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Stability and Consistency Analysis for Central Difference Scheme for Advection Diffusion Partial Differential Equation

Abel Kurura Tinega[1], Charles Otieno Ndede[2]

Faculty of Pure and Applied Sciences, Department of Mathematics Kisii University, P.O BOX 408-40200, Kisii, Kenya

Abstract: ***The paper studies stability and consistency analysis for one-dimensional advection diffusion equation using the Central Difference Scheme (CDS). Taylor's series expansion is used to expand the finite difference approximations in the CDS scheme and it is found out that the scheme is consistent with the model equation. Von-Neumann Method is used to analyze the stability of the CDS scheme developed and the scheme is found to be conditionally stable.***

Keywords: Advection diffusion equation (ADE), Partial Differential Equations (PDEs), Central difference scheme (CDS), Forward time backward space and centered space (FTBSCS) and forward time and centered Space (FTCS) and Stability condition

1. Introduction

Apart from very special cases, PDEs can only be solved numerically; the construction of their numerical solutions is a fundamental task in science and engineering. Among three classical numerical methods that are widely used for numerical solving of PDEs the finite difference method is the oldest one and is based upon the application of a local Taylor expansion to approximate the differential equations by difference ones defined on the chosen computational grid. The difference equations that approximate differential equations in the system of PDEs form its finite difference approximation which together with discrete approximation of initial or/and boundary conditions is called finite difference scheme.

1.1 The Model Equation

This research examines the stability and consistency of central difference scheme for solving the one dimension advection-diffusion equation

$$\frac{\partial u(x,t)}{\partial t}+c\frac{\partial u(x,t)}{\partial t}=D\frac{\partial^2 u(x,t)}{\partial t^2} \qquad (1)$$

which is frequently used to model the physical processes of advection and diffusion in a one dimensional system such as one involving fluid flow. The parameter v is the viscosity coefficient and c is the phase speed, and both are assumed to be positive. It is a parabolic type partial differential equation and is derived on the principle of conservation of mass using Fick's law (Socolofsky and Jirka 2002). Stability analysis of finite difference schemes for the Navier-Stokes equations is obtained (Rigal 1979).Stability and convergence in fluid flow problems is presented (Morton 1971). Stability analysis of finite difference schemes for the advection-diffusion equation is studied (Chan 1984). A comparison of some numerical methods for the advection-diffusion equation is presented (Thongmoon and Mckibbin 2006). Stability analysis of finite difference schemes for the advection diffusion equation is presented (Chan 1984). An analytical solution of the advection diffusion equation for a ground level finite area source is presented (Park and Baik 2008). An explicit finite difference scheme for solving the advection diffusion equation is studied. Numerical solution of the ADE is obtained by using FTBSCS and FTCS techniques for prescribed initial and boundary data. Numerical results for both the schemes are compared in terms of accuracy by error estimation with respect to exact solution of the ADE and also the numerical features of the rate of convergence are presented graphically (Azad *et al.* 2015).

2. Properties of Numerical Methods

Many techniques are available for numerical simulation work and in order to quantify how well a particular numerical technique performs in generating a solution to a problem, there are four fundamental criteria that can be applied to compare and contrast different methods. The concepts are accuracy, consistency, stability and convergence. The method of Finite Difference Method is one of the most valuable methods of approximating numerical solution of PDEs. Before numerical computations are made, these four important properties of finite difference equations must be considered.

Accuracy is a measure of how well the discrete solution represents the exact solution of the problem. Two quantities exist to measure this, the local or truncation error, which measures how well the difference equations match the differential equations, and the global error which reflects the overall error in the solution. This is not possible to find unless the exact solution is known.

A finite difference scheme is stable if the errors made at one time step of the calculation do not cause the errors to be magnified as the computations are continued. A neutrally stable scheme is one in which errors remain constant as the computations are carried forward. If the errors decay and eventually damp out, the numerical scheme is said to be stable. If, on the contrary, the errors grow with time the numerical scheme is said to be unstable. When a truncation error goes to zero, a finite difference equation is said to be consistent or compatible with a partial differential equation.

