

The Numerical Range for $z_1V + z_2V^*$

D. Tsedenbayar¹, L. Khadkhuu²

¹School of Applied Mathematics, Mongolian University of Science and Technology, Department of Mathematics, Ulaanbaatar -14191, P.O.Box 46/520, Mongolia

²School of Mathematics, National University of Mongolia, Department of Mathematics, Ulaanbaatar-14191, P.O.Box 145/520, Mongolia

Abstract: Let V denote the classical Volterra operator on $L^2[0,1]$ and let z_1, z_2 be an arbitrary complex numbers. We investigated the numerical range and the numerical radius of $z_1V + z_2V^*$. In particular, we determine the numerical range of V without using the known results.

Keywords: Numerical range, Numerical radius, Volterra operator

1. Introduction

Denote by V the classical Volterra operator

$$(Vf)(x) = \int_0^x f(t)dt, f \in L^2[0,1].$$

The adjoint of the Volterra operator is

$$(V^*f)(x) = \int_x^1 f(t)dt.$$

The Volterra operator is compact, quasinilpotent and accretive. In [3] considered the operator norm of operators $z_1V + z_2V^*$. For a bounded linear operator A on a complex Hilbert space H , the numerical range $W(A)$ is the image of the unit sphere of H under the quadratic form $x \rightarrow (Ax, x)$ associated with the operator. More precisely,

$$W(A) = \{(Ax, x) : x \in H, \|x\|=1\}.$$

It is well known that numerical range of an operator is convex (The Toeplitz-Hausdorff theorem) and spectrum is contained in the closure of its numerical range. The numerical radius of an operator A is defined by

$$\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

(see [2])

We will need the following theorem.

Theorem 1. ([1], p.268) If A is a bounded operator on H and $\theta \in [-\pi, \pi]$, put $\lambda_\theta = \max \sigma(B_\theta)$, where

$$B_\theta = \frac{1}{2}(e^{-i\theta}A + e^{i\theta}A^*) = B_\theta^*.$$

$$\overline{W(A)} = \bigcup_{\theta \in [-\pi, \pi]} H_\theta$$

where the half-space H_θ is defined by

$$H_\theta = \{z \in C : \operatorname{Re}(e^{-i\theta}z) \leq \lambda_\theta\}.$$

According to Theorem 1, if $\lambda_\theta \in C^1[-\pi, \pi]$ then $x \cos \theta + y \sin \theta = \lambda_\theta$ is envelope curves. Because, if

$0 < \theta < \pi$, then $\sin \theta > 0$ and $y \leq \frac{\lambda_\theta}{\sin \theta} - x \cot \theta$. Observe

that, if $-\pi < \theta < 0$, then $\sin \theta < 0$ and $y \geq \frac{\lambda_\theta}{\sin \theta} - x \cot \theta$.

By the calculation, implies that

$$\begin{cases} x = \lambda_\theta \cos \theta - \lambda'_\theta \sin \theta \\ y = \lambda_\theta \sin \theta + \lambda'_\theta \cos \theta. \end{cases} \quad (1)$$

The aim of this paper is to study the numerical range and the numerical radius of operators $z_1V + z_2V^*$, where z_1, z_2 are arbitrary complex numbers. In particular, we determine the numerical range of V without using the known results.

2. The Results

We consider the numerical range and numerical radius of operators $z_1V + z_2V^*$.

Theorem 2. Let z_1 and z_2 be arbitrary complex numbers and $A = z_1V + z_2V^*$. If $z_1 \neq z_2$ ($z_1 = r_1e^{i\alpha}, z_2 = r_2e^{i\beta}$) then closure of $W(A)$ is the convex hull of the following curves

$$\begin{cases} x = \frac{r_1 \sin \alpha - r_2 \sin \beta}{2(\psi + \pi k_0)} \\ \quad + \frac{(r_2^2 - r_1^2)(r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)) \sin \theta}{2(\psi + \pi k_0)^2 (r_1^2 + r_2^2 + 2r_1 r_2 \cos(2\theta - (\alpha + \beta)))} \\ y = \frac{r_2 \cos \beta - r_1 \cos \alpha}{2(\psi + \pi k_0)} \\ \quad - \frac{(r_2^2 - r_1^2)(r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)) \cos \theta}{2(\psi + \pi k_0)^2 (r_1^2 + r_2^2 + 2r_1 r_2 \cos(2\theta - (\alpha + \beta)))}, \end{cases} \quad (2)$$

where $\theta \in [-\pi, \pi]$, $\psi = \arg(e^{-i\theta}z_1 + e^{i\theta}z_2)$,

$$\lambda_\theta = \max_k \frac{\operatorname{Im}(e^{-i\theta}(z_1 - z_2))}{2(\psi + \pi k)} = \frac{r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)}{2(\psi + \pi k_0)}$$

and $k_0 \in \{-1, 0, 1\}$. The numerical radius of A is

$$\omega(A) = \sup_{-\pi \leq \theta \leq \pi} \sqrt{\lambda_\theta^2 + \lambda'_\theta{}^2}.$$

