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# Forced Thermal Ratchets and Stochastic Resonance in Inhomogeneous Underdamped Periodic Potential System

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**Abstract:** ***In this work we numerically solve the 1-d Langevin equation to obtain the trajectories of particles moving in a sinusoidal potential in a medium that also offers a spatially periodic damping coefficient with a phase difference with respect to the potential. In addition, the system is also driven by a weak periodic force. In a limited range of the system's parameters, two solutions of the trajectories are obtained at low noise strength (temperature) and in the deterministic limit (no noise). These solutions are of small amplitude and large amplitudes with respect to the forcing. The occurrence of stochastic resonance (SR) in a similar system has been explained as due to the existence of these two solutions. Also, due to the frictional asymmetry, the system also exhibits the phenomenon of ratchet effect (RE). In this work, we attempt to obtain both SR and RE simultaneously in the same domain of parameter space considered.***

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## 1. Introduction

Brownian particles diffusing in periodic structures and driven by not so obvious periodic forcing abound in nature. Such systems arise in biological domains [1], electronic setups [2], and Josephson structures [3], to name a few. In Feynman's famous thought experiment [4], initially proposed by Smoluchowski [5], the conclusion was made that no matter what the anisotropy of the periodic structure is, directed transport or the so called ratchet effect (RE) is not possible for a system in equilibrium. In order to get a net current in isothermal conditions, the system has to be driven out of equilibrium in conjunction with broken symmetry. Such criteria can generate a plethora of noise induced phenomena apart from the ratchet effect. Consequently various models of ratchets have been proposed, see [6]. Another one of those is a phenomenon called stochastic resonance (SR). Initially proposed in the early 1980's by Benzi [7], to explain the recurrence of the ice-ages in the earth, SR was quantified by the peaking behavior of the signal-to-noise ratio. This climatic change was modeled as a bistable system. Although, SR was later found to be an incorrect explanation to the ice-age phenomenon, it however divulged in other bistable systems [8] and was met with great success. Soon questions were raised whether SR could also be observed in periodic (sinusoidal) systems which were multistable. It was only a few years ago that this question met a positive response through a numerical work [9]. The authors this time used a different quantifier of SR - Sekimoto's Stochastic Energetics definition of input energy [10]. Input energy is defined as the work done by the drive (signal/weak periodic force) on the system. In this paper, we consider the system to be 1d so the path taken by the particles is linear. At low temperatures (low noise strengths) and in the deterministic regime (no noise applied), it was seen that the particles undergo oscillations either with large amplitudes (LA) as well as larger phase difference with respect to the periodic forcing of suitable frequency or with smaller amplitudes (SA) and smaller phase lag. These oscillations are ascribed to have the status of dynamical states with their apparent resemblance to the wells of bistability. They were later also subsequently observed in other related numerical works in [11–13]. At higher temperatures however, their distinct identities becomes a blur owing to the transitions between them. Moreover only in the parameter space where the two states of trajectories coexist, SR was found to occur [9, 13, 14]. In this small paper, we aim to show that both the phenomenon of RE and SR can be observed simultaneously in the same domain of parameter space that we have chosen to explore. This same work had been carried out in a similar inhomogeneous system but in a different parameter space [14]. By an inhomogeneous system, we mean that the spatially periodic friction coefficient offered by the medium and the periodic potential where the particles move, are out of phase with one another by a phase difference $\theta$. Such inhomogeneity arises in biological systems [15] and can also be designed artificially [16].

## 2. The Model

We consider the motion of an ensemble of underdamped non-interacting Brownian particles each of mass m in a periodic potential $V(x) = -V_0 \sin(kx)$, with peak value $V_0$ and wave number $k$. As mentioned in the introduction, the medium in which the particle moves is taken to be inhomogeneous in a way that it offers a spatially periodic friction with coefficient

$$\gamma(x) = \gamma_0(1 - \lambda \sin(kx + \theta)) \qquad (2.1)$$

that lags the potential by a phase difference $\theta$. This choice of friction coefficient introduces the necessary asymmetry for the system to generate a net current. Here, $\lambda$ is the inhomogeneity parameter and it specifies the strength of the friction coefficient and $\gamma_0$ is the average value of the friction coefficient over one period.

