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Fibonacci, Lucas and Chebyshev Polynomials

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Abstract: ***Fibonacci and Lucas polynomials are special cases of Chebyshev polynomial. There are two types of Chebyshev polynomials, a Chebyshev polynomial of first kind and a Chebyshev polynomial of second kind. Chebyshev polynomial of second kind can be derived from the Chebyshev polynomial of first kind. Chebyshev polynomial is a polynomial of degree n and satisfies a second order homogenous differential equation. We consider the difference equations which are related with Chebyshev, Fibonacci and Lucas polynomials. Thus Chebyshev polynomial of second kind plays an important role in finding the recurrence relations with Fibonacci and Lucas polynomials.***

Keywords: Chebyshev polynomial, Fibonacci polynomial, Lucas polynomial

1. Introduction

Fibonacci and Lucas polynomials are special cases of Chebyshev polynomials.

We consider the sequences generated by homogenous second order difference equation [2]

$u_{n+1} = au_n + bu_{n-1}$, $n \geq 1$ and u_0 = and u_1 are constant coefficients. Consider the case when a= 1, b=1, then $u_{n+1} = u_n + u_{n-1}$, then the generalized Fibonacci Numbers H_n are produced.

Further $u_0 = 0$, $u_1 = 1$ leads to Fibonacci Numbers F_n and $u_0 = 2$ and $u_1 = 1$ gives Lucas Numbers L_n

If a and b are polynomials in x, then a sequence of polynomials is generated.

In particular, if a = 2x and b = -1, we get Chebyshev polynomial of First kind $T_n(x)$
For $T_0 =1$, $T_1 = x$, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$
If $u_0 = 1$, $u_1 = 2x$, we get Chebyshev polynomial of second kind $U_n(x)$,
$U_{n+1}(x) = 2x\, U_n(x) - U_{n-1}(x)$
If a =x and b=1 with initial conditions $U_0 = 0$, $U_1 = 1$, we get Fibonacci polynomial $F_n(x)$,
$F_{n+1}(x) = x\, F_n(x) + F_{n-1}(x)$, $F_0(x) = 1$, $F_1(x) = 1$
If a = x and b = 1 with the initial conditions $U_0 = 2$, $U_1 = 1$, we get Lucas polynomial $L_n(x)$,
$L_{n+1}(x) = x\, Ln(x) + L_{n-1}(x)$, $L_0(x) = 2$, $L_1(x) = 1$
The Fibonacci and Lucas Numbers are related by Chebyshev polynomials by the equation
$F_{n+1}(x) = i^{-n}\, U_n(i/2)$ and $L_n(x) = 2i^{-n}\, T_n(i/2)$
$T_n\, (T_m\, (i/2) = i^{mn} /2\, (L_{mn)}$
$U_n\, (T_m\, (i/2) = i^{mn}\, F_{m(n+1)} / F_m$

Second order linear recurrence sequences,
a) $T_n(x) = 1/2\{ (x + \sqrt{x}^2 - 1)^n + \{ (x - \sqrt{x}^2 - 1)^n \}$
b) $U_n(x) =(1/ 2\sqrt{x}^2- 1)\{ (x + \sqrt{x}^2 - 1)^{n+1} - \{ (x - \sqrt{x}^2 - 1)^{n+1} \}$
c) $F_n(x)= (1/ 2^n \sqrt{x}^2+ 4) \{ (x + \sqrt{x}^2 + 4)^n - \{ (x - \sqrt{x}^2 + 4)^n \}$

d)$L_n(x) = (1/ 2^n (x+ \sqrt{x}^2 + 4)^n + \{ (x - \sqrt{x}^2 + 4)^n$ }Recurrence Relations: If x = Cos θ, then we have the following Lemmas.