If $z_1 = z_2 = z$, then

$$W(A) = [0, z] \text{ and } \omega(A) = |z|.$$

Proof. Let z_1, z_2 be distinct complex numbers. Denote $A = z_1V + z_2V^*$ in Theorem 1, then spectral problem $A^*Af = \lambda f$ is

$$e^{-i\theta}(z_1V + z_2V^*)f + e^{i\theta}(\overline{z_1}V^* + \overline{z_2}V)f = 2\lambda f, \lambda \in \mathbb{R}. \quad (3)$$

This integral equation to a differential equation by applying the operator $D = \frac{d}{dx}$. Thus,

$$i\text{Im}(e^{-i\theta}(z_1 - z_2))f'(t) = \lambda f'(t)$$

If $z_1 \neq z_2$ then general solution is

$$f(t) = Ce^{i \frac{\text{Im}(e^{-i\theta}(z_1 - z_2))}{\lambda} t}$$

where, $\lambda \in \mathbb{R}$ and C -constant. Now, we insert $x = 0$ and $x = 1$ into (3), account that

$$(\overline{z_1}e^{i\theta} + z_2e^{-i\theta}) \int_0^1 f(t)dt = 2\lambda f(0)$$

and

$$(z_1e^{-i\theta} + \overline{z_2}e^{i\theta}) \int_0^1 f(t)dt = 2\lambda f(1)$$

respectively. If $\overline{z_1}e^{i\theta} + z_2e^{-i\theta} = 0$, then $f(0) = f(1) = 0$, implies that $f(t) = 0$. It is a contradiction. If or $\overline{z_1}e^{i\theta} + z_2e^{-i\theta} \neq 0$, then we get

$$f(1) = \frac{e^{-i\theta}z_1 + e^{i\theta}z_2}{e^{i\theta}z_1 + e^{-i\theta}z_2} f(0) = e^{2i\psi} f(0),$$

where

$$\psi = \arg(e^{-i\theta}z_1 + e^{i\theta}\overline{z_2}).$$

So,

$$\frac{\text{Im}(e^{-i\theta}(z_1 - z_2))}{\lambda} = 2\psi + 2\pi k, k \in \mathbb{Z}.$$

Therefore,

$$\lambda_\theta = \max_k \lambda_k = \max_k \frac{\text{Im}(e^{-i\theta}(z_1 - z_2))}{2(\psi + \pi k)}.$$

Let $z_1 = r_1e^{i\alpha}, z_2 = r_2e^{i\beta}$. Observe that,

$$\text{Im}(e^{-i\theta}(z_1 - z_2)) = r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha),$$

$$e^{-i\theta}z_1 + e^{i\theta}\overline{z_2} = r_1e^{-i(\theta - \alpha)} + r_2e^{i(\theta - \beta)},$$

$$\psi + \pi k = \arctan\left(\frac{r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)}{r_1 \cos(\theta - \alpha) - r_2 \cos(\theta - \beta)}\right) + \pi k,$$

$$\lambda_\theta = \frac{r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)}{2(\psi + \pi k_0)}, (\text{where } k_0 \in \{-1, 0, 1\})$$

and

$$\begin{aligned} (\psi + \pi k_0)' &= \left[\arctan\left(\frac{r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)}{r_1 \cos(\theta - \alpha) - r_2 \cos(\theta - \beta)}\right) \right]' \\ &= \frac{r_2^2 - r_1^2}{r_1^2 + r_2^2 + 2r_1r_2 \cos(2\theta - (\alpha + \beta))}. \end{aligned}$$

It follows from (1), closure of $W(A)$ is convex hull of the following curves

$$\begin{cases} x = \frac{r_1 \sin \alpha - r_2 \sin \beta}{2(\psi + \pi k_0)} \\ \quad + \frac{(r_2^2 - r_1^2)(r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)) \sin \theta}{2(\psi + \pi k_0)^2 (r_1^2 + r_2^2 + 2r_1r_2 \cos(2\theta - (\alpha + \beta)))} \\ y = \frac{r_2 \cos \beta - r_1 \cos \alpha}{2(\psi + \pi k_0)} \\ \quad - \frac{(r_2^2 - r_1^2)(r_2 \sin(\theta - \beta) - r_1 \sin(\theta - \alpha)) \cos \theta}{2(\psi + \pi k_0)^2 (r_1^2 + r_2^2 + 2r_1r_2 \cos(2\theta - (\alpha + \beta)))} \end{cases}$$

where $\theta \in [-\pi, \pi]$.