In addition, the potential is rocked by a weak periodic time-dependent forcing, weak in the sense that it is unable to cause the particle to hop from the initial well to neighbouring wells by itself, $F(t) = F_0 \cos(\omega t)$, with $\omega = 2\pi/\tau$ as the rocking frequency and $\tau$ as the rocking period. The equation of motion of the particle subjected to a thermal Gaussian white noise $\xi(t)$ at temperature $T$ is given by the Langevin equation,

$$m\frac{d^2x}{dt^2}=-\gamma(x)\frac{dx}{dt}+\frac{\partial V(x)}{\partial x}+F(t)+\sqrt{\gamma(x)T}\,\xi(t) \quad (2.2)$$

with

$$<\xi(t)>=0, <\xi(t)\xi(t')>=\delta(t-t'). \quad (2.3)$$

For simplicity and convenience the equation is transformed in to dimensionless units [17] by setting $m = 1$, $V_0 = 1$and $k = 1$, with reduced variables denoted again by the same symbols. Thus, the Langevin equation takes the form

$$\frac{d^2x}{dt^2}=-\gamma(x)\frac{dx}{dt}+\frac{\partial V(x)}{\partial x}+F(t)+\sqrt{\gamma(x)T}\,\xi(t) \quad (2.4)$$

where the potential is reduced to $V(x) = -\sin(x)$ and the friction coefficient to $\gamma(x) = \gamma_0(1 - \lambda \sin(x+\theta))$ (2.5)

Equation (2.4) is numerically solved to obtain the trajectories $x(t)$ of the particle for various initial conditions. For initial positions $x(t=0)$ the period $-\frac{\pi}{2} \leq x < 3\frac{\pi}{2}$ is divided uniformly into $n = 100$ parts (and hence $n$ initial positions)and the initial velocity v(t = 0) is set equal to zero throughout in the following.

For each trajectory, corresponding to one initial position $x(0)$, the work done by the force

$F(t)$ on the system, or the input energy, is calculated as [10]:

$$W(0,N\tau)=\int_0^{N\tau}\frac{\partial U(x(t),t)}{\partial t}\,dt, \quad (2.6)$$

where $N$ is a large integer denoting the number of periods taken to reach the final point of the trajectory. The effective potential $U$ is given by

$$U(x(t),t)=V(x(t))-x(t)F(t). \quad (2.7)$$

The mean input energy per period, for a particular initial condition, is therefore given by

$$\bar{W}=W(0,N\tau) \quad (2.8)$$

Similarly, the mean velocity, for one initial condition is calculated as

$$\bar{v}=x(t=N\tau)/N\tau \quad (2.9)$$

and the overall mean (net) velocity or the ratchet current, $<\bar{v}>$, is calculated as the ensemble average over all $n$ initial conditions. In the same way, the mean input energy over the ensemble is denoted as $<\bar{W}>$. As mentioned earlier, we use $<\bar{W}>$ as the quantifier of SR in this work.

## 3. Numerical Results

We show in Fig. 1, the symmetry breaking between the potential $V(x)$ and the friction coefficient $\gamma(x)$ for two values of phase difference $\theta = 0.25\pi$ and $0.75\pi$. As mentioned earlier, this symmetry breaking is necessary for observing RE although it is not essential for observing SR. SR has been shown earlier in a similar system but with constant friction coefficient [9]. One expects maximum current when the phase difference between the potential and the friction coefficient is maximum i.e. at $\theta = 0.5\pi$, but it has been shown earlier [14] that this is not so. It has been argued there that although $\theta = 0.5\pi$ exhibits maximum asymmetry, yet maximum current occurs at a smaller $\theta$ value. This is just because the effective friction shows a monotonic increase with $\theta$ in the range $0 < \theta < \pi$.

