Response Analysis of Multiple-Input Multiple-Output Frequency and Singular Value Decomposition

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Abstract: Recently, developments in linear control theory have concentrated on the multiple-input multiple-output control systems (MIMO). Many systems, particularly in technologically advanced areas such as aerospace systems, are represented by models with several inputs, with each input having several outputs. Such cross-coupling makes the use of single-input, single-output (SISO) methods complicated. In this paper, we introduced the basic closed-loop expressions for MIMO. We then discussed 2-norms, stability, and uncertainty. The standard design problem defined and H∞ solutions are worked out.

1. Introduction

In parallel with developments in MIMO systems, the past twenty years have seen a renewed emphasis on frequency response. The ability of the state framework to handle uncertainty, especially nonparametric uncertainty, proved deficient. In contrast, uncertainty fits quite naturally in an invariant systems, the coefficients are constants. Thus all equations ca...

\[ U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_m(t) \end{bmatrix}, \quad X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{bmatrix}, \quad Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_p(t) \end{bmatrix} \]

Where \( U \) is the inputs, \( X \) is state variables and \( Y \) is the outputs. The state variable representation can be arranged in the form of ‘n’ first order differential equations

\[ \frac{dX_1(t)}{dt} = \dot{X}_1(t) = a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n + b_{11}U_1 + b_{12}U_2 + \cdots + b_{1m}U_m \]
\[ \frac{dX_2(t)}{dt} = \dot{X}_2(t) = a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n + b_{21}U_1 + b_{22}U_2 + \cdots + b_{2m}U_m \]
\[ \vdots \]
\[ \frac{dX_n(t)}{dt} = \dot{X}_n(t) = a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n + b_{n1}U_1 + b_{n2}U_2 + \cdots + b_{nm}U_m \]

For the linear time invariant systems, the coefficients \( a_{ij} \), \( b_{ij} \) are constants. Thus all equations can be written in vector matrix form as

\[ \dot{X}(t) = AX(t) + BU(t) \]

Where \( X(t) \) = State vector matrix of order \( n \times 1 \)
\( U(t) \) = Input vector matrix of order \( m \times 1 \)

\[ Y_1(t) = c_{11}X_1 + c_{12}X_2 + \cdots + c_{1n}X_n + d_{11}U_1 + d_{12}U_2 + \cdots + d_{1m}U_m \]
\[ \vdots \]
\[ Y_p(t) = c_{p1}X_1 + c_{p2}X_2 + \cdots + c_{pn}X_n + d_{p1}U_1 + d_{p2}U_2 + \cdots + d_{pm}U_m \]

\[ A = \text{System matrix or Evolution matrix of order } n \times n \]
\[ B = \text{Inputmatrix or controlmatrix of order } n \times m \]
For the linear time invariant systems, the coefficients $c_{ij}, b_{ij}$ are constants. Thus all output equations can be written in vector matrix form as

$$Y(t) = CX(t) + DU(t)$$

Where

$Y(t)$ = Output vector matrix of order $p \times 1$
$C$ = Output matrix or observation matrix of order $p \times n$
$D$ = Direct transmission matrix of order $p \times m$

The two vector equations together is called the State model of the linear system

$$X(t) = AX(t) + BU(t) \quad \text{state equation}$$

$$Y(t) = CX(t) + DU(t) \quad \text{output equation} \ [2, 4].$$

### 1.2 Systems in Terms of Transfer Function

Taking the Laplace transform of the state equation, we see that

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$X(s) = (sI - A)^{-1}BU(s) \quad \text{if} \ x(0) = 0$$

and $Y(t) = [C(sI - A)^{-1}B + D]U(s) \equiv G(s)U(s)$

The matrix quantity $G(s)$ is the transfer function matrix (TFM) to make this explicit

$$G(s) = \begin{bmatrix}
    g_{11}(s) & \ldots & g_{1m}(s) \\
    \vdots & \ddots & \vdots \\
    g_{p1}(s) & \ldots & g_{pm}(s)
\end{bmatrix}$$

Where $y_k(s) = \sum_{j=1}^{m} g_{jk}(s)u_j(s), k = 1, \ldots, p \ [9]$. 

