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The Numerical Range for $z_1V+z_2V^*$

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Abstract: ***Let V denote the classical Volterra operator on $L^2[0,1]$ and let z_1, z_2 be an arbitrary complex numbers. We investigated the numerical range and the numerical radius of $z_1V+z_2V^*$. In particular, we determine the numerical range of V without using the known results.***

Keywords: Numerical range, Numerical radius, Volterra operator

1. Introduction

Denote by V the classical Volterra operator

$$(Vf)(x)=\int_0^x f(t)dt,\ f\in L^2[0,1].$$

The adjoint of the Volterra operator is

$$(V^*f)(x)=\int_x^1 f(t)dt.$$

The Volterra operator is compact, quasinilpotent and accretive. In [3] considered the operator norm of operators $z_1V+z_2V^*$. For a bounded linear operator A on a complex Hilbert space H, the numerical range $W(A)$ is the image of the unit sphere of H under the quadratic form $x\to(Ax,x)$ associated with the operator. More precisely,

$$W(A)=\{(Ax,x):x\in H,\|x\|=1\}.$$

It is well known that numerical range of an operator is convex (The Toeplitz-Hausdorf theorem) and spectrum is contained in the closure of its numerical range. The numerical radius of an operator A is defined by

$$\omega(A)=\sup\{|\lambda|:\lambda\in W(A)\}.$$

(see [2])

We will need the following theorem.

Theorem 1.([1], p.268) If A is a bounded operator on H and $\theta\in[-\pi,\pi]$, put $\lambda_\theta=\max\sigma(B_\theta)$, where $B_\theta=\frac{1}{2}(e^{-i\theta}A+e^{i\theta}A^*)=B_\theta^*$. Then

$$\overline{W(A)}=\bigcap_{\theta\in[-\pi,\pi]}H_\theta$$

where the half-space H_θ is defined by

$$H_\theta=\{z\in C:\mathrm{Re}(e^{-i\theta}z)\le\lambda_\theta\}.$$

According to Theorem 1, if $\lambda_\theta\in C^1[-\pi,\pi]$ then $x\cos\theta+y\sin\theta=\lambda_\theta$ is envelope curves. Because, if $0<\theta<\pi$, then $\sin\theta>0$ and $y\le\frac{\lambda_\theta}{\sin\theta}-x\cot\theta$. Observe that, if $-\pi<\theta<0$, then $\sin\theta<0$ and $y\ge\frac{\lambda_\theta}{\sin\theta}-x\cot\theta$.

By the calculation, implies that

$$\begin{cases}x=\lambda_\theta\cos\theta-\lambda_\theta'\sin\theta\\ y=\lambda_\theta\sin\theta+\lambda_\theta'\cos\theta.\end{cases}\qquad(1)$$

The aim of this paper is to study the numerical range and the numerical radius of operators $z_1V+z_2V^*$, where z_1,z_2 are arbitrary complex numbers. In particular, we determine the numerical range of V without using the known results.

2. The Results

We consider the numerical range and numerical radius of operators $z_1V+z_2V^*$.

Theorem 2. Let z_1 and z_2 be arbitrary complex numbers and $A=z_1V+z_2V^*$. If $z_1\ne z_2$ $(z_1=r_1e^{i\alpha}, z_2=r_2e^{i\beta})$ then closure of $W(A)$ is the convex hull of the following curves

$$\begin{cases}x=\dfrac{r_1\sin\alpha-r_2\sin\beta}{2(\psi+\pi k_0)}+\dfrac{(r_2^2-r_1^2)(r_2\sin(\theta-\beta)-r_1\sin(\theta-\alpha))\sin\theta}{2(\psi+\pi k_0)^2(r_1^2+r_2^2+2r_1r_2\cos(2\theta-(\alpha+\beta)))}\\ y=\dfrac{r_2\cos\beta-r_1\cos\alpha}{2(\psi+\pi k_0)}-\dfrac{(r_2^2-r_1^2)(r_2\sin(\theta-\beta)-r_1\sin(\theta-\alpha))\cos\theta}{2(\psi+\pi k_0)^2(r_1^2+r_2^2+2r_1r_2\cos(2\theta-(\alpha+\beta)))},\end{cases}\qquad(2)$$

where $\theta\in[-\pi,\pi]$, $\psi=\arg(e^{-i\theta}z_1+e^{i\theta}\overline{z_2})$,

$$\lambda_\theta=\max_k\frac{\mathrm{Im}(e^{-i\theta}(z_1-z_2))}{2(\psi+\pi k)}=\frac{r_2\sin(\theta-\beta)-r_1\sin(\theta-\alpha)}{2(\psi+\pi k_0)}$$

and $k_0\in\{-1,0,1\}$. The numerical radius of A is

$$\omega(A)=\sup_{-\pi\le\theta\le\pi}\sqrt{\lambda_\theta^2+\lambda_\theta'^2}.$$