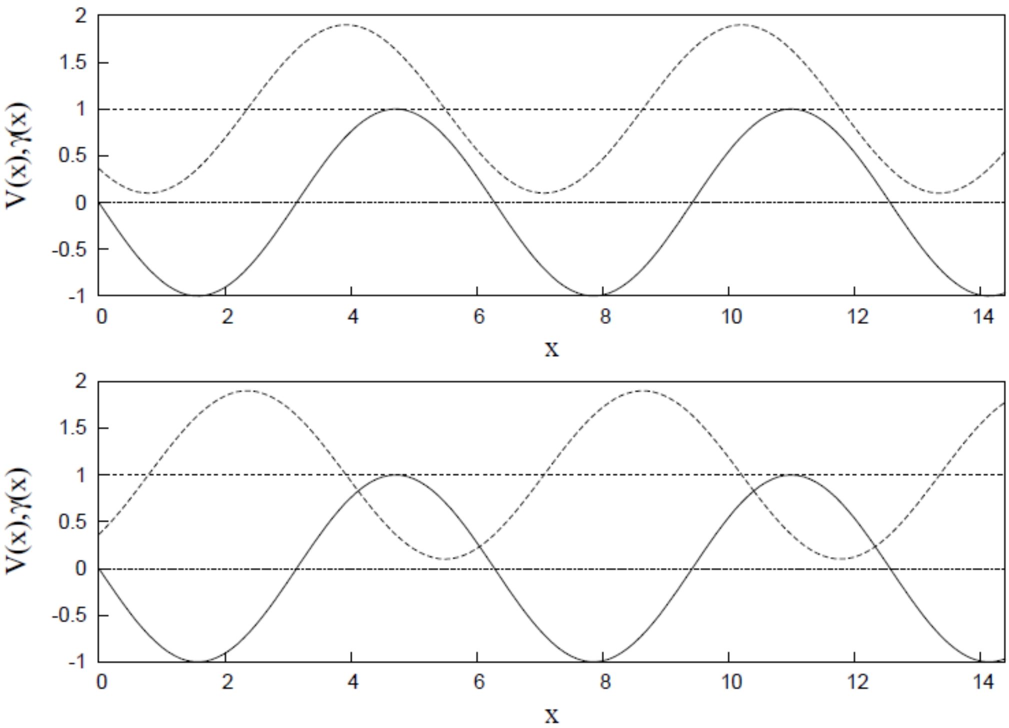


**Figure 1:** Oscillating shape of the potential $V(x)$ (in unbroken lines) and the phase-shifted friction coefficient $\gamma(x)$ (broken lines) with $\lambda = 0.9$ and $\gamma_0 = 1$. The corresponding horizontal lines denote the average value of $V(x)$ and $\gamma(x)$. The top figure is when $\theta = 0.25\pi$ and the bottom figure is when $\theta = 0.75\pi$

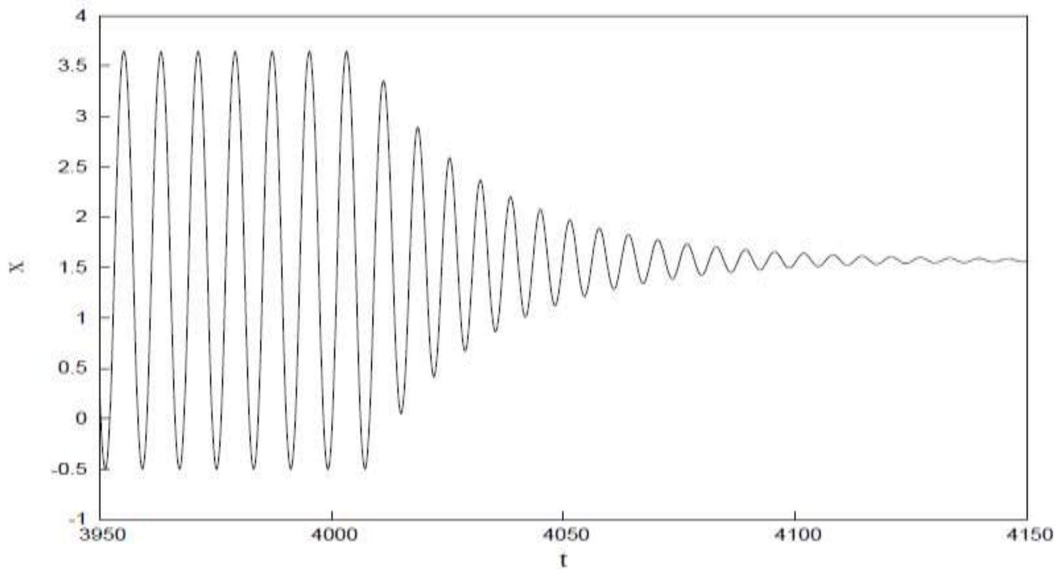

**Figure 2:** Variation of $x$ as a function of t showing the underdamped characteristic after the forcing has been removed after 500 periods with $F_0 = 0.2$, $T = 0$, $\tau = 8$, $\theta = 0.5\pi$, $\gamma_0 = 0.1$, $\lambda = 0$ and $x(t=0)= 0$.

