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A Bayesian Solution to the Sleeping Beauty Problem Aided by Exemplification from the Solution of the Bertrand's Box Problem and Elimination of the Confusion and Discrepancy Therein

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Abstract: ***The paper discusses and aims to prove that the thirder position with respect to the Sleeping Beauty Problem is the correct one, i.e. the probability asked in the problem is 1/3. Bayesian Analysis and an analogical correspondence drawn with the solved but similar Bertrand's Box Problem constitute the twofold approach, employed to address the problem. The two methods, one objective and one subjective, both conclude by meticulous and unambiguous arguments and definitions, that the probability is indeed 1/3; also stressing how seemingly inconspicuous observations, alter the priori probabilities of an experiment and disrupt the original partitioning of events.***

Keywords: Sleeping Beauty Problem; Bayesian Probability; Partitioning of Sample Space; Priori Probabilities; Bertrand's Box Problem

1. Literature Survey

The problem was originally formulated in unpublished work in the mid 1980s by Arnold Zuboff (work that was later published as "One Self: The Logic of Experience"[1]), followed by a paper by Adam Elga and is related to problems of imperfect recall such as the "paradox of the absent minded driver". The name "Sleeping Beauty" was given to the problem by Robert Stalnaker and was first used in extensive discussion in the Usenet newsgroup *rec.puzzles* in 1999.

This problem continues to produce ongoing debate.

Thirder position

The thirder position argues that the probability of heads is 1/3. Adam Elga argued for this position originally as follows: Suppose Sleeping Beauty is told and she comes to fully believe that the coin landed tails. By even a highly restricted principle of indifference, her credence that it is Monday should equal her credence that it is Tuesday since being in one situation would be subjectively indistinguishable from the other. In other words, P(Monday | Tails) = P(Tuesday | Tails), and thus

P(Tails and Tuesday) = P(Tails and Monday).

Consider now that Sleeping Beauty is told upon awakening and comes to fully believe that it is Monday. She knows the experimental procedure doesn't require the coin to actually be tossed until Tuesday morning, as the result only affects what happens after the Monday interview. Guided by the objective chance of heads landing being equal to the chance of tails landing, it should therefore hold that P(Tails | Monday) = P(Heads | Monday), and thus

P(Tails and Tuesday) = P(Tails and Monday) = P(Heads and Monday).

Since these three outcomes are exhaustive and exclusive for one trial, the probability of each is one-third by the previous two steps in the argument.

Halfer position

David Lewis responded to Elga's paper with the position that Sleeping Beauty's credence that the coin landed heads should be 1/2. Sleeping Beauty receives no new non-self-locating information throughout the experiment because she is told the details of the experiment. Since her credence before the experiment is P(Heads) = 1/2, she ought to continue to have a credence of P(Heads) = 1/2 since she gains no new relevant evidence when she wakes up during the experiment. This directly contradicts one of the thirder's premises, since it means P(Tails | Monday) = 1/3 and P(Heads | Monday) = 2/3.

Nick Bostrom argues that Sleeping Beauty does have new evidence about her future from Sunday: "that she *is* now in it," but does not know whether it is Monday or Tuesday, so the halfer argument fails. In particular, she gains the information that it is not both Tuesday and the case that Heads was flipped.

Double halfer position

The double halfer position[1] argues that both P(Heads) and P(Heads | Monday) equal 1/2. Mikal Cozic, in particular, argues that context-sensitive propositions like "it is Monday" are in general problematic for conditionalization and proposes the use of an imaging rule instead, which supports the double halfer position.

2. Introduction

The **Sleeping Beauty problem** is a puzzle in decision theory in which an ideally rational epistemic agent is to be woken once or twice according to the toss of a coin, and asked her degree of belief for the coin having come up heads.

The problem was originally formulated in unpublished work in the mid 1980s by Arnold Zuboff (work that was later published as "One Self: The Logic of Experience"[1]), followed by a paper by Adam Elga[2] and is related to problems of imperfect recall such as the "paradox of the absent minded driver".