Consistency requires that the original equations can be recovered from the algebraic equations. Obviously this is a minimum requirement for any discretization.

A solution of a set of algebraic equations is convergent if the approximate solution approaches the exact solution of the set of PDEs for each value of the independent variable as the mesh sizes approaches zero i.e the grid spacing and time step goes to zero.

2.1 Stability of the numerical Schemes

Stability considerations are very important in getting the numerical solution of a differential equation using finite difference methods. The solution of the finite difference equation is said to be stable, if the error do not grow exponentially as we progress from one step to another. The matrix method is employed in the analysis of stability. The theory behind matrix method is that the modulus of the eigen values of the amplification matrix should be less than or equal to unity. The partial derivatives in (1) are approximated with the following finite difference approximations;

$$\frac{\partial u(x,t)}{\partial x}=\frac{U_{i+1,j}-U_{i,j}}{2(\Delta x)} \quad (2)$$

$$\frac{\partial u(x,t)}{\partial t}=\frac{U_{i,j+1}-U_{i,j}}{(\Delta t)} \quad (3)$$

$$\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{(\Delta x)^2} \quad (4)$$

Substituting the partial derivatives in (1) with finite approximations in equations (2), (3) and (4) we get

$$\frac{U_{i,j+1}-U_{i,j}}{(\Delta t)}+c\frac{U_{i+1,j}-U_{i,j}}{2(\Delta x)}=D\frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{(\Delta x)^2} \quad (5)$$

If we take c = D = 1 and multiply both sides of (5) by Δt and

let $\phi=\frac{\Delta t}{\Delta x}$ and $\mu=\frac{\Delta t}{(\Delta x)^2}$ we get

$$2U_{i,j+1}+(-2+4\mu)U_{i,j}+(\phi-2\mu)U_{i+1,j}+(-\phi-2\mu)U_{i-1,j}=0 \quad (6)$$

2.2 Von-Neumann Stability for the Central Difference Scheme

The primary observation in the Fourier or Von-Neumann method is that the numerical scheme is linear and therefore it will have a solution in the form $u_{i,j}=\lambda^t e^{Ix\psi}$. Thus, a numerical scheme is stable provided that $|\lambda|\le 1$ and unstable whenever $|\lambda|>1$, Shanthakumar (1989). We can apply this method by substituting the trivial solution in finite difference method at the time t by $x>0$ when $I=\sqrt{(-1)}$, $|\lambda|\le 1$ (Douglas (1955) and Lapidus and Pinder (1982).

We assume $u_{i,j}=\lambda^t e^{Ix\psi}$ and substitute into (6) , we have

$$2\lambda^{t+1}e^{Ix\psi}+(-2+4\mu)\lambda^t e^{Ix\psi}+(\phi-2\mu)\lambda^t e^{I(x-1)\psi}+(-\phi-2\mu)\lambda^t e^{I(x-1)\psi}=0 \quad (7)$$

Dividing (7) by $\lambda^t e^{Ix\psi}$, we get

$$2\lambda+(-2+4\mu)+(\phi-2\mu)e^{I\psi}+(-\phi-2\mu)e^{-I\psi}=0 \quad (8)$$

Making λ the subject of the formula (8)

$$\lambda=\frac{(2-4\mu)+(2\mu-\phi)e^{I\psi}+(\phi+2\mu)e^{-I\psi}}{2} \quad (9)$$

By Eulers formula

$$\left.\begin{aligned} e^{I\psi}&=\cos\psi+i\sin\psi \\ e^{-I\psi}&=\cos\psi-i\sin\psi \end{aligned}\right\} \quad (10)$$

Substituting (10) into (9) we

$$\lambda=\frac{(2-4\mu)+(2\mu-\phi)(\cos\psi+i\sin\psi)+(\phi+2\mu)(\cos\psi-i\sin\psi)}{2} \quad (11)$$

Upon simplification of (11), we get

$$\lambda=\frac{2+4\mu\cos\psi-4\mu}{2},\lambda>0 \quad (12)$$

where λ is the amplification factor. For stable situation we require $|\lambda|\le 1$.

