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Response Analysis of Multiple-Input Multiple-Output Frequency and Singular Value Decomposition

Abdelrahim B. Hamid[1], Omer Abdal-Azeem Almahi[2]

[1]Associate Professor, University of Gezira, Sudan

[2]Lecturer, University of King Saud, Saudi Arabia

Abstract: ***Recently, developments in linear control theory have concentrated on the multiple-input multiple-output control systems (MIMO). Many systems, particularly in technologically advanced areas such as aerospace systems, are represented by models with several inputs, with each input having several outputs. Such cross-coupling makes the use of single-input, single-output (SISO) methods complicated. In this paper, we introduced the basic closed-loop expressions for MIMO. We then discussed 2-norms, stability, and uncertainty. The standard design problem defined and H^2 solutions are worked out.***

1. Introduction

In parallel with developments in MIMO systems, the past twenty years have seen a renewed emphasis on frequency response. The ability of the state framework to handle uncertainty, especially nonparametric uncertainty, proved deficient. In contrast, uncertainty fits quite naturally in an input-output setting such as frequency response. The state framework has not been cast aside; rather connections have been made between it and the frequency response approach.

Developments in frequency response design have been greatly abetted by the theory of H^∞. Quadratic performance indices do lead to integrals in the frequency domain, through Parseval's integral. However, engineers are usually concerned with specifications expressed pointwise in the frequency domain, not with the averages yielded by integrals. The H^∞ theory provides a direct handle on this type of specification[10].

1.1 Modeling MIMO Linear Time-Invariant

Consider a MIMO system defined by

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ : \\ : \\ U_m(t) \end{bmatrix}, X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ : \\ : \\ X_n(t) \end{bmatrix}, Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ : \\ : \\ Y_p(t) \end{bmatrix}$$

Where U is the inputs, X is state variables and Y is the outputs. The state variable representation can be arranged in the form of 'n' first order differential equations

$$\frac{dX_1(t)}{dt} = \dot{X}_1(t) = a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n + b_{11}U_1 + b_{12}U_2 + \cdots + b_{1m}U_m$$
$$\frac{dX_2(t)}{dt} = \dot{X}_2(t) = a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n + b_{21}U_1 + b_{22}U_2 + \cdots + b_{2m}U_m$$
$$\vdots$$
$$\frac{dX_n(t)}{dt} = \dot{X}_n(t) = a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n + b_{n1}U_1 + b_{n2}U_2 + \cdots + b_{nm}U_m$$

For the linear time invariant systems, the coefficients a_{ij}, b_{ij} are constants. Thus all equations can be written in vector matrix form as

$$\dot{X}(t) = AX(t) + BU(t)$$

Where $X(t)$= State vector matrix of order n×1
$U(t)$= Input vector matrix of order m×1
A= System matrix or Evolution matrix oforder n×n
B = Inputmatrix or controlmatrix oforder n×m

Similarly the output variables can be expressed

$$Y_1(t) = c_{11}X_1 + c_{12}X_2 + \cdots + c_{1n}X_n + d_{11}U_1 + d_{12}U_2 + \cdots + d_{1m}U_m$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$Y_p(t) = c_{p1}X_1 + c_{p2}X_2 + \cdots + c_{pn}X_n + d_{p1}U_1 + d_{p2}U_2 + \cdots + d_{pm}U_m$$

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For the linear time invariant systems, the coefficients c_{ij} , d_{ij} are constants. Thus all output equations can be written in vector matrix form as

$$Y(t) = CX(t) + DU(t)$$

Where
$Y(t)$= Output vector matrix of order p×1
C= Output matrix or observation matrix oforder p×n
D= Direct transmissionmatrix oforder p×m

The two vector equations together is called theState model of the linear system

$$\dot{X}(t) = AX(t) + BU(t) \quad \textit{state equation}$$
$$Y(t) = CX(t) + DU(t) \;\; \textit{output equation}$$ [2], [4].

