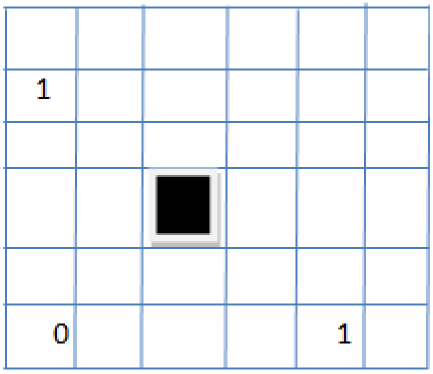

Figure 3.1: Closed boxes of sides $\left(\frac{1}{2^n}\right)$ cover □ ^{2}, where n=2, m=2.

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