$$|\lambda|=\left|\frac{2+4\mu\cos\psi-4\mu}{2}\right|,\lambda>0 \quad (13)$$

We determine stability for the largest and smallest value of the amplification factor λ.

a) For the largest value of λ we take $\psi=0^o$, Substituting into (13) ,we get

$$|\lambda|=\left|\frac{2+4\mu-4\mu}{2}\right|=|1|=1 \quad (14)$$

b) For the smallest value of λ we take $\psi = 90^o$, Substituting into (13) ,we get

$$|\lambda| = \left|\frac{2-4\mu}{2}\right| = |1-2\mu| \quad (15)$$

Obviously $|\lambda|$ will always be less than 1 for the equations (14) and (15). The above cases are always satisfied as the left inequality of Equations (10) and (11) requires. Thus the central difference scheme (6) is stable for all values of $\mu > 0$, i.e conditionally stable.

3. Consistency of Numerical Schemes

Consistency requires that the original equation can be recovered from the algebraic Equations. Obviously this is a minimum requirement for any discretization. In the following we illustrate how this can be done in terms of a Taylor expansion of the discretized of both the Advection Diffusion equation for CDS scheme developed in (6).Thus, showing the consistency and stability of the finite difference scheme is sufficient for convergence. Doyo and Gofe and Doyo (2016) considered the convergence rates and stability of the Forward Time, Centered Space (FTCS) and Backward Time Centered Space (BTCS) schemes for solving one-dimensional, time-dependent diffusion equation with Neumann boundary condition. The derivation of the schemes and development of a computer program to implement them were presented. The consistency and the stability of the schemes were described and used numerical problems to determine convergence rates of the schemes. It was found that both methods are first order accurate in the spatial dimension. The Gerschgorin's Theorem to determine the stability of the methods (Michae,2011), and showed that An Alternating Direction Explicit Scheme is stable if the modulus of the Eigenvalues of the Amplification Matrix should be less than or equal to one. The method is unconditionally stable. Since finite difference discretization converges at the rate of the Truncation Error (TE) (determined by the order of the spatial and temporal discretization) if the exact solution is smooth enough, the exact solution are expanded at the mesh points of the scheme with a Taylor series and insert the Taylor expansions in the scheme to calculate the TE (difference between the resulting equation and the original PDE) and determine its order in the approximation used. Then, it is seen that as the discrete step sizes approach to zero, their TE also approaches to zero which indicates that the difference approximations are consistent.

We expand every term of the CDS scheme in equation (6) using Taylors series expansion

$$2U_{i,j+1,} = U^n_{i,j} + 2(\Delta t u_t) + \frac{1}{2!}.2(\Delta t)u_{tt} + \frac{1}{3!}.2(\Delta t)^3 u_{ttt} + ... \quad (16)$$

$$(-2+4\mu)U_{i,j} = -2U^n_{i,j} + 4\mu U^n_{i,j} \quad (17)$$

$$\left.\begin{aligned}(\phi-2\mu)U_{i+1,j} &= (\phi-2\mu)\left\{U^n_{i,j} + (\Delta x u_x) + \frac{1}{2!}(\Delta x)^2 u_{xx} + \frac{1}{3!}(\Delta x)^3 u_{xx} + ...\right\}\\ &= \phi U^n_{i,j} + \phi(\Delta x u_x) + \frac{1}{2}\phi(\Delta x)^2 u_{xx} - 2\mu U^n_{i,j} - 2\mu(\Delta x u_x) - \mu(\Delta x)^2 u_{xx} + ...\end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned}-(\phi+2\mu)U_{i-1,j} &= -(\phi+2\mu)\left\{U^n_{i,j} - (\Delta x u_x) + \frac{1}{2!}(\Delta x)u_{xx} + \frac{1}{3!}(\Delta x)^3 u_{xx} + ...\right\}\\ &= -\phi U^n_{i,j} + \phi(\Delta x u_x) - \frac{1}{2}.\phi(\Delta x)^2 u_{xx} - 2\mu U^n_{i,j} + 2\mu(\Delta x u_x) - \mu(\Delta x)^2 u_{xx} + ...\end{aligned}\right\} \quad (19)$$

