On The Category of Symplectic Yang-Mills Fields

Lutanda Panga G[1], **Mateso Tailoshi B.**[2], **Patrick Azere Phiri**[3], **Kabunda Kasakwa H**[4], **Bwalya Mulumba I.**[5]

[1, 2]Department of Mathematics and Informatic, University of Lubumbashi, Lubumbashi, D.R.Congo

[3]Department of Mathematics, Copperbelt University, Kitwe, Zambia

[4, 5]Département de Mathématiques - Physique, Institut Supérieur Pédagogique de Lubumbashi, D.R.Congo

Abstract: ***The main goal of this paper consists to built the category of symplectic Yang-Mills fields by mean of Abstract Differential Geometry (ADG) mechanisms. We realize this category as a subcategory of the category of symplectic vector sheaves and also as a subcategory of the category of Yang-Mills fields.***

Keywords: vector sheaf, symplectic vector sheaf, A-connection, Yang-Mills field, symplectic A-connection, symplectic Yang-Mills field.

1. Introduction

In his two volumes "Modern Differential Geometry in Gauge Theories" [1], [2], Mallios expressed Yang-Mills theory via sheaf materials. He considered as application the case of Riemannian vector sheaf (E, ρ) over a topological space X, i.e, a vector sheaf equipped with a Riemannian A-form ρ and defined a Riemannian Yang-Mills field $\{(E, \nabla); \rho\}$ on X as a Riemannian vector sheaf endowed with an A-connection ∇. Based of the previous considerations he discussed in details the case of Maxwell field which plays an important role in electrodymanic. Treating the case of symplectic sheaves is our battle-horse. In our previous articles, we explore the symplectic case by considering a vector sheaf endowed with a symplectic A-form and a symplectic A-connection.

In this paper, we use the technical machinery of Abstract Differential Geometry (ADG); indeed the sheaf-theoretic approach to explore the subcategory of the category of vector sheaves named the category of symplectic Yang-Mills fields. The objects and the arrows of this category are respectively symplectic Yang-Mills fields and symplectic Yang-Mills fields morphisms.

2. Category of Vector Sheaves

Definition 2.1 Let E and X be two topological spaces and π : E → X a local homeomorphism between them, the triple (E, π, X) is called a sheaf over X.

We notice that E and X are respectively named the sheaf space and the base space of the sheaf (E, π, X).

Definition 2.2 Given a sheaf (E, π, X) and $x \in X$, the set, denoted E_x, defined by $E_x = \pi^{-1}(\{x\})$ is the fiber over x.

Given $E \equiv (E, \pi, X)$ a sheaf over X and U an open subset of X, a local section of the sheaf E over U is a continuous map $s : U \to E$ such that $\pi \circ s = id_U$.

We designate by $E(U) \equiv \Gamma(U, E)$ the set of all local sections of E over U.

Definition 2.3 Given two sheaves $E \equiv (E, \pi, X)$ and $F \equiv (F, \lambda, X)$ over a same topological space X. A continuous map $\phi: E \to F$ such that $\pi = \lambda \circ \phi$ is called a sheaf morphism. [3]

A sheaf morphism preserves fibers i.e for a given sheaf morphism $\phi: E \to F$ one has $\phi(E_x) \subset F_x$ with $x \in X$.

Proposition 2.4 The composition of two sheaf morphisms is a sheaf morphism.

Proof. It is obvious.

The category of sheaves on a topological space X, designated by $\boldsymbol{Sh_X}$, admits sheaves over X as objects and sheaf morphisms as arrows X.

Definition 2.5 Let X be a topological space, $A \equiv (A, \tau, X)$ is called a sheaf of $\mathbb{C}$-algebras over X if the following holds :

(i) A is a sheaf of rings on X,
(ii) the fiber over x, $A_x = \tau^{-1}(x)$, is a $\mathbb{C}$-algebra, for any $x \in X$,
(iii) The scalar multiplication $\cdot : \mathbb{C} \times A \to A$, $(a,\alpha) \mapsto a\cdot\alpha \in A_x$ with $\tau(a) = x \in X$, is continuous.

$A \equiv (A, \tau, X)$ is called the structure sheaf of generalized arithmetics or coordinates (see [4]). In the remaining, we will consider $A \equiv (A, \tau, X)$ a sheaf of $\mathbb{C}$-algebras with identity element and we assume that A is associative and commutative.