If $z_1 = z_2 = z$, then

$$W(A) = [0, z] \text{ and } \omega(A) = |z|.$$

Proof. Let z_1, z_2 be distinct complex numbers. Denote $A = z_1 V + z_2 V^*$ in Theorem 1, then spectral problem $A^* A f = \lambda f$ is

$$e^{-i\theta}(z_1 V + z_2 V^*)f + e^{i\theta}(\overline{z_1} V^* + \overline{z_2} V)f = 2\lambda f, \ \lambda \in R. \qquad (3)$$

This integral equation to a differential equation by applying the operator $D = \frac{d}{dx}$. Thus,

$$i\,\mathrm{Im}\left(e^{-i\theta}(z_1 - z_2)\right)f(t) = \lambda f'(t)$$

If $z_1 \neq z_2$ then general solution is

$$f(t) = Ce^{i\frac{\mathrm{Im}(e^{-i\theta}(z_1 - z_2))}{\lambda}t}$$

where, $\lambda \in R$ and C-constant. Now, we insert $x = 0$ and $x = 1$ into (3), account that

$$(\overline{z_1} e^{i\theta} + z_2 e^{-i\theta})\int_0^1 f(t)dt = 2\lambda f(0)$$

and

$$(z_1 e^{-i\theta} + \overline{z_2} e^{i\theta})\int_0^1 f(t)dt = 2\lambda f(1)$$

respectively. If $\overline{z_1} e^{i\theta} + z_2 e^{-i\theta} = 0$, then $f(0) = f(1) = 0$, implies that $f(t) = 0$. It is a contradiction. If $\overline{z_1} e^{i\theta} + z_2 e^{-i\theta} \neq 0$, then we get

$$f(1) = \frac{e^{-i\theta} z_1 + e^{i\theta} \overline{z_2}}{e^{i\theta} \overline{z_1} + e^{-i\theta} z_2} f(0) = e^{2i\psi} f(0),$$

where

$$\psi = \arg(e^{-i\theta} z_1 + e^{i\theta} \overline{z_2}).$$

So,

$$\frac{\mathrm{Im}(e^{-i\theta}(z_1 - z_2))}{\lambda} = 2\psi + 2\pi k, \ k \in Z.$$

Therefore,

$$\lambda_\theta = \max_k \lambda_k = \max_k \frac{\mathrm{Im}(e^{-i\theta}(z_1 - z_2))}{2(\psi + \pi k)}.$$

Let $z_1 = r_1 e^{i\alpha}$, $z_2 = r_2 e^{i\beta}$. Observe that,

$$\mathrm{Im}(e^{-i\theta}(z_1 - z_2)) = r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha),$$

$$e^{-i\theta} z_1 + e^{i\theta} \overline{z_2} = r_1 e^{-i(\theta - \alpha)} + r_2 e^{i(\theta - \beta)},$$

$$\psi + \pi k = \arctan\left(\frac{r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)}{r_1 \cos(\theta - \alpha) - r_2 \cos(\theta - \beta)}\right) + \pi k,$$

$$\lambda_\theta = \frac{r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)}{2(\psi + \pi k_0)}, \text{(where } k_0 \in \{-1, 0, 1\})$$

and

$$(\psi + \pi k_0)' = \left[\arctan\left(\frac{r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)}{r_1 \cos(\theta - \alpha) - r_2 \cos(\theta - \beta)}\right)\right]' =$$

$$= \frac{r_2^2 - r_1^2}{r_1^2 + r_2^2 + 2r_1 r_2 \cos(2\theta - (\alpha + \beta))}.$$

It follows from (1), closure of $W(A)$ is convex hull of the following curves

$$\begin{cases} x = \frac{r_1 \sin\alpha - r_2 \sin\beta}{2(\psi + \pi k_0)} + \frac{(r_2^2 - r_1^2)(r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha))\sin\theta}{2(\psi + \pi k_0)^2 (r_1^2 + r_2^2 + 2r_1 r_2 \cos(2\theta - (\alpha + \beta)))} \\ y = \frac{r_2 \cos\beta - r_1 \cos\alpha}{2(\psi + \pi k_0)} - \frac{(r_2^2 - r_1^2)(r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha))\cos\theta}{2(\psi + \pi k_0)^2 (r_1^2 + r_2^2 + 2r_1 r_2 \cos(2\theta - (\alpha + \beta)))}, \end{cases}$$

where $\theta \in [-\pi, \pi]$.