Lemma1.1: $T_{n+1}(x) = x\, T_n(x) - (1- x^2)U_{n-1}\, (x)$

Proof: $T_{n+1}(x) = T_{n+1}(\cos \theta) = \cos (n+1)\, \theta = \cos n\theta \cos \theta - \sin n\theta \sin \theta$
$= T_n(\cos \theta) \cos \theta - U_{n-1}(\cos \theta) \sin^2 \theta$
$= x\, T_n(x) - (1- x^2)\, U_{n-1}\, (x)$

Lemma1.2: $U_n(x) = U_n(\cos \theta) = \sin (n+1)\, \theta / \sin \theta$
$=(\sin n\, \theta \cos \theta + \cos n\, \theta \sin \theta) / \sin \theta$
$= \cos \theta\, U_{n-1}(x) + \cos n\, \theta$
$= x\, U_{n-1}(x) + T_n\, (x)$
Following points should be noted.

Remarks
1) The relationship between the sequences $F_n(x)$ and $L_n(x)$ is like the relationship between sine and cosine. e.g. $F_{2n} = F_n\, L_n$ which is similar to sin 2x = 2sinx cosx
2) If x=1, then we get Fibonacci and Lucas Numbers, F_n and L_n
3) Chebyshev polynomial of first kind $T_n(x)$ are orthogonal with respect to weight
$1/ \sqrt{1} - x^2$
4) Chebyshev polynomial of second kind $U_n(x)$ are orthogonal with respect to weight $\sqrt{1} - x^2$

Lemma1.3 : $\int_{-1}^{1}$ Tn (x) Tm (x) dx$\frac{1}{\sqrt{1-x2}}$
$= \pi$, if n= m=0
$= \pi/2$, if $n = m \neq 0 \quad x^2$
$= 0$, if $n \neq m$

Lemma1.4:
$\int_{-1}^{1}$ Un (x) Um (x) $\sqrt{1} - x^2$ dx
$= \pi /2$, if $n = m$ recurrence relations
$=0$, if $n \neq m$

Identity 1.5: $T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$
Proof: $T_n(x) =\{ (x - \sqrt{x}^2 -- 1)^n + (x + \sqrt{x}^2 -- 1)^n\} / 2$, $T_0(x) = 1$, $T_1(x) = x$
Substituting, $r = (x + \sqrt{x}^2 -- 1)$ and $s = (x - \sqrt{x}^2 -- 1)$, we get
Hence $T_n(x) = (r^n + s^{n)}) / 2$, $r + s = 2x$, $r.s = 1$, $r - s = 2 \sqrt{x}^2- 1$ (*)
$U_n(x) = \{ (x + \sqrt{x}^2 -- 1)^{n+1} - \{ (x - \sqrt{x}^2 -- 1)^{n+1}/ \{ (x + \sqrt{x}^2 -- 1) - \{ (x - \sqrt{x}^2 -- 1)\}$
$= (r^{n+1} - s^{n+1}) / (r-s)$
(**)
$T_n(x)^2 =[(r^n + s^n) / 2]^2 = r^{2n} + 2r^n\, s^n + s^{2n} / 4 = r^{2n} + s^{2n} +2(rs)^n / 4 = (r^{2n} + s^{2n} +2)/4$

$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$

Identity 1.6: $T_{m+n}(x) + T_{m-n}(x) = 2\, T_m(x)\, T_n(x)$

Proof: $T_{m+n}(x) + T_{m-n}(x) = (r^{m+n} + s^{m+n}/2) + (r^{m-n} + s^{m-n}/2) = r^m(r^n + r^{-n})/2 + s^m(s^n + s^{-n})/2$

$= r^m (r^n + s^n)/2 + s^m (s^n + r^n)/2$

$= (r^n + s^n)/2 \,.\, (r^m + s^m)$

$= T_n(x)\, 2T_m(x)$

$= 2T_m(x)\, T_n(x)$

Particular cases of Identity 1.6

When m = n, we have

i) $T_{2n}(x) = 2\, T_n(x)^2 - T_0(x) = 2\, T_n(x)^2 - 1$

When m = n+1, we have

ii) $T_{2n+1}(x) = 2\, T_{n+1}(x)\, T_n(x) - T_1(x) = 2\, T_{n+1}(x)\, T_n(x) - x$