Sleeping Beauty volunteers to undergo the following experiment and is told all of the following details: On Sunday she will be put to sleep. Once or twice, during the experiment, Beauty will be awakened, interviewed, and put back to sleep with an amnesia-inducing drug that makes her forget that awakening. A fair coin will be tossed to determine which experimental procedure to undertake: if the coin comes up heads, Beauty will be awakened and interviewed on Monday only. If the coin comes up tails, she will be awakened and interviewed on Monday and Tuesday. In either case, she will be awakened on Wednesday without interview and the experiment ends.

Any time Sleeping Beauty is awakened and interviewed she will not be able to tell which day it is or whether she has been awakened before. During the interview Beauty is asked: "What is your credence now for the proposition that the coin landed heads?".

The paper aims to offer a conclusive resolution of the “Sleeping Beauty problem”, by a novel twofold approach : First, a discreet application of Bayes’ Theorem and second, by comparison and drawing analogies to the solved but once, equally tantalizing “Bertrand’s Box Problem”.

The paper strives to satisfactorily depict, why the thirder position with regard to the problem, is the correct one. The paper also extensively discusses, why the problem is debated, perplexing and perhaps counterintuitive. The latter part discusses at length, how the other incorrect positions taken by some mathematicians, with regard to the problem, are analogous to the typical, common wrong responses given by most people with regards to similar, conclusively solved problems, notably the eminent Bertrand’s Box Problem, The Monty-Hall Problem and The Three Prisoners’ Problem.

The paper also stresses how discreet and very specific selection and accurate description of “Events” is significant in application of the Bayes’ Theorem, and how often, the effect, than an observation, has on the experiment viz. The priori probabilities, is ignored, leading to an incorrect, though perhaps, intuitive solution.

The paper argues and concludes, by extensive and holistic discussion, that the probability, asked in the problem is 1/3.

3. The Problem

The elusive Sleeping Beauty Problem is stated as follows :

“ Sleeping Beauty volunteers to undergo the following experiment and is told all of the following details: Once or twice, during the experiment, Beauty will be awakened, interviewed, and put back to sleep with an amnesia-inducing drug that makes her forget that awakening. A fair coin will be tossed to determine which experimental procedure to undertake : if the coin comes up heads, Beauty will be awakened and interviewed on Monday only. If the coin shows up tails, she will be awakened and interviewed on Monday and Tuesday. In either case, she will be awakened on Wednesday without interview and the experiment ends.

Any time Sleeping Beauty is awakened and interviewed she will not be able to tell which day it is or whether she has been awakened before, During the interview, Beauty is asked : “What is your credence now for the proposition that the coin landed heads?”

4. The Solution

The Mathematical Formulation and Calculations

Baye’s Theorem (for 2 priori events) :

$$P(E_1|A) = \frac{P(A|E_1)\cdot P(E_1)}{P(A|E_1)\cdot P(E_1) + P(A|E_2)\cdot P(E_2)}$$

...Eq. (i)

where E_1 and E_2 are disjoint, exhaustive and nonzero.

Now, consider the following events:

A : Sleeping Beauty is awake that very day or Sleeping Beauty was awakened on that particular day from Monday & Tuesday

E_1 : The coin landed Heads = ½

E_2 : The coin landed Tails = ½

Clearly, **E_1** and **E_2** are disjoint, exhaustive and nonzero, hence, satisfying all the requisite criteria of Bayes’ Theorem

$P(A|E_1)$ = Probability that she was awakened on a day from amongst Monday and Tuesday if the coin came Heads = Total No. Of favourable cases/Total No. Of possible cases = Total No. Of awakenings resultant of a Head/Total no. Of possible awakenings =

$$\frac{\boldsymbol{n\,(Monday)}}{\boldsymbol{n\,(Monday \cup Tuesday)}} = \frac{\mathbf{1}}{\mathbf{2}}$$