1.2 Systems in Terms of Transfer Function

Taking the Laplace transform of the state equation, we see that

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$X(s) = (sI - A)^{-1}BU(s) \qquad if \;\; x(0) = 0$$
$$and\; Y(t) = [C(sI - A)^{-1}B + D]U(s) \equiv G(s)U(s)$$

The matrix quantity G(s) is the transfer function matrix (TFM) to make this explicit

$$G(s) = \begin{bmatrix} g_{11}(s) & \dots & g_{1m}(s) \\ \vdots & & \vdots \\ g_{p1}(s) & \dots & g_{pm}(s) \end{bmatrix}$$

Where $y_k(s) = \sum_{j=1}^{m} g_{kj}(s)u_j(s)\, , k = 1, \dots ,p$ [9].

1.3 Short Mathematical Review (Matrices, Complex and Vectors)

Definition (1): For two vectors $u = [u_1 u_2 \dots u_n]^T$ and $v = v1v2\dots vnT$ in $\mathbb{C}n, uk, vk \in \mathbb{C}$ The Euclidean inner product defined by

$$u.v = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n = \bar{u}^T v$$

$where\;\; \bar{u} = [\bar{u}_1 \bar{u}_2 \dots \bar{u}_n]^T$ the conjugate of u .

The Euclidean length (or magnitude) of a vector $u \in \mathbb{C}^n$ is defined by

$$\|u\| = (u.u)^{\frac{1}{2}} = \sqrt{|u_1|^2 + |u_2|^2 + \dots + |u_n|^2}$$

The Euclidean norm of u, defined by

$$\|u\|_2 = \sqrt{u^H u} = \sqrt{\sum_{i=1}^{n} (a_i{}^2 + b_i{}^2)}$$

Where u^H is the Complex conjugate transpose of u i.e. $u^H = (u_1^*, \dots, u_2^*) = a^T - jb^T$. We define the Hermitian of a complex matrix$M \in \mathbb{C}^{p\times m}$ as the complex-conjugate transposes of$M$, that is,M^H is computed by$M^H = A^T - jB^T$. A complex-valued matrix M is called Hermitian if $= M^H$. A nonsingular, complex-valued matrix is called unitary if$M^{-1} = M^H$. a complex-valued matrix M is unitary if its column vectors are mutually orthonormal (which satisfy$u^H v = 0$, are said to be orthogonal). The spectral norm of a matrix $M \in \mathbb{C}^{p\times m}$, denoted$\|M\|_2$, is defined by

$$\|M\|_2 = \max_{\|u\|_2=1} \|Mu\|_2$$ [7], [5].

2. Results and Discussion

2.1 Frequency Response for MIMO Plants

Consider the stable, linear, time-invariant system. The input and output of the system hose transfer function is G(s), If the input u(t) is a sinusoidal signal, the steady-state output will also be a sinusoidal signal of the same frequency, but with possibly different magnitude and phase angle.

In MIMO Suppose that we have in mind a complex exponential input, as below,

$$u(t) = \tilde{u}e^{-j\omega t}$$

Where $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_m)^T$ is a fixed (complex) vector inC^m. This input is applied to our stable LTI system G (frequency response $G(s = j\omega)$) the steady state

$$y_k(t) = \sum_{j=1}^{m} g_{kj}(j\omega)\, \tilde{u}_j\, e^{j\omega t} \qquad ; k = 1, \dots, p$$

The vector output y(t) at steady state as follows

$$y(t) = \tilde{y}e^{j\omega t} \quad , \quad \tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_p)^T \in C^p$$

Where

$$\tilde{y}_k = \sum_{j=1}^{m} g_{kj}(j\omega)\, \tilde{u}_j \qquad ; k = 1, \dots, p$$

In generally

$$\tilde{y} = G(j\omega)\tilde{u}$$

2.2 The Singular Value Decomposition

2.2.1 The Singular Values of a Matrix

In matrix $M \in \mathbb{C}^{p\times m}$ with rank=k the singular values of M, computed by

$$\sigma_i(M) = \sqrt{\lambda_i(M^H M)} = \sqrt{\lambda_i(MM^H)} > 0 \;, \qquad i = 1, \dots, k$$

Where$\lambda_i(\cdot)$is "the ith nonzero eigenvalue of " .It is common to index and rank the singular values as follows:

$$\sigma_1(M) \geq \sigma_2(M) \geq \dots \geq \sigma_k(M) > 0$$ [2].