When m = n-1, we have

iii) $T_{2n-1}(x) = 2\, T_{n-1}(x)\, T_n(x) - T_{-1}(x) = 2\, T_{n-1}(x)\, T_n(x) - x$

Identity1.7: $T_{n+m}(x) = T_n(x)\, U_m(x) - T_{n-1}(x)\, U_{m-1}(x)$

Proof: $T_{n+m}(x) = r^{n+m} + s^{n+m}/2$

$T_n(x)\, U_m(x) - T_{n-1}(x)\, U_{m-1}(x)$

$= (r^n + s^n/2)\, (r^{m+1} - s^{m+1)}/r\text{-}s - (r^{n-1} + s^{n-1}/2)\, (r^m - s^m/r\text{-}s)$

$= (r^{m+n-1} - s^{m+n-1} + r^{m+1} s^n - r^n s^{m+1}) - (r^{m+n-1} - s^{m+n-1} + r^m s^{n-1} - r^{n-1} s^m) / 2\,(r\text{-}s)$

$= r^{n+m}\,(r\text{-}s) - s^{n-m}(s\text{-}r) + r^m s^n(r - s^{-1}) - s^m r^n(s - r^{-1})\, /2(r\text{-}s)$

$= r^{n+m}\,(r\text{-}s) - s^{n-m}\,(s\text{-}r)\, /2(r\text{-}s)$

$= r^{n-m}\,(r\text{-}s) + s^{n-m}\,(r\text{-}s)\, /\, 2(r\text{-}s)$

$= r^{n-m} + s^{n-m}/2$

$= T_{n-m}$

Particular cases of Identity 1.7

When n = m, we have

i) $T_{2n}(x) = T_n(x)\, U_n(x) - T_{n-1}(x)\, U_{n-1}(x)$

When n = m = 1, we have

ii) $T_2(x) = x.1 - 1.2 = x - 2$

Chebyshev Polynomial of second kind

$U_n(x)\, U_m(x) = \sum_{k=0}^{n} U_{m-n+2k}\,(x)$

$U_{m+2}(x) = U_2(x)\, U_m(x) - U_m(x) - U_{m-2}(x)\,,\ m \geq 2$

$\quad = U_m(x)\,(U_2(x) - 1) - U_{m-2}(x)$

$U_{n+m}(x) = U_n(x)\, T_m(x) + T_{n+1}(x)\, U_{m-1}(x)$

We prove the following identities using (*) and (**)

Identity 1.8 $U_{2n+1}(x) = U_n(x)\, 2T_{n+1}(x)$

Proof: We have $U_{2n+1}(x) = r^{2n+2} - s^{2n+2}/r\text{-}s$

$= (r^{n+1} - s^{n+1})\,(r^{n+1} + s^{n+1})\,/\,r\text{-}s$

$= [r^{n+1} - s^{n+1}/r\text{-}s][r^{n+1} + s^{n+1}]$

$= U_n(x)\, 2T_{n+1}(x)$

Identity 1.9 $T_{2n+2}(x) = 2T_{n+1}(x)^2 - 1$

Proof: $T_{2n+2}(x) = r^{2n+2} + s^{2n+2}/2$

$= (r^{n+1} + s^{n+1})^2 - 2r^{n+1}s^{n+1}/2$

$= (2T_{n+1}(x))^2/2 - (rs)^{n+1}$

$= 2T_{n+1}(x)^2 - 1$

Identity 2 $U_{2n+1}(x)\, T_{2n+2}(x) = 4U_n(x)\, T_{n+1}(x)^3 - 2\, U_n(x)\, T_{n+1}(x)$