### 1.3 Short Mathematical Review (Matrices, Complex and Vectors)

Definition (1): For two vectors $u = [u_1 u_2 \ldots u_n]^T$ and $v = [v_1 v_2 \ldots v_n]^T$ in $\mathbb{C}^n$, $u \cdot v = u_1v_1 + u_2v_2 + \ldots + u_nv_n = \bar{u}^Tv$

where $\bar{u} = [\bar{u}_1 \bar{u}_2 \ldots \bar{u}_n]^T$ is the conjugate of $u$.

The Euclidean length (or magnitude) of a vector $u \in \mathbb{C}^n$ is defined by

$$||u|| = (u\cdot u)^{1/2} = \sqrt{|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2}$$

The Euclidean norm of $u$, defined by

$$||u||_2 = \sqrt{u^*u} = \left(\sum_{i=1}^{n} (a_i^2 + b_i^2)\right)^{1/2}$$

Where $u^H$ is the Complex conjugate transpose of $u$, i.e. $u^H = (u_1^*, \ldots, u_n^*) = a^T - jb^T$. We define the Hermitian of a complex matrix $M \in \mathbb{C}^{p \times m}$ as the complex-conjugate transposes of $M$, that is $M^H$ is computed by $M^H = A^T - jB^T$.

A complex-valued matrix $M$ is called Hermitian if $M = M^H$. A nonsingular, complex-valued matrix is called a unitary if $M^{-1} = M^H$, a complex-valued matrix $M$ is unitary if its column vectors are mutually orthonormal (which satisfy $u^H v = 0$, are said to be orthogonal). The spectral norm of a matrix $M \in \mathbb{C}^{p \times m}$, denoted $||M||_2$, is defined by

$$||M||_2 = \max_{||u||_2 = 1} ||Mu||_2 \ [7, 5].$$

### 2. Results and Discussion

#### 2.1 Frequency Response for MIMO Plants

Consider the stable, linear, time-invariant system. The input and output of the system is a transfer function $G(s)$, if the input $u(t)$ is a sinusoidal signal, the steady-state output will also be a sinusoidal signal of the same frequency, but with possibly different magnitude and phase angle.

In MIMO Suppose that we have in mind a complex exponential input, as below,

$$u(t) = \bar{u}e^{-j\omega t}$$

Where $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)^T$ is a fixed (complex) vector in $\mathbb{C}^m$. This input is applied to our stable LTI system $G$ (frequency response $G(s = j\omega)$) the steady state

$$y_k(t) = \sum_{j=1}^{m} g_{kj}(j\omega) \bar{u}_j e^{j\omega t} \ \ ; k = 1, \ldots, p$$

The vector output $y(t)$ at steady state as follows

$$y(t) = \bar{y}e^{j\omega t} \quad , \quad \bar{y} = (\bar{y}_1, \ldots, \bar{y}_p)^T \in \mathbb{C}^p$$

Where

$$\bar{y}_k = \sum_{j=1}^{m} g_{kj}(j\omega) \bar{u}_j \ \ ; k = 1, \ldots, p$$

In generally

$$\bar{y} = G(j\omega)\bar{u}$$

#### 2.2 The Singular Value Decomposition

##### 2.2.1 The Singular Values of a Matrix

In matrix $M \in \mathbb{C}^{p \times m}$ with rank $k$ the singular values of $M$, computed by

$$\sigma_i(M) = \sqrt{\lambda_i(M^HM)} = \sqrt{\lambda_i(\overline{M^HM})} > 0 \quad , \quad i = 1, \ldots, k$$