Substituting (16),(17),(18) and (20) into (6) and simplifying, we get

$$2(\Delta t)u_t + 2\phi(\Delta t)u_x - 2\mu(\Delta x)^2 u_{xx} + ... = 0 \quad (20)$$

Let $\Delta x = \Delta y = \Delta t = h$. then $\phi = \frac{\Delta t}{\Delta x} = 1$ and $\mu = \frac{\Delta t}{(\Delta x)^2} = \frac{1}{h}$.

The equation (20) becomes

$$2hu_t + 2hu_x - 2h^2 u_{xx} + ... = 0 \quad (21)$$

Dividing equation (21) by $2h$ throughout gives

$$u_t + u_x - u_{xx} + = 0 \quad (22)$$

It is noted that the equation (22) is the recovered PDE that is (Advection diffusion equation).Since the Advection diffusion equation has been recovered from the algebraic equation of the CDS scheme (6) developed; we therefore conclude that the scheme is consistent with the Advection diffusion PDE.

4. Convergence

Since convergence is difficult to prove directly, we use an equivalent result known as the Lax Equivalence Theorem which states that, 'For a given properly posed linear consistent finite difference approximation to Partial differential equation (PDE), stability is necessary and sufficient for convergence (Randall,1998)'. Lax has proved that under appropriate conditions a consistent scheme is convergent if and only if it is stable. According to Lax - Richtmeyer Equivalence Theorem: 'Given a properly posed linear initial value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence'(Richtmeyer and Morton,1967).

5. Conclusion

The Equation (14) and (15) satisfies the stability conditions. The condition on the right is always satisfied as the left inequality requires. All the eigenvalues in Equations (14) and (15) are bounded by 1. Thus the CDS (6) scheme is conditionally stable. The Advection diffusion PDE is also recovered from the CDS scheme in equation (6).It can be concluded that the stability of the CDS developed for the one-dimensional Advection diffusion equation that conditionally stable. And also consistent with the Advection diffusion PDE.

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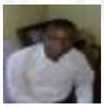
Author Profile

Dr. Abel Kurura Tinega was born at Tombe Market, Nyamira County, Kenya. He holds a Bachelor of Education Science degree, with specialization in Mathematics and Physics, from Kenyatta University (KU), Nairobi, Kenya, and a Master of Science degree in Applied Mathematics from Masinde Muliro University of Science and Technology (MMUST), Kakamega, Kenya and a PhD in applied Mathematics at Jaramogi Oginga Odinga University of Science and Technology (JOOUST), Bondo, Kenya. ***Affiliation:*** Jaramogi Oginga Odinga University of Science and Technology (JOOUST), BONDO, KENYA. He is currently a full time Mathematics lecturer at Kisii National Polytechnic, Kenya and part-time lecturer at Kisii University. He has participated in many conferences including Kenya Association of Mathematics Education (KAME), published many papers in refereed journals and has been a trainer in Strengthening of Mathematics and science For Secondary Education in Kenya (SMASSE).He has much interest in the study of numerical methods and their respective applications in modeling physical phenomenon in Mathematics, sciences and engineering.



Charles Otieno Ndede was born at Oyugis Market, Homabay County, Kenya. He holds a Bachelor of Education Arts degree, with specialization in Mathematics and Economics, from Moi University, Eldoret, Kenya, and a Master of Business Administration degree from Kisii University of Science, CPA(K), KASNEB, Kenya and is currently pursuing MSC in applied Mathematics at Jomo Kenyatta University of Agriculture and Technology, (JKUAT), JUJA, Kenya. ***Affiliation:*** Jomo Kenyatta University of Agriculture and Technology, (JKUAT), JUJA, Kenya. He is currently a banker with Kenya commercial bank. He has participated in many conferences has much interest in the study of numerical methods and their respective applications in modeling physical phenomenon in Mathematics, sciences and engineering.