In this work we focus in the underdamped regime where $\gamma_0 < \omega$. In the presence of damping, the oscillations die out in time but in order to sustain the oscillations, a periodic force of frequency $\omega$ must be applied to the system. Figure 2 shows the oscillations of the particle for some interval of time in the presence of the forcing. Once the forcing has been removed (around 500 periods), the oscillations die out exponentially with decaying amplitudes - a characteristic of underdamped systems. In Fig. 2, we apply a forcing of period $\tau = 8$ i.e. $\omega \approx 0.785$ and $\gamma_0 = 0.1$, with the rest of the parameters as mentioned in the caption.

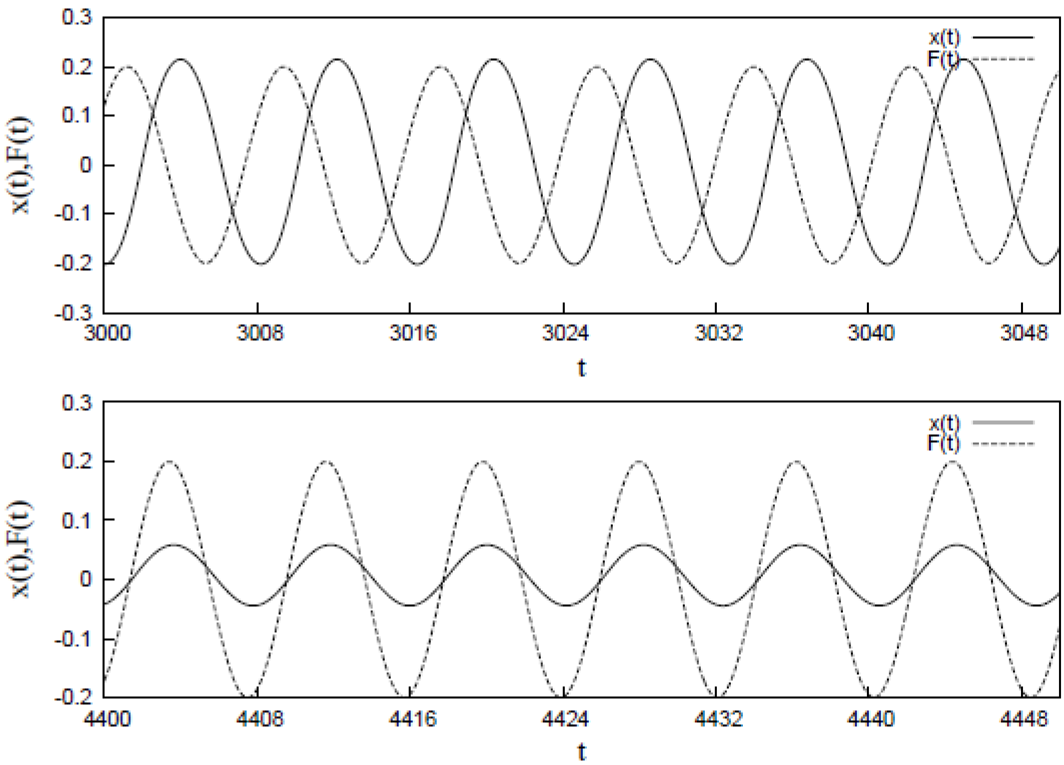


**Figure 3:** Variation of $x$ as a function of $t$ showing the LA state (top figure) and the SA state (bottom figure) both in unbroken lines with the periodic force of amplitude $F_0 = 0.2$ in broken lines. Here, $T = 0$, $\tau = 8.2$, $\theta = 0.5\pi$, $\gamma_0 = 0.1$, $\lambda = 0.9$ and $x(t=0)= 0$ (LA) and $x(t=0)= \pi$ (SA)

As mentioned earlier, in the absence of noise (T = 0) and at low noise strengths, and within a limited range of the parameters $\tau$ and $F_0$, the underdamped sinusoidal potential system exhibits two solutions - LA and SA. These two states of particle's trajectory apart from having different amplitudes and phase differences with respect to the periodic drive, also have different energy losses (hysteresis loss). Figure 3 shows a typical LA and SA state with their corresponding hysteresis loops in Fig.4. Physically, the hysteresis loop area corresponds to the heat lost to the