By (1), implies that $x^2 + y^2 = \lambda_\theta^2 + \lambda_\theta'^2$, we get

$$\omega(A) = \sup_{-\pi \le \theta \le \pi} \sqrt{\lambda_\theta^2 + \lambda_\theta'^2}.$$

Let $z_1 = z_2 = z = re^{i\alpha}$ $(-\pi < \alpha \le \pi)$. Actually,

$$\lambda_\theta = \begin{cases} 0, & \text{if } \mathrm{Re}(e^{i\alpha} e^{-i\theta}) \le 0 \\ r\mathrm{Re}(e^{i\alpha} e^{-i\theta}), & \text{if } \mathrm{Re}(e^{i\alpha} e^{-i\theta}) > 0 \end{cases}$$

or

$$\lambda_\theta = \begin{cases} 0, & \text{if } |\theta - \alpha| > \frac{\pi}{2} \\ r\cos(\theta - \alpha), & \text{if } |\theta - \alpha| \le \frac{\pi}{2}. \end{cases}$$

If $|\theta - \alpha| > \frac{\pi}{2}$, then $x = y = 0$.

If $|\theta - \alpha| \le \frac{\pi}{2}$, then

$$\begin{cases} x = r\cos(\theta - \alpha)\cos\theta + r\sin(\theta - \alpha)\sin\theta = r\cos\alpha \\ y = r\cos(\theta - \alpha)\sin\theta - r\sin(\theta - \alpha)\cos\theta = r\sin\alpha. \end{cases}$$

We choose $f_0(t) = \frac{t - \frac{1}{2}}{\| t - \frac{1}{2} \|}$ and $f_1(t) = 1$. Note that, $(Af_0, f_0) = 0$ and $(Af_1, f_1) = z$. Therefore, $W(A) = [0, z]$ and $\omega(A) = |z|$. The completes the proof.

Corollary 1. Let $|z_1| = |z_2| = r$ and $z_1 \neq z_2$. $(z_1 = re^{i\alpha}, z_2 = re^{i\beta}, \alpha \neq \beta)$ Then

$$W(A) = \left[-r \cdot \frac{\sin\frac{|\alpha - \beta|}{2}}{\pi - \frac{|\alpha - \beta|}{2}} e^{i\frac{\alpha + \beta}{2}}, \ r \cdot \frac{\sin\frac{\alpha - \beta}{2}}{\frac{\alpha - \beta}{2}} e^{i\frac{\alpha + \beta}{2}} \right] \qquad (4)$$

and

$$\omega(A) = \|A\| = \begin{cases} r\cdot\dfrac{\sin\frac{\alpha-\beta}{2}}{\frac{\alpha-\beta}{2}}, & \text{if} \quad |\alpha-\beta| \le \pi \\ r\cdot\dfrac{\sin\frac{|\alpha-\beta|}{2}}{\pi-\frac{|\alpha-\beta|}{2}}, & \text{if} \quad |\alpha-\beta| > \pi. \end{cases} \tag{5}$$

Proof. Put $r_1 = r_2 = r \quad (\alpha \ne \beta)$ in (1), we get

$$\begin{cases} x = \dfrac{r(\sin\alpha - \sin\beta)}{2(\psi + \pi k_0)} = r\cdot\dfrac{\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}}{\psi+\pi k_0} \\ y = \dfrac{r(\cos\beta - \cos\alpha)}{2(\psi + \pi k_0)} = r\cdot\dfrac{\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}}{\psi+\pi k_0} \end{cases} \tag{6}$$

and

$$z = x + iy = r\cdot\frac{\sin\frac{\alpha-\beta}{2}}{\psi+\pi k_0} e^{i\frac{\alpha+\beta}{2}}.$$

Observe that,

$$\psi = \arg\left(\cos\left(\theta - \frac{\alpha+\beta}{2}\right)e^{i\frac{\alpha-\beta}{2}}\right)$$

i.e,

$$\psi = \begin{cases} \dfrac{\alpha-\beta}{2}, & \text{if } |\theta - \frac{\alpha+\beta}{2}| \le \frac{\pi}{2} \\ -\pi + \dfrac{\alpha-\beta}{2}, & \text{if } |\theta - \frac{\alpha+\beta}{2}| > \frac{\pi}{2},\ \alpha > \beta \\ \pi + \dfrac{\alpha-\beta}{2}, & \text{if } |\theta - \frac{\alpha+\beta}{2}| > \frac{\pi}{2},\ \alpha < \beta. \end{cases} \tag{7}$$

and $k_0 = 0$. Now, we insert (6) and $k_0 = 0$ into (5), and put $f_1(t) = e^{i(\alpha-\beta)t}$, $f_2(t) = e^{i(\alpha-\beta\pi)}$, $f_3(t) = e^{i(\alpha+\beta\pi)}$ desired (3) and (4), respectively.

Corollary 2. The numerical range of the Volterra operator is the set lying between the curves

$$\varphi \in [0,2\pi] \mapsto \frac{1-\cos\varphi}{\varphi^2} \pm i\frac{\varphi - \sin\varphi}{\varphi^2}.$$

(Also see [4], p.113)

Corollary 3. For the classical Volterra operator V, it holds:

i) $\omega(V) = \frac{1}{2}$.

ii) $W(\mathrm{Re}V) = [0, \frac{1}{2}]$.

iii) $W(\mathrm{Im}V) = [-\frac{1}{\pi}, \frac{1}{\pi}]$.

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