Definition 2.6 Given X a topological space and $A \equiv (A,\tau, X)$ a sheaf of $\mathbb{C}$-algebras over X, a triplet (E, π, X) is called a sheaf of A-modules if the following holds :

(i) E is a sheaf of abelian groups,
(ii) The fiber over x, $E_x = \pi^{-1}(x)$, is a left A_x-module,
(iii) The exterior multiplication: $A \times E \to E$, $(a,z) \mapsto a\cdot z \in E_x$ with $\tau(a) = \pi(z) = x \in X$, is continuous. [1]

Note that for two sheaves of A-modules $E \equiv (E, \pi, X)$ and $F \equiv (F, \lambda, X)$ over X, a sheaf morphism $\phi : E \to F$ is called a morphism of A-modules. [3]

The category whose objects are the A-modules over a topological space X and whose arrows are the morphisms of A-modules over X is a sub-category of the category of sheaves over X. We designate it by $\boldsymbol{A\text{-}Mod_X}$.

Definition 2.7 A vector sheaf $E \equiv (E, \pi, X)$ on a topological space X is a locally free A-module of finite rank n over X, i.e for any open subet U of X, $E|_U = A^n|_U = (A|_U)^n$.

The rank of a vector sheaf E is the positive interger n so that $E|_U = A^n|_U$. An open set $U \subset X$ such that $E|_U = A^n|_U$ is called a local gauge of the A-module E.

A morphism of vector sheaves is a morphism of locally free A-modules.

The category of vector sheaves over a topological space X is denoted $\boldsymbol{VectSh_X}$. This category is a subcategory of $\boldsymbol{A\text{-}Mod_X}$ the category of A-modules over X

3. Category Symplectic Vector Sheaves

Definition 3.1 Let E be a vector sheaf on a given topological space X. A sheaf morphism $\sigma : E \bigoplus E \rightarrow A$ which is
(i) A-bilinear,
(ii) skew-symmetric,
(iii) nondegenerate
is called a symplectic A-form on E. [5, p.59]

A vector sheaf E of even rank on a topological space X equipped with a symplectic A-form σ is said to be a symplectic vector sheaf on X which is denoted by (E,σ).

Definition 3.2 Given two symplectic vector sheaves (E,σ) and (E',σ') on a topological space X. A morphism sheaf $\varphi : E \rightarrow E'$ such that $\sigma = \sigma' \circ (\varphi, \varphi)$ i.e for any sections s, t $\in E(U)$, $(\sigma' \circ (\varphi, \varphi))(s,t) = \sigma'(\varphi(s),\varphi(t)) = \sigma(s,t)$ with U an open subset of X, is called an A-symplectomorphism .

An A-symplectomorphism preserves fibers i.e for two symplectic vector sheaves $\varphi : E \rightarrow E'$ one has $\varphi(E_x) \subset E'_x$ with $x \in X$.

If E=E', $\varphi : E \rightarrow E$ such that $\sigma = \sigma \circ (\varphi, \varphi)$ is an A-symplectomorphism of E.

Proposition 3.3 The composition map of two A-symplectomorphisms is an A-symplectomorphisms.

Proof. For three given symplectic vector sheaves (E,σ), (E',σ') and (E'',σ'') on a topological space X, one consider the A-symplectomorphisms $\varphi : E \rightarrow E'$ and $\varphi' : E' \rightarrow E''$. For every $x \in X$, since φ and φ' are fiber preserving maps i.e $\varphi(E_x) \subset E'_x$ and $\varphi'(E'_x) \subset E''_x$, it is obvious that $\varphi' \circ \varphi(E_x) \subset E''_x$. Thus the composition map $\varphi' \circ \varphi$ is a fiber preserving map.
As $\sigma = \sigma' \circ (\varphi, \varphi)$ and $\sigma' = \sigma'' \circ (\varphi', \varphi')$, it stands that
$\sigma = [\, \sigma'' \circ (\varphi', \varphi')] \circ (\varphi, \varphi)$
$\quad = \sigma'' \circ (\varphi' \circ \varphi , \varphi' \circ \varphi)$.
It follows that the map $\varphi' \circ \varphi$ is an A-symplectomorphism

Note that for a given symplectic vector sheaf (E,σ), the identity morphism $1_E : E \rightarrow E$ is an A-symplectomorphism, it preserves fiber i.e $1_E(E_x) \subset E_x$ for any $x \in E$ and $\sigma \circ (1_E , 1_E) = \sigma$.
According to the fact that the composition of sheaf morphisms is associative it follows that the composition of A-symplectomorphisms is also associative.