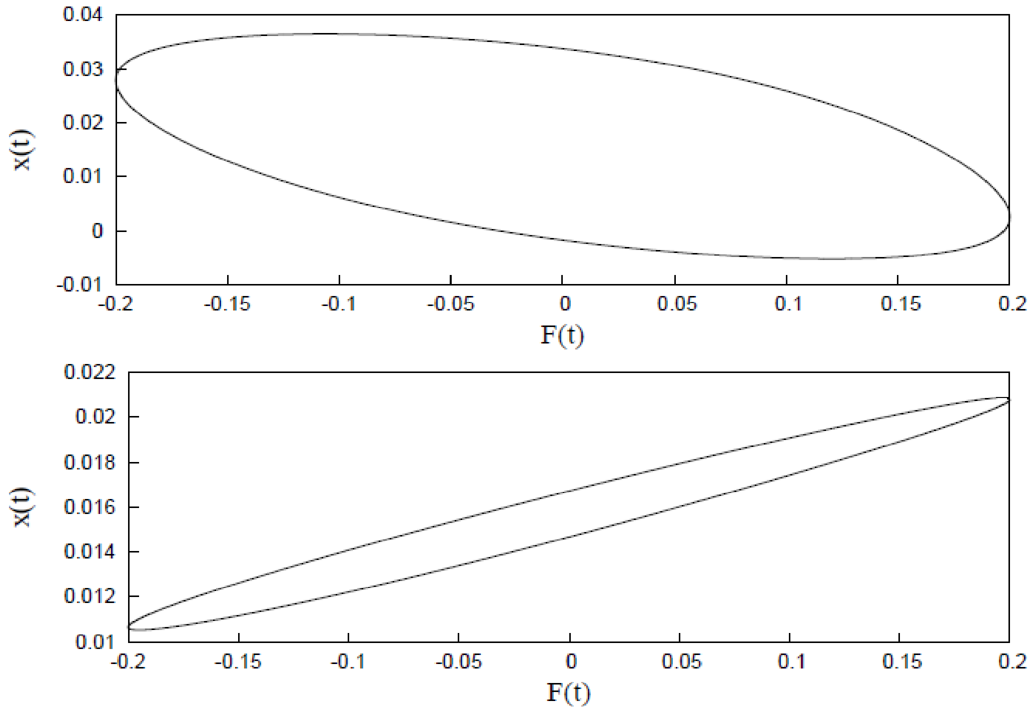


**Figure 4:** The corresponding hysteresis loops for the LA state (top figure) and the SA state (bottom figure) with the same parameters as that of Fig. 3

bath. It is numerically equal to the work done by the periodic force. Clearly, the area enclosed by the loop for SA is smaller than that for the LA.

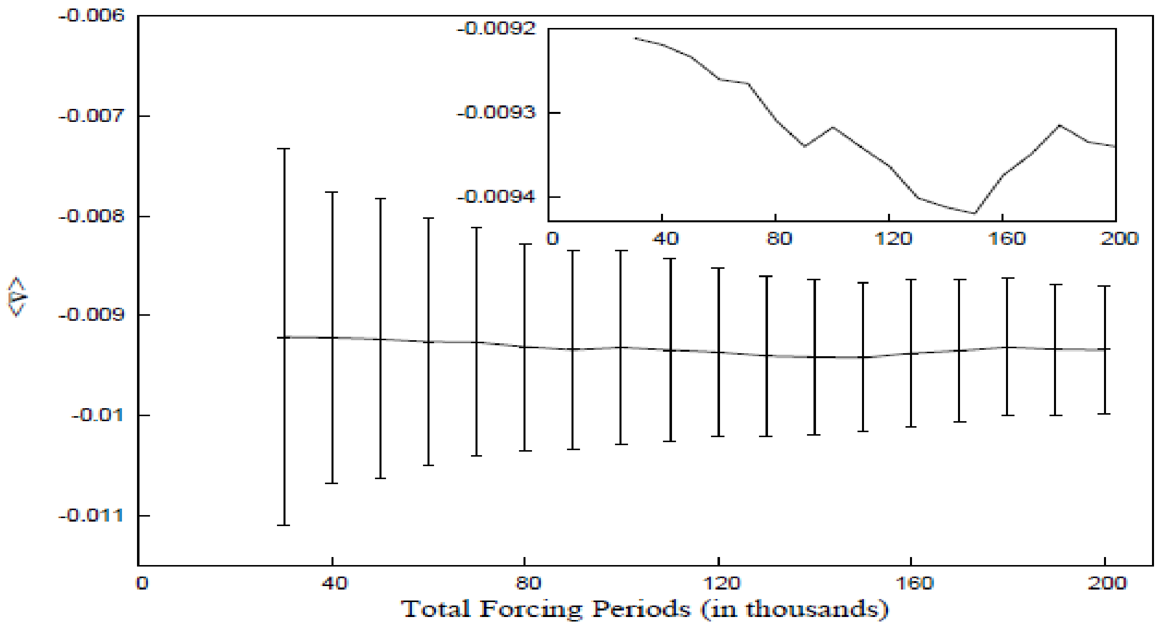


**Figure 5:** Variation of $< \bar{v} >$ as a function of forcing periods with corresponding error bars when $T = 0.2$, $\tau = 7.7$, $\theta = 0.5\pi$, $\gamma_0 = 0.07$, $\lambda = 0.9$ and $F_0 = 0.2$. The inset is the same plot without errorbars.