Where $\lambda_i$ is the $i$th nonzero eigenvalue of $M$. It is common to index and rank the singular values as follows:

$$\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_k(M) > 0 \ [2].$$

##### 2.2.2 The Singular Value Decomposition

For any matrix $M \in \mathbb{C}^{p \times m}$ there exist the matrices $U \in \mathbb{C}^{p \times p}$, $V \in \mathbb{C}^{m \times m}$ such that

$$M = U \Sigma^H V$$

Where $U$ is a unitary matrix of eigenvectors of $MM^H$ (left singular vectors)

$V$ is a unitary matrix of eigenvectors of $M^HM$ (right singular vectors)

$\Sigma$ is a real "diagonal" matrix of the non-negative singular value i.e.

$$\Sigma_{p \times m} = \begin{bmatrix}
    \sigma_1 & 0 & \cdots & 0 \\
    0 & \sigma_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \sigma_k \\
\end{bmatrix}_{k \times (m-k)}$$

And $M^{-1} = V^{-1}U^H$ where $V^{-1} = \text{diag}\left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_m}\right)$

[8].
2.2.3 Some Properties (Define the maximum and minimum amplifications)

1) \( \sigma = \sigma_{\text{max}}(M) = \max_{\|u\|_2 = 1} \|Mu\|_2 = \|M\|_2 = \frac{1}{\sigma_{\text{min}}(M^{-1})} \)

2) \( \sigma = \sigma_{\text{min}}(M) = \min_{\|u\|_2 = 1} \|Mu\|_2 = \|M^{-1}\|_2 = \frac{1}{\sigma_{\text{max}}(M^{-1})} \)

2.3 The SVD and MIMO Frequency Response

The SVD of the matrix \( G(s = j\omega) \) for each frequency \( s = j\omega \)
\[ G(j\omega) = U(j\omega)\Sigma(j\omega)V^{H}(j\omega) \]
Where \( \Sigma \in \mathbb{R}^{p \times m} \) (diagonal elements) \( \sigma_k \geq 0 \) of \( \Sigma \) are the singular values of \( G(j\omega) \)
\[ \sigma_i = \sqrt{\lambda_i(G^{H}G)} \quad \text{or} \quad \sigma_i = \sqrt{\lambda_i(GG^{H})} \]

The matrix \( U(j\omega) \in \mathbb{C}^{n \times m} \) whose column vectors (eigenvectors) \( u_i(j\omega) \) are the left singular vectors of \( G(ju) \) i.e. \( G^{H}Gu_i = \sigma_i^2 u_i \)

The matrix \( V(j\omega) \in \mathbb{C}^{p \times m} \) whose column vectors (eigenvectors) \( v_i(j\omega) \) are the right singular vectors of \( G(j\omega) \) i.e. \( GG^{H}v_i = \sigma_i^2 v_i \).

2.4 Analysis Singular Value Plots (SV Plots)

Once we calculate the maximum and minimum singular values of \( G(j\omega) \) for a range of frequencies \( \omega \), we can plot a Bode plot (decibels versus rad/sec in log-log scale). Figure 1 shows a hypothetical SV plot. This “gain-band” of the plant at each frequency is described by two curves, not one. It is crucial to interpret the information contained in the SV plot correctly. At each frequency \( \omega \) we assume that the input is a unit complex exponential \( u(t) = \bar{u}e^{-j\omega t} \) then, assuming that we have reached steady state, we know that the output is also a complex exponential with the same frequency \( y(t) = \bar{y}e^{j\omega t} \) [4], [6].

![Figure 1: A hypothetical SV plot](image)

Now, by looking at an SV plot, we can say that, at a given frequency
1- The largest output size is \( \|\bar{y}\|_{2,\text{max}} = \sigma_{\text{max}} G(j\omega) \), for \( \|\bar{u}\|_2 = 1 \)
2- The smallest output size is \( \|\bar{y}\|_{2,\text{min}} = \sigma_{\text{min}} G(j\omega) \), for \( \|\bar{u}\|_2 = 1 \)

This allows us to discuss qualitatively the size of the plant gain as a function of frequency
\( G_\omega \) is said to be large if \( \bar{G}_\omega \gg 1 \)
and \( G_\omega \) is said to be small if \( \bar{G}_\omega \gg 1 \) [5].