2.2.2 The Singular Value Decomposition

For any matrix $M \in \mathbb{C}^{p\times m}$ there exist the matrices$U \in \mathbb{C}^{p\times p}$, $V \in \mathbb{C}^{m\times m}$ such that

$$M = U\Sigma V^H$$

Where $U$is a unitary matrix of eigenvectors ofMM^H(left singular vectors)
$V$is a unitary matrix of eigenvectors of$M^H M$ (right singular vectors)
Σ= is a real "diagonal" matrix of the non-negative singular value i.e.

$$\Sigma_{p\times m} = \left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & 0_{k\times(m-k)} \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \sigma_k & \\ & & & & \\ & 0_{(p-k)\times k} & & & 0_{(p-k)\times(m-k)} \end{array}\right]$$

And $M^{-1} = V\Sigma^{-1}U^H$ where $\Sigma^{-1} = diag\left\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \dots, \frac{1}{\sigma_m}\right\}$ [8].

2.2.3 Some Properties (Define the maximum and minimum amplifications)

1) $\bar{\sigma} = \sigma_{max}(M) = \max_{\|u\|_2=1} \|Mu\|_2 = \|M\|_2 = \frac{1}{\sigma_{min}(M^{-1})}$

2) $\underline{\sigma} = \sigma_{min}(M) = \min_{\|u\|_2=1} \|Mu\|_2 = \frac{1}{\|M^{-1}\|_2} = \frac{1}{\sigma_{max}(M^{-1})}$

2.3 The SVD and MIMO Frequency Response

The SVD of the matrix $G(s = j\omega)$ for each frequency s = jω

$$G(j\omega) = U(j\omega)\Sigma(\omega)V^H(j\omega)$$

Where $\Sigma \in \mathbb{R}^{p\times m}$ thediagonal elements $\sigma_k \geq 0$ of Σ are the singular values of $G(j\omega)$

$$\sigma_i = \sqrt{\lambda_i(G^H G)} \quad or \quad \sigma_i = \sqrt{\lambda_i(G G^H)}$$

The matrix $U(j\omega) \in \mathbb{C}^{m\times m}$ whose column vectors (eigenvectors) $u_j(j\omega)$are the left singular vectors of $G(j\omega)$ $i.e.$ $G^H G v_j = \sigma_i^2 v_j$

The matrix $V(j\omega) \in \mathbb{C}^{p\times p}$ whose column vectors (eigenvectors) $v_j(j\omega)$ are the rightsingular vectors of $G(j\omega)$ $i.e.$ $GG^H u_j = \sigma_i^2 u_j$[3].

2.4 Analysis Singular Value Plots (SV Plots)

Once we calculate the maximum and minimum singular values of $G(j\omega)$ for a range of frequencies ω, we can plot a Bode plot (decibels versus rad/sec in log-log scale). Figure 1 shows a hypothetical SV plot. this "gain-band" of the plant at each frequency is described by two curves, not one. It is crucial to interpret the information contained in the SV plot correctly. At each frequency ωwe assume that the input is a unit complex exponential $u(t) = \tilde{u}e^{-j\omega t}$ then, assuming that we have reached steady state, we know that the output is also a complex exponential with the same frequency $y(t) = \tilde{y}e^{j\omega t}$ [4], [6].

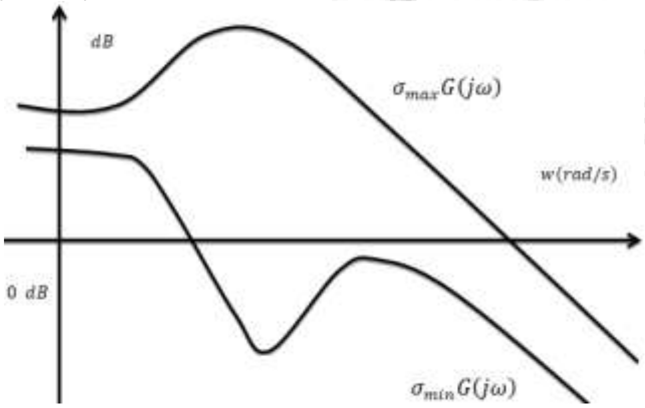

Figure 1: A hypothetical SV plot

Now, by looking at an SV plot, we can say that, at a given frequency

1- The largest output size is $\|\tilde{y}\|_{2,max} = \sigma_{max}G(j\omega)$, for $\|\tilde{u}\|_2 = 1$

2- The smallest output size is $\|\tilde{y}\|_{2,min} = \sigma_{min}G(j\omega)$, for $\|\tilde{u}\|_2 = 1$

This allows us to discuss qualitatively the size of the plant gain as a function of frequency

G_ω is said to be large if $\underline{\sigma}(G_\omega) \gg 1$
and G_ω is said to be small if $\bar{\sigma}(G_\omega) \gg 1$ [5].