In the presence of noise, in order to get better statistics, we take 200000 periods of the forcing. As seen in Fig. 5, $< \bar{v} >$ as a function of the forcing periods, for $\tau = 7.7$ and $T = 0.2$, shows larger standard deviations if the number of periods is small. So in order to reduce the error bars, we resort to larger number of forcing periods. The inset of Fig. 5 shows the plot without the error bars.

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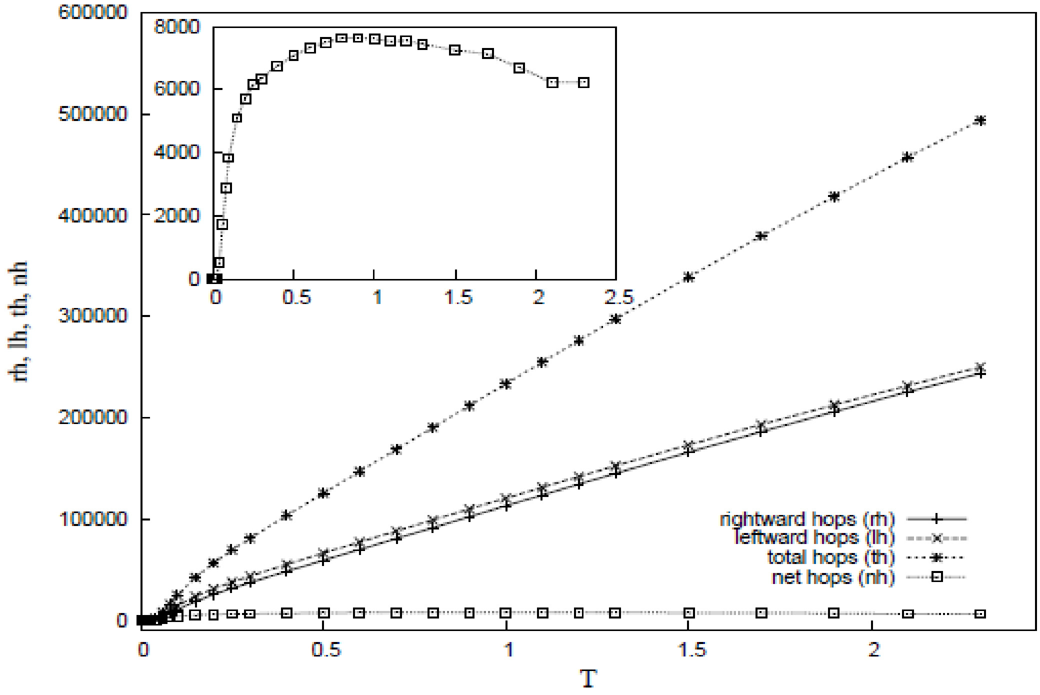


**Figure 6:** Variation of rightward hops, leftward hops, total hops and net hops as a function of temperature. Here, $T = 0.2$, $\tau = 7.7$, $\theta = 0.5\pi$, $\gamma_0 = 0.07$, $\lambda = 0.9$ and $F_0 = 0.2$. The inset shows the magnified plot of the net hops as a function of T.

Figure 6 as a function of temperature, shows the number of hoppings of the particles towards the right (**rh**) of the initial well, the number of hoppings towards the left (**lh**) of the initial well, the total number of hoppings (**th**), and the net (difference) hoppings (**nh**). As seen from the figure and for the parameters as mentioned in the caption, there is a monotonic increase in rh, lh and th as a function of temperature. It is also seen that the particles at every temperature has more number of hops to the left of the initial well than to the right. This means that the system shows a ratchet effect towards the left. The inset in Fig. 6 shows the magnified plot of the net hops as a function of temperature. This difference in hoppings is the ratchet current.

Below we present the main results of this paper. For the parameter values considered i.e. $\theta = 0.75\pi$, $\gamma_0 = 0.08$, $\lambda = 0.9$, and $F_0 = 0.2$, we show in Fig. 7 the variation of $< \bar{v} >$ as a function of noise strengths for various driving periods. As seen in Fig. 7, the system generates a current towards the left for the parameters considered.