2.5 Computing Directional Information

2.5.1 Maximum Amplification Direction Analysis

1) Compute the SVD of \( G(j\omega) \), where \( \omega \) is Select a specific frequency.

2) Find \( \sigma_{\text{max}} G(j\omega) \) (maximum singular value).

3) Find \( \sigma_{\text{max}} G(j\omega) \) (maximum right singular vector).

Write in polar form
\[ [u_{\text{max}}(\omega)]_i = \frac{a_i e^{j\psi_i}}{i} \]
\[ u_{\text{max}}(\omega) = 1, ... , m \]
where \( a_i \) and \( \psi_i \) are real functions of \( \omega \)

4) Find \( \sigma_{\text{max}} G(j\omega) \) (maximum left singular vector).

Write in polar form
\[ [u_{\text{max}}(\omega)]_i = \frac{b_i e^{j\phi_i}}{i} \]
\[ u_{\text{max}}(\omega) = 1, ... , p \]
where \( b_i \) and \( \phi_i \) are real functions of \( \omega \)

5) Construct the real sinusoidal input signals that correspond to the direction of maximum amplification and to predict the output sinusoids that are expected at steady state.

The input vector \( u(t) \) is defined by
\[ u_i(t) = \left| a_i \right| \sin(\omega t + \phi_i) \]
\[ i = 1, ..., m \]

We can utilize the implications of the SVD to predict the steady-state output sinusoids as
\[ y_i(t) = \sigma_{\text{max}}(\omega) \left| b_i \right| \sin(\omega t + \phi_i) \]
\[ i = 1, ..., p \]

That all parameters needed to specify the output sinusoids are already available from the SVD[1], [10].

2.6 Norms

The Euclidean norm of a vector \( X \) is:
\[ \|X\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \]
\[ = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \]
\[ = \left( \int_{-\infty}^{\infty} X^T(t)X(t)dt \right)^{1/2} \]
To calculate the norm of \( G(s) = \frac{1}{s^2 + 3s + 2} \)
we compute:
\[ \|G\|_2 = \left| \frac{s^2 + 3(s + 3) + 4(s + 2)(s + 2)}{(s + 1)(s + 2)(s + 1)(s + 2)} \right| \]
\[ = \left| \frac{-3s^2 + 21}{(s + 1)(s + 2)(s + 1)(s + 2)} \right| \]
If we integrate about a contour enclosing the positive direction, then the sum of residues at \( s = -1 \) and \( s = -2 \):
\[ \|G\|_2 = \frac{18}{(1)(2)(3)} + \frac{9}{(-1)(3)(4)} = \frac{9}{4} \]
Therefore,
\[ \|G\|_2^2 = \frac{9}{4} \]

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A different type of norm is the induced norm, which applies to operators and is essentially a type of maximum gain. But for matrices induced Euclidean norm is:

$$\|y\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 |u(j\omega)|^2 d\omega.$$  

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma[G(j\omega)] |u(j\omega)|^2 d\omega$$  

$$\leq \sup_{\omega} \sigma[G(j\omega)]^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} |u(j\omega)|^2 d\omega$$  

$$\leq \left[ \sup_{\omega} \sigma[G(j\omega)] \right]^2 \|u\|_2^2.$$  

Which can be approached arbitrary closely by proper choice of $u(j\omega).$ Essentially, we pick $u(j\omega)$ to be the eigenvector of $G^*(j\omega)G(j\omega)$ corresponding to the largest eigenvalue[2].

3. Conclusion

The cross-coupling makes the use of single-input, single-output (SISO) methods complicated. Thus, we introduced the basic closed-loop expressions for MIMO. We then discussed 2-norms, stability, and uncertainty. The standard design problem defined and $H^2$ solutions are worked out. And we concentrated the spectrum of $u(j\omega)$ at the frequency where $\sigma$ is the largest, and may be some arbitrary frequency large if $\sigma$ has no maximum but a supremum.

References