2.5 Computing Directional Information

2.5.1 Maximum Amplification Direction Analysis

1) Compute the SVD of $(j\omega)$, where ω is Select a specific frequency.
2) find $\sigma_{max}G(j\omega)$ (maximum singular value)
3) find $v_{max}(\omega)$ (maximum right singular vector) Write in polar form
   $[v_{max}(\omega)]_i = |a_i|e^{j\psi_i}$, $i = 1, \ldots, m$ where a_i and ψ_i are really functions of ω
4) find $u_{max}(\omega)$ (maximum left singular vector) Write in polar form
   $[u_{max}(\omega)]_i = |b_i|e^{j\Phi_i}$, $i = 1, \ldots, p$ where b_i and Φ_i are really functions of ω
5) Constructthe real sinusoidal input signals that correspond to the direction of maximum amplification and to predict the output sinusoids that are expected at steady state. The input vector u(t) is defined by
   $u_i(t) = |a_i|\sin(\omega t + \psi_i)$ $\quad i = 1, \ldots, m$

We can utilize the implications of the SVD to predict the steady-state output sinusoids as
$y_i(t) = \sigma_{max}(w)|b_i|\sin(\omega t + \Phi_i)$ $\quad i = 1, \ldots, p$

That all parameters needed to specify the output sinusoids are already available from the SVD[1], [10].

2.6 Norms

The Euclidean norm of a vector X is :

$$\|X\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \left(X^T X\right)^{1/2}.$$

$$= \left[\int_{-\infty}^{\infty} X^T(t)X(t)dt\right]^{1/2}.$$

To calculate the norm of $G(s) = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & -(s+2) \\ -2 & s+2 \end{bmatrix}$,

we compute:

$$trG^T(-s)G(s) = \frac{(s+3)(-s+3)+4+(s+2)(s+2)-(-s+2)(s+2)}{(s+1)(s+2)(-s+1)(-s+2)}$$

$$= \frac{-3s^2+21}{(s+1)(s+2)(-s+1)(-s+2)}$$

If we integrate about a contour enclosing the positive direction, then the sum of residues at s = -1 and s = -2 :

$$\|G\|_2^2 = \frac{18}{(1)(2)(3)} + \frac{9}{(-1)(3)(4)} = \frac{9}{4}.$$

Therefore, $\|G\|_2^2 = 3/2.$

A different type of norm is the induced norm, which applies to operators and is essentially a type of maximum gain. But for matrices induced Euclidean norm is:

$$\|y\|_2{}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 |u(j\omega)|^2 d\omega.$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\sigma}[G(j\omega)]\|u(j\omega)\|^2 d\omega$$

$$\leq \sup_{\omega}[\bar{\sigma}[G(j\omega)]]^2 \frac{1}{2\pi}\int_{-\infty}^{\infty}\|u(j\omega)\|^2$$

$$\leq \left[\sup_{\omega}\bar{\sigma}[G(j\omega)]\right]^2 \|u\|_2{}^2.$$

Which can be approached arbitrary closely by proper choice of $u(j\omega)$. Essentially, we pick$u(j\omega)$ to be the eigenvector of $G^*(j\omega)G(j\omega)$ corresponding to the largest eigenvalue[2].

3. Conclusion

The cross-coupling makes the use of single-input, single-output (SISO) methods complicated. Thus, we introduced the basic closed-loop expressions for MIMO. We then discussed 2-norms, stability, and uncertainty. The standard design problem defined and H^2solutions are worked out. And we concentrated the spectrum of $u(j\omega)$ at the frequency where $\bar{\sigma}$ is the largest, and may be some arbitrary frequency large if $\bar{\sigma}$ has no maximum but a supremum.

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