Similarily,

$P(A|E_2)$ = Probability that she was awakened on a day from amongst Monday and Tuesday if the coin came Tails =

$$\frac{\boldsymbol{n\,(Monday \cup Tuesday)}}{\boldsymbol{n\,(Monday \cup Tuesday)}} = \frac{\mathbf{2}}{\mathbf{2}} = \mathbf{1}$$

Now putting the values in ...Eq. (i), we get

$$\mathbf{P(E_1|A)} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+1\cdot\frac{1}{2}} = \frac{1}{3}$$

Subjective, Intuitive & Argumentative Explanation

As is typical with intriguing problems of probability and as is characteristic of Bayesian Analysis, an observation alters the expected probabilities of an experimental setup by disrupting the intuitive symmetry, weighted proportionate distribution or equipartitions. A famous, analogous example, albeit a solved one, but nonetheless no less debated, is the classic Bertrand's Box Problem.

Here, because the Sleeping Beauty is awake, that very fact biases (tilts odds) towards the priori case which would have led to a more abundance of awakenings. Therefore, because of the fact that she is awakened, the cause which would have favoured being awakened more (numerously) automatically has its priori probability increased and the one which would have led to lesser chance or frequency of awakenings (here Heads) proportionately decreases. Hence the ½ : ½ symmetry which had existed until the very instant, the beauty was awakened got disrupted at that precise moment and became 1/3 and 2/3 resp for Heads & Tails, because being awakened was favoured comparatively or proportionately more by the Tails case than the Heads. (Because Tails led to 2 awakenings on the stipulated experiment days while Heads led to 1).

This is an argument quite similar to that given for the Monty Hall Problem's Solution.

5. The Comparative Analogical Analysis

We begin the analysis of this tantalizing problem and the controversy pertaining to it by referencing to the root of confusion and error in another eminent problem, which, though simpler and resolved much earlier with a well-known solution had perplexed many of those who tried to intuitively approach it and also gave rise to many of other more perplexing contemporary probability puzzles.

That aforementioned problem is the well-known "Bertrand's Box Problem", which can be briefly stated as follows :

"There are three identical boxes, one with two gold coins, second with one gold coin and one silver coin, and the third one with two silver coins. One box is chosen at random and a coin is drawn from it, again at random, and is revealed to be a gold coin. What is the probability that the other coin is also a gold coin?"

The answer to the above problem, at first glance and often, even after repeated approaches seems to be clearly 1/2., intuitively. However, the answer is 2/3. A simple application of the Bayes' Theorem clearly testifies for the latter answer. Hence, we seek what might have been the misconception or error in our intuitive approach, which resulted in many of us quoting the former wrong answer with a high degree of confidence.

Let us analyse the intuitive approach that most of us take. The display of one coin as a gold coin eliminates the possibility of the third box being the selected box. Now we are left with two boxes. Most of us might unwarily be tempted or spontaneously be drawn and compelled to judge that both boxes have now equal and ½ probability of being the chosen box, as was the case before the drawing of the box and the one gold coin, as both contained gold coins.

Nevertheless, although it is true that, either of the boxes being chosen was equally likely in the beginning, prior to the draw, the draw of the coin has altered it. This is more explicitly evident in and illustrated as it turning the probability of the third box from 1/3 to 0. Similarly, the two remaining boxes are NOT equally likely to be the chosen box, as the revelation has changed their probabilities too, like it has annulled that of the third box; The occurrence of the event of observation of withdrawal of one coin, the fact that the one coin drawn is gold, has biased the odds and re-apportioned probability in favour of the one with greater number of gold coins, i.e. Box1. That means, because one coin is gold, it is more likely that the box chosen, from which it was drawn is the box with more number of gold coins. Hence, the box being Box 1 has a probability of 2/3 and it being Box 2 has probability 1/3.