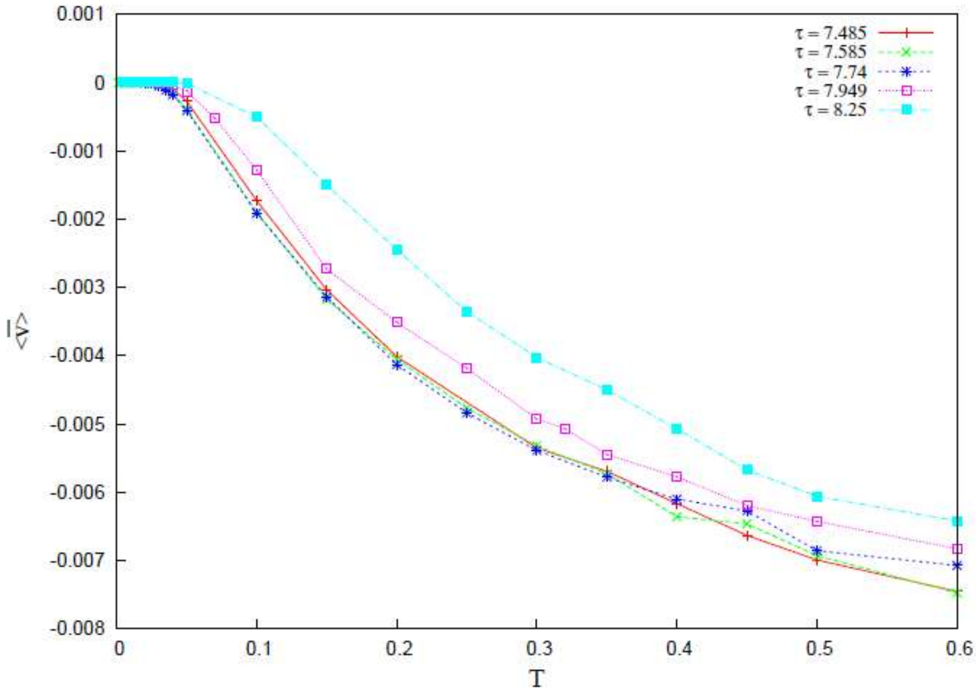


**Figure 7:** Variation of $< \bar{v} >$ as a function of temperature $T$ for various values of $\tau$ as indicated in the figure with $\theta = 0.75\pi$, $\gamma_0 = 0.08$, $\lambda = 0.9$, and $F_0 = 0.2$.

Figure 8 shows the variation of $<\overline{W}>$ as a function of noise strength for the same parameter values as in Fig. 7. We see that there are three behaviors of the variation of $<\overline{W}>$ as a function of temperature $T$:

a) monotonic decrease when $\tau = 7.485$
b) when $\tau = 7.585$, there is an increase in $<\overline{W}>$ as $T$ is increased which then peaks and subsequently a decrease as $T$ is increased further
c) When $\tau = 7.74,\ 7.949,\ 8.25$, there is an initial dip of $<\overline{W}>$ as $T$ is increased followed by a peak in $<\overline{W}>$ as $T$ is increased and subsequent decrease in $<\overline{W}>$ as $T$ is increased further.

The peaking of $<\overline{W}>$ as a function of $T$ is the signature of stochastic resonance. However, it must be pointed out that if the peak occurs at low noise values, the particles will remain in their initial well and the notion of SR in the sinusoidal potential is not justified. Also, when the peak occurs at large noise values, the effects of the periodic force is dampened by the noise. SR is the response of the system to a weak periodic force at an optimal value of noise. Thus, from Fig. 8, we conclude that the signature of SR is seen when $\tau = 7.949$ at $T \approx 0.2$.