The category designated by $\boldsymbol{SVectSh_X}$, whose objects are the symplectic vector sheaves over X and whose morphisms are the A-symplectomorphisms, is called the category of symplectic vector sheaves.

It is obvious that $\boldsymbol{SVectSh_X}$ is a subcategory of $\boldsymbol{VectSh_X}$ the category of vector sheaves over X.

4. Category of Symplectic Yang-Mills Fields

Definition 4.1 Let E be an A-module on a topological space X and $\partial : A \rightarrow E$ a sheaf morphism. The triplet (A,∂,E) constitutes a differential triad, if it satisfies the following conditions :

(i) ∂ is a $\mathbb{C}$-linear morphism,
(ii) For any s,t $\in$ A(U), $\partial(s.t) = \partial(s).t + s.\partial(t)$. (1) [2]

The notion of differential triad in Abstract Differential Geometry (ADG) replaces charts and atlases in Classical Differential Geometry (CDG). It plays the role of the structure sheaf germs of smooth functions. See for instance [7] and [8].

Definition 4.2 Let (A,∂,Ω) be a differential triad on a topological space X and E an A-module on X.A sheaf morphism

$$\nabla \in Hom_{\mathbb{C}}(E,E \otimes \Omega) \qquad (2)$$

such that for any $\alpha \in A(U)$ and $s \in E(U)$,

$$\nabla(\alpha s) = \alpha\nabla(s) + \partial(s) \otimes \alpha \qquad (3)$$

is called an A-connection of the vector sheaf E, where U is an open subset of X (see [9]).

In Abstract Differential Geometry, the concept of A-connection is the fundamental one, about which the whole theory revolves (see [10] p.7).

A Yang-Mills field is the pair (E,∇) where E is a vector sheaf and ∇ is an A-connection of E (see[11] p.64).

Definition 4.3 Given two Yang-Mills fields (E,∇) and (E',∇'), a sheaf morphism $\varphi : E \rightarrow E'$ such that
$\nabla' \circ \varphi = (\varphi \otimes id_\Omega) \circ \nabla$ is a Yang-Mills field morphism.

If $\varphi : (E,\nabla) \rightarrow (E',\nabla')$ is a Yang-Mills field morphism, the A-connections ∇ and ∇' are said to be φ-related.

In particular, for a given Yang-Mills field (E,∇) , the identity morphism $id_E : (E,\nabla) \rightarrow (E,\nabla)$ is a Yang-Mills field morphism i.e $\nabla \circ id_E = (id_E \otimes id_\Omega) \circ\nabla$.

It is obvious to establish that if (E,∇), (E',∇') and (E'',∇'') are Yang-Mills fields such that ∇ and ∇' are φ-related and ∇' and ∇'' are φ'-related, then ∇ and ∇'' are $\varphi' \circ \varphi$-related.

We denote by $\boldsymbol{YM_X}$ the category of Yang-Mills fields. Given (E,∇) and (E',∇') $\in$ Ob($\boldsymbol{YM_X}$) i.e two Yang-Mills fields, a sheaf morphism $\varphi \in Hom_A$(E,E') such that $\nabla' \circ \varphi = (\varphi \otimes id_\Omega) \circ \nabla$ belongs to Mor($\boldsymbol{YM_X}$).

Definition 4.4 Let (E,σ) be a symplectic vector sheaf on a topological space X, ∇ an A-connection on E is said to be a symplectic A-connection on E if $\nabla\sigma = 0$ i.e for any U open subset of X and sections s,t,r $\in$ E(U), one has :

$$\partial[\sigma(t,r)](s)= \sigma(\nabla(s)(t),r)+\sigma(t,\nabla(s)(r)). \qquad (4)$$

Definition 4.5 Let (E,σ) be a symplectic vector sheaf on a given topological space X and ∇ a symplectic A-connection on E, the pair (E,∇) is called a symplectic Yang-Mills field. [6 p.1102]

We will denote a symplectic Yang-Mills field (E,∇) by {(E,∇); σ}.