The revelation of one gold coin has not only narrowed down our choices from three boxes to two but has changed the fact that all three boxes were equally likely to be chosen; All three now have different and unequal probabilities of being chosen because each contains a different number of gold coins, which being the observation has become the selective or biasing factor in the play and altered the probability distribution.

This explanation and reasoning is intuitively much more satisfying and relatable.

We can re-phrase the above explained reason in a more generalised form as "The occurrence of a event has biased the previously equal odds in favour of the other event whose occurrence had been more favourable to occurrence of the first." Or we can put it as "The occurrence of the event has biased the odds in favour of the event that was more likely to cause it or where it was more likely to occur or following whose occurrence, the former's occurrence was rendered more likely".

Now coming back to the case at hand i.e. the Sleeping Beauty Problem.

Here, it would be quite pertinent and convenient to draw certain analogies and correspondences between this problem and the Bertrand's Box Problem explained above, prior to the commencement our solution of the former.

We shall relate being awoken to getting a gold coin in the random draw and connect a greater abundance of gold coins in Box Problem to more abundance of awakenings in the Sleep Experiment Problem. To some proponents of rival solutions, it may seem that given Beauty's non awareness of her previous awakenings would invalidate the concept of "abundance" or "frequency of occurrence" of awakenings; This shall be addressed and shown to be misleading automatically, in due course, without explicit mentioning.

As the withdrawal of one coin and its revelation to be gold disrupted the equitable balance of the boxes, in favour of the one with more such (gold) coins, Similarly, the very fact that Sleeping Beauty is awake on that particular day, regardless of whatever day it might be, tilts the odds in favour of the coin toss having resulted in that case, which would have given more awakenings or, in other words, that case (heads or tails) which would have favoured a higher frequency of occurrence of awakenings i.e. Tails.
By a similar, simple mathematical workout as with the Box Problem, we find that the probability of heads having occurred is 1/3 and that of tails is 2/3; the same answer as in the Box Problem. Here the very occurrence of an awakening biasing in favour of the case which gives more awakenings (i.e. Tails which gives 2 awakenings, over the other case of heads which results in only one awakening just like the 2 and 1 gold coins in the first and second box respectively in the previous problem).

Another factor might have been if the day of awakening was a Wednesday; but this factor has no bias for either of the two cases the coin toss may result in. It does NOT play a role here. Here, the role of Beauty being awoken on a Wednesday is NOT pertinent to the final asked question whose answer is requisite, because it has no relation with the event of coin toss, which forms the principal object of the final question whose answer is the requisite final answer. Hence, it needs not to be taken into consideration for furnishing an answer to this particular final question.

Hence, the answer of the Sleeping Beauty Problem is 1/3. The probability of heads occurring should be answered by Beauty as 1/3 given her knowledge of the experiment rules, which were explained to her, well in advance.

As an additional note to those who expound that the heavy restrictions imposed on the test subject Sleeping Beauty and her oblivion would restrict her into giving the elementary, natural answer of 1/2 i.e. the only possible sample space of a fair, independent coin toss, which is the only knowledge she has; It is essential and indispensible t remember and emphasize that the test subject Beauty does NOT forget the basic, primary, first rules and terms of the experiment conveyed to her initially, prior to her being sedated and put to sleep i.e. the number of awakenings if a head appears and the number of awakenings if a tail comes up. Hence, without garnering any statistical information, based on the conditions and rules of the experiment essentially conveyed to her, Sleeping Beauty can prove the odds to be 1/3 in favour of head having come up.

6. Conclusion

The answer to the problem is 1/3, i.e. the credence of Sleeping Beauty, when interviewed, about the coin having landed heads should be 1/3.

7. Future Scope

Both components of the approach are entirely novel. It also depicts, how crucial, proper and accurate selection, definition and distinction of events and sample space partitions is, in Bayesian Probability. In the analogical components, it depicts how counterintuitive problems tend to mislead us and how to avoid them. The research has possible implications in stimulating new investigations in varied fields from decision theory to psychology and logic to quantum states.

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