Proof: $U_{2n+1}(x)\, T_{2n+2}(x) = 2\, U_n(x) T_{n+1}(x)\, \{2T_{n+1}(x)^2 - 1\}$

$= 4U_n(x)\, T_{n+1}(x)^3 - 2\, U_n(x)\, T_{n+1}(x)$

Identity 2.1 $T_{2n+1}(x)\, U_{2n+2}(x) = 1/2[U_{4n+3}] + T_1(x)$

Proof: $T_{2n+1}(x)\, U_{2n+2}(x) = \{ r^{2n+1} + s^{2n+1}/2\}\{ r^{2n+3} - s^{2n+3}/r\text{-}s\}$

$= r^{4n+4} - s^{4n+4} + r^{2n+1}s^{2n+1}(r^2 - s^2)\, \}\, /\, 2(r\text{-}s)$

$= 1/2[r^{4n+4} - S^{4n+4}/r\text{-}s] + [(rs)^{2n+1}(r\text{-}s)(r+s)]\, /2(r\text{-}s)$

$= 1/2[U_{4n+3]} + 1.(r+s)/2$

$= 1/2[U_{4n+3}] + T_1(x)$

Identity 2.2 $T_{2n+1}(x) = 2T_{n+1}(x)\, Tn(x) - T_1(x)$

Identity 2.3 $U_{m+n} + U_{m-n} = U_{m+1}\,(2T_n)$

$= r^{m+1}(r^n + r^{-n}) - s^{m+1}(s^n + s^{-n})\,/\,r\text{-}s$

$= r^{m+1}\,(r^n + s^n) - s^{m+1}(r^n + s^n)\,/r\text{-}s$

$= [r^{m+1} - s^{m+1}/r\text{-}s].\,(r^n + s^n)$

$= U_m\,.2T_n$

Similar identities in Fibonacci and Lucas polynomials can be proved.

Identity 2.4: $F_{m+n}(x) + F_{m-n}(x) = F_m(x)\, L_n(x)(\sqrt{\ } x^2+4)$

Identity 2.5: $L_{m+n}(x) + L_{m-n}(x) = L_m(x) L_n(x)$

W. Zhang [9] has proved the following lemmas for Chebyshev polynomial.

Lemma 2.6: $U_n\,(T_m(x)) = U_{m(n+1)-1}\,(x)\, /\, U_{m-1}\,(x)$

Lemma 2.7: $T_n\,(T_m\,(x)) = T_{mn}\,(x)$

We prove similar Lemmas in Fibonacci and Lucas polynomial.

Lemma 2.7: $F_n\,(L_m\,(x)) = F_{mn}\,(x)\, /\, 2^n\, F_m\,(x)$

Proof: $L_m(x) = 1/2^m\, \{\, (x + \sqrt{x^2+4})^m + (x - \sqrt{x^2+4})^m\}$

$L_m(x)^2 = \{\, ((x + \sqrt{x^2+4})^m + (x - \sqrt{x^2+4})^m / 2^m\, \}$

$= \{\, (x + \sqrt{x^2+4})^{2m} + (x - \sqrt{x^2+4})^{2m} - 8\, /\, 2^m\, \}$

Now adding 4 both sides we get,

$L_m(x)^2 + 4 = \{\, (x + \sqrt{x^2+4})^{2m} + (x - \sqrt{x^2+4})^{2m} - 8 + 16\, /\, 2^m\, \}$

$= \{\, (x + \sqrt{x^2+4})^{2m} + (x - \sqrt{x^2+4})^{2m} + 8\, /\, 2^m\, \}$

$= \{\, (x + \sqrt{x^2+4})^m - (x - \sqrt{x^2+4})^m / 2^m\, \}^2$

Taking square root of both sides we get,

$\sqrt{L_m(x)^2+4} = (x + \sqrt{x^2+4})^m - (x - \sqrt{x^2+4})^m / 2^m$

Now $L_m(x) + \sqrt{L_m(x)^2+4} = (x + \sqrt{x^2+4})^m / 2^m.2$

$L_m(x) - \sqrt{L_m(x)^2+4} = (x - \sqrt{x^2+4})^m / 2^m.2$

As $F_n(L_m(x)) = 1/2^n \sqrt{\ } (L_m(x)^2+4)\ \{\, (L_m(x) + \sqrt{L_m(x)^2+4}) - L_m(x) - \sqrt{L_m(x)^2+4}\, \}$