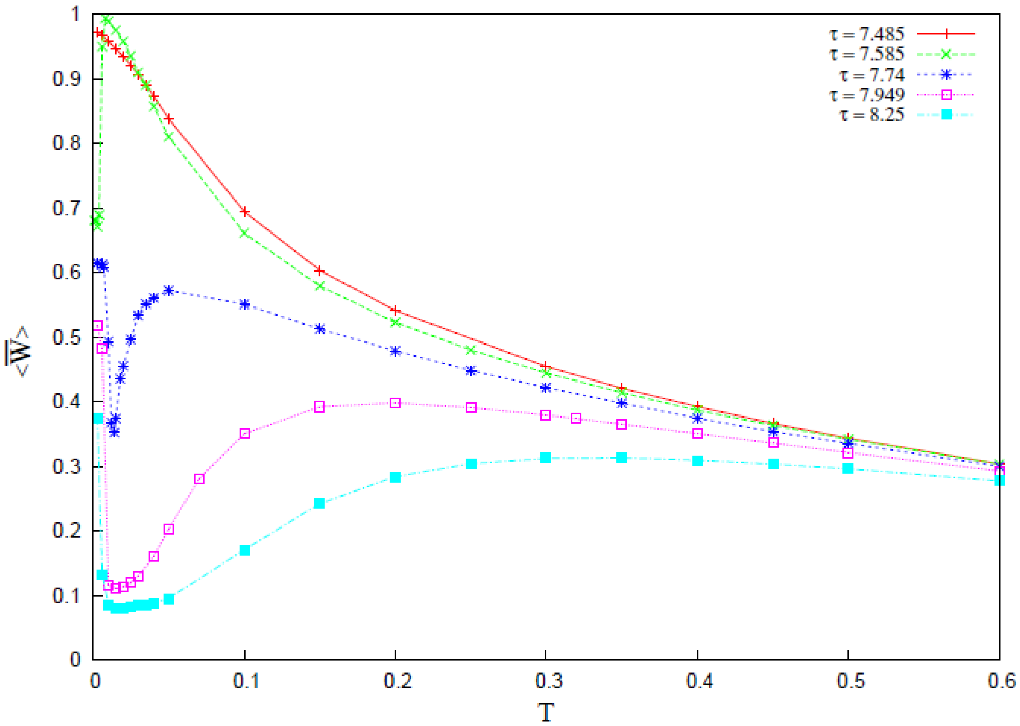


**Figure 8:** Variation of $<\overline{W}>$ as a function of temperature $T$ for various values of $\tau$ as indicated in the figure with $\theta = 0.75\pi$, $\gamma_0 = 0.08$, $\lambda = 0.9$, and $F_0 = 0.2$.

## 4. Discussion and Conclusion

In this small work, we have chosen a spatially periodic friction coefficient that is out-of-phase with respect to the sinusoidal potential. The asymmetry in conjunction with a weak periodic drive of suitable frequency yields both RE and SR simultaneously in the same range of parameter space considered. We can see that RE occurs in a wider range of drive periods considered as compared to SR. For this system, we also notice that the drive period which shows SR does not necessarily exhibit maximum current at the temperature where SR is observed. This work can provide idea to experimentalists who can design structures that can be used to separate micro-sized particles having different diffusion coefficients.

## References

[1] J. K. Douglass et. al, Nature (London) 315, 337 (1993).
[2] S. Fauve, and F. Heslot, Phys. Lett. A 97, 5 (1983).
[3] C. M. Falco, Am. J. Phys. 44, 733 (1976).
[4] R. P. Feynman, R. B. Leighton, and M. Sands, The Feynman Lectures on Physics, Vol. 1, Chapter 46.
[5] M. Smoluchowski, Physik. Aeitschr. 13, 1069 (1912).
[6] P. Reimann, Phys. Rep. 361, 57 (2002).
[7] R. Benzi, A. Sutera, and A. Vulpiani J. Phys. A 14, L453 (1981).
[8] D. Dan, and A. M. Jayannavar, Physica A 345 (2005).
[9] S. Saikia, A.M. Jayannavar, and M.C. Mahato, Phys. Rev. E 83, 061121 (2011).
[10] K. Sekimoto, J. Phys. Soc. Jpn. 66, 1234 (1997).
[11] W.L. Reenbohn, and M.C. Mahato, Phys. Rev. E 88, 032143 (2013).
[12] W. L. Reenbohn, S. S. Pohlong, and Mangal C. Mahato, Phys. Rev. E 85, 031144 (2012).
[13] W.L. Reenbohn, and M.C. Mahato, Phys. Rev. E 91, 052151 (2015).
[14] D. Kharkongor, W. L. Reenbohn, and Mangal C. Mahato, Phys. Rev. E 94, 022148 (2016).
[15] R. H. Luchsinger, Phys. Rev. E 62, 272 (2000).
[16] L. P. Faucheux, and A. Libchaber, Phys. Rev. E 49, 6 (1994).
[17] E.A. Desloge, Am. J. Phys. 62, 601 (1994). England (1987).