By (1), implies that $x^2 + y^2 = \lambda_\theta^2 + \lambda_\theta'^2$, we get

$$\omega(A) = \sup_{-\pi \leq \theta \leq \pi} \sqrt{\lambda_\theta^2 + \lambda_\theta'^2}.$$

Let $z_1 = z_2 = z = re^{i\alpha}$ ($-\pi < \alpha \leq \pi$). Actually,

$$\lambda_\theta = \begin{cases} 0, & \text{if } \text{Re}(e^{i\alpha}e^{-i\theta}) \leq 0 \\ r \text{Re}(e^{i\alpha}e^{-i\theta}), & \text{if } \text{Re}(e^{i\alpha}e^{-i\theta}) > 0 \end{cases}$$

$$\lambda_\theta = \begin{cases} 0, & \text{if } |\theta - \alpha| > \frac{\pi}{2} \\ r \cos(\theta - \alpha), & \text{if } |\theta - \alpha| \leq \frac{\pi}{2}. \end{cases}$$

If $|\theta - \alpha| > \frac{\pi}{2}$, then $x = y = 0$.

If $|\theta - \alpha| \leq \frac{\pi}{2}$, then

$$\begin{cases} x = r \cos(\theta - \alpha) \cos \theta + r \sin(\theta - \alpha) \sin \theta = r \cos \alpha \\ y = r \cos(\theta - \alpha) \sin \theta - r \sin(\theta - \alpha) \cos \theta = r \sin \alpha. \end{cases}$$

We choose $f_0(t) = \frac{t - \frac{1}{2}}{\|t - \frac{1}{2}\|}$ and $f_1(t) = 1$. Note that,

$(Af_0, f_0) = 0$ and $(Af_1, f_1) = z$. Therefore, $W(A) = [0, z]$ and $\omega(A) = |z|$. The completes the proof.

Corollary 1. Let $|z_1| = |z_2| = r$ and $z_1 \neq z_2$. ($z_1 = re^{i\alpha}, z_2 = re^{i\beta}, \alpha \neq \beta$) Then

$$W(A) = \left[-r \cdot \frac{\sin \frac{|\alpha - \beta|}{2}}{\pi - \frac{|\alpha - \beta|}{2}} e^{i \frac{\alpha + \beta}{2}}, r \cdot \frac{\sin \frac{\alpha - \beta}{2}}{2} e^{i \frac{\alpha + \beta}{2}} \right] \quad (4)$$

and

$$\omega(A) = \|A\| = \begin{cases} r \cdot \frac{\sin \frac{\alpha - \beta}{2}}{\alpha - \beta}, & \text{if } |\alpha - \beta| \leq \pi \\ r \cdot \frac{\sin \frac{|\alpha - \beta|}{2}}{\pi - \frac{|\alpha - \beta|}{2}}, & \text{if } |\alpha - \beta| > \pi. \end{cases} \quad (5)$$

Proof. Put $r_1 = r_2 = r$ ($\alpha \neq \beta$) in (1), we get

$$\begin{cases} x = \frac{r(\sin \alpha - \sin \beta)}{2(\psi + \pi k_0)} = r \cdot \frac{\sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}}{\psi + \pi k_0} \\ y = \frac{r(\cos \beta - \cos \alpha)}{2(\psi + \pi k_0)} = r \cdot \frac{\sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}}{\psi + \pi k_0} \end{cases} \quad (6)$$

and

$$z = x + iy = r \cdot \frac{\sin \frac{\alpha - \beta}{2}}{\psi + \pi k_0} e^{i \frac{\alpha + \beta}{2}}.$$

Observe that,

$$\psi = \arg \left(\cos \left(\theta - \frac{\alpha + \beta}{2} \right) e^{i \frac{\alpha - \beta}{2}} \right)$$

i.e.,

$$\psi = \begin{cases} \frac{\alpha - \beta}{2}, & \text{if } \left| \theta - \frac{\alpha + \beta}{2} \right| \leq \frac{\pi}{2} \\ -\pi + \frac{\alpha - \beta}{2}, & \text{if } \left| \theta - \frac{\alpha + \beta}{2} \right| > \frac{\pi}{2}, \alpha > \beta \\ \pi + \frac{\alpha - \beta}{2}, & \text{if } \left| \theta - \frac{\alpha + \beta}{2} \right| > \frac{\pi}{2}, \alpha < \beta. \end{cases} \quad (7)$$

and $k_0 = 0$. Now, we insert (6) and $k_0 = 0$ into (5), and put $f_1(t) = e^{i(\alpha - \beta)t}$, $f_2(t) = e^{i(\alpha - \beta)\pi}$, $f_3(t) = e^{i(\alpha + \beta)\pi}$ desired (3) and (4), respectively.

Corollary 2. The numerical range of the Volterra operator is the set lying between the curves

$$\varphi \in [0, 2\pi] \mapsto \frac{1 - \cos \varphi}{\varphi^2} \pm i \frac{\varphi - \sin \varphi}{\varphi^2}.$$

(Also see [4], p.113)

Corollary 3. For the classical Volterra operator V , it holds:

- i) $\omega(V) = \frac{1}{2}$.
- ii) $W(\operatorname{Re} V) = [0, \frac{1}{2}]$.
- iii) $W(\operatorname{Im} V) = [-\frac{1}{\pi}, \frac{1}{\pi}]$.

Acknowledgments

The authors would like to thank Prof. Michal Zajac for useful discussions during the preparation of this article. The authors grateful for the referee's valuable comments that improved the paper.

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