$= \{\, (x + \sqrt{x^2+4})^{mn} - (x - \sqrt{x^2+4})^{mn}\, \} / \{\, (x + \sqrt{x^2+4})^m - (x - \sqrt{x^2+4})^m\, \}$

$= F_{mn}(x) / F_m\,(x)$

Lemma 2.8 $L_n\,(L_m\,(x)) = L_{mn}\,(x)$

Proof: LHS $= L_n\, \{1/2^m\, (L_m(x) + \sqrt{L_m(x)^2+4})^m + (L_m(x) - \sqrt{L_m(x)^2+4})^m\}$

$= 1/2^{mn}\,(x + \sqrt{x^2+4})^{mn} + (x - \sqrt{x^2+4})^{mn}$

$= L_{mn}(x)$

Similar recurrence equations in Fibonacci and Lucas Numbers are the following.
Here we consider x=1, in Fn(x) and Ln(x)
1) $F_n L_{m-n} + F_{m-n} L_n = 2F_m$
2) $L_m L_n + L_m + L_{n+1} = 5 F_{m+n+1}$
3) $F_{m+1} L_{n+1} + F_m L_n = L_{m+n+1}$

Finally, we quote differential Equations for Chebyshev polynomials and Fibonacci and Lucas polynomial are as follows.

1) $d/dx (T_n(x)) = n U_{n-1}(x)$
2) $d/dx (U_n (x)) = \{ (n+1) T_{n+1}(x) - x U_n(x) \} / x^2 - 1$
3) $d/dx (F_n(x)) = \{n L_n(x) - x F_n(x) \} / x^2 - 4$
4) $d/dx (L_n(x)) = n F_n (x)$

2. Conclusion

Chebyshev polynomials play an important role in finding the Fibonacci and Lucas polynomials. They are the special classes derived from the Chebyshev polynomials. We also observed that recurrence relations in Chebyshev polynomials of first kind and second kind are parallel to in Fibonacci and Lucas polynomials.

3. Applications

1) Chebyshev polynomials are the examples of orthogonal polynomials which are related to De Moivre's formula, can be defined recursively.
2) Theory of orthogonal polynomials is mostly applied in analysis as an important tool in the approximation of functions.
3) The roots of Chebyshev polynomials of first kind, are used as nodes in polynomial interpolation.
4) However there are very important applications in coding theory.
5) As an application in coding theory and cryptography Lucas polynomials may be useful in the generation of irreducible polynomials of higher degree.

References

[1] Approximation Theory, Wikipedia.
[2] Buchman R.G., Fibonacci Numbers, Chebyshev polynomials Generalizations and difference equations, Fibonacci Quarterly, Vol 1(4), 1963
[3] Chebyshev Polynomials, Wikipedia.
[4] Fibonacci Polynomials, from MathWorld- a Wolfram Web Resource.
[5] Lucas sequences and Chebyshev polynomials, Matt Baker's Math Blog.
[6] Paul Bruckman, Elementary Problems 45-2, Fibonacci Quarterly, 2004.
[7] T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley, 2001.
[8] Yang Li, On Chebyshev Polynomials, Fibonacci Polynomials, and Their Derivatives,, Journal of Applied Mathematics, Volume2014(2014), Article ID 451953.,
[9] W. Zhang, Some Identities involving Fibonacci Numbers and Lucas Numbers, Fibonacci Quarterly, Vol2, 2004, 149-152.