Consider two symplectic Yang-Mills fields {(E,∇);σ}, {(E',∇'); σ'}, a sheaf morphism φ : E → E' is called a symplectic Yang-Mills fields morphism if :
(i) φ is an A-symplectomorphism,
(ii) ∇ and ∇' are φ-related.

Proposition 4.6 Given a symplectic Yang-Mills field {(E,∇);σ} and a symplectic vector sheaf (E',σ'), an A-symplectomorphism φ : E → E' induces the existence of symplectic A-connection on E'.

Proof. From the definition of ∇ a symplectic A-connection on E i.e

$$\partial[\sigma(t,r)](s)= \sigma(\nabla(s)(t),r)+\sigma(t,\nabla(s)(r)) \qquad (5)$$

for any U open subset of X and sections s,t,r $\in$ E(U), and by using the fact that φ : E → E' is an A-symplectomorphism i.e

$$\sigma = \sigma' \circ (\varphi , \varphi) . \qquad (6)$$

As φ : E → E' is an A-symplectomorphism, it follows that there exists an A-endormorphism $\nabla'(\varphi(s))$ of E' such that $\varphi \circ \nabla(s) = \nabla'(\varphi(s)) \circ \varphi$.

Hence $\nabla'(\varphi(s)) = \varphi \circ \nabla(s) \circ \varphi^{-1}$ and we can write

$$\begin{aligned}
\partial[\sigma(t,r)](s) &= \sigma(\nabla(s)(t),r) + \sigma(t,\nabla(s)(r)) \\
&= \sigma'(\varphi(\nabla(s)(t)), \varphi (r)) + \sigma'(\varphi (t),\varphi(\nabla(s)(r))) \\
&= \sigma'(\varphi \circ \nabla(s)(t), \varphi (r)) + \sigma'(\varphi(t), \varphi \circ \nabla(s)(r)) \\
&=\sigma'(\nabla'(\varphi(s))\circ \varphi(t),\varphi(r))+\sigma'(\varphi(t),\nabla'(\varphi(s)) \circ \varphi(r)) \\
&= \sigma'(\nabla'(\varphi(s))\varphi(t),\varphi(r))+\sigma'(\varphi(t),\nabla'(\varphi(s))\, \varphi(r)) \\
&= \partial[\sigma'(\varphi (t), \varphi (r))]\, \varphi (s).
\end{aligned}$$

Proposition 4.7 Let {(E,∇);σ} be a symplectic Yang-Mills fields, the identity morphism 1_E : E → E is a symplectic Yang-Mills fields morphism.

Proof. Since id_E : E → E is an A-symplectomorphism i.e $\sigma = \sigma \circ (id_E, id_E)$ and id_E : (E,∇) → (E,∇) is a Yang-Mills field morphism i.e $\nabla \circ id_E = (id_E \otimes id_\Omega) \circ\nabla$, it follows that id_E is aYang-Mills field morphism.

Proposition 4.8 The composition map of two symplectic Yang-Mills fields morphisms is a symplectic Yang-Mills fields morphism.

Proof. According to the fact that the composition of two A-symplectomorphisms is an A-symplectomorphism and the composition of two Yang-Mills fields morphisms is a Yang-Mills fields morphism, one gets the result.

The subcategory together of the category of symplectic vector sheaves $\boldsymbol{SVectSh_X}$ and of the category of Yang-Mills fields $\boldsymbol{YM_X}$, whose objects are symplectic Yang-Mills fields and whose morphisms are symplectic Yang-Mills fields morphisms, denoted $\boldsymbol{SYM_X}$, is named the category of symplectic Yang-Mills fields.

5. Conclusion

We describe a sheaf-theoretic approach to the category of symplectic Yang-Mills fields. The proposition 4.6 leads us to define a symplectic A-connetion ∇'on a symplectic vector sheaf (E',σ') from an A-symplectomorphism between a symplectic Yang-Mills field {(E,∇);σ} and (E',σ'), it permits to define morphisms of the category of symplectic Yang-Mills fields $\boldsymbol{SYM_X}$.

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