Dark Matter and Holographic Dark Energy Models in Different Scenarios of the Universe

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Abstract: *In this paper we examine homogeneous and anisotropic Bianchi type-V universe filled with interacting Dark matter and Holographic dark energy. Here we express a scale factor in emergent, logamediate, and intermediate scenarios, under which the universe expands differently. It is shown that for suitable choice of interaction between dark matter and holographic dark energy there is no coincidence problem (unlike ΛCDM). Also, in all the resulting models the anisotropy of expansion dies out very quickly and attains isotropy after some finite time. The physical and geometrical aspects of the models are also discussed.*

Keywords: Dark energy; Bianchi Type V Space-time; Emergent; Logamediate; Intermediate; Deceleration Parameter

1. Introduction

The most attractive subject in cosmology is the accelerating expansion of the universe which is based on the recent astronomical observations as Type Ia supernovae (SNe) Riess et al. [1], Wilkinson Microwave Anisotropy Probe (WMAP) Observations Spergel [2], cosmic microwave background (CMB) anisotropy Tegmark et al.[3] and large scale structure (LSS) Enqvist et al. [4]. This implies that there is a mysterious component in the universe, which has a large negative pressure called dark energy (DE).

An approach to the problem of DE arises from holographic principle that states that the number of degrees of freedom related directly to entropy scales with the enclosing area of the system. As an application of holographic principle in cosmology, it was studied by Li [5] that consequence of excluding those degrees of freedom of the system which will never be observed by that effective field theory gives rise to IR cut-off L at the future event horizon. Based on cosmological state of holographic principle, proposed by Fischler and Susskind [6], the Holographic model of Dark Energy (HDE) has been proposed and studied widely in the literature Setare and Shafei [7]. In Huan and Gong [8] using the type Ia supernova data, the model of HDE is constrained once when c is unity and another time when c is taken as free parameter. It is concluded that the HDE is consistent with recent observations, but future observations are needed to constrain this model more precisely.

Holographic dark energy models have been tested and constrained by various astronomical observations Zhang and Wu [9]; Shen et al. [10]; Chang et al. [11]. A special class are models in which holographic DE is allowed to interact with DM Pavón and Zimdahl [12]; Wang et al. [13-14]; Carvalho and Saa [15];Perivolaropoulos [16]; Gong [17]; Gong and Zhang [18];Huang and Li [19]; Nojiri and Odintsov [20]; Guberina et al. [21-22]; Guo et al. [23-25]; Hu and Ling [26]; Li et al. [27]; Setare [28-29]; Sadjadi [30]; Banerjee and Pavón [31]; Kim et al. [32]; Zimdahl and Pavón [33]; Zimdahl [34]; Adhav et al.[35]. Furthermore, the holographic dark energy model has been extended to include the spatial curvature contribution, i.e. the holographic dark energy model in non-flat space Sarkar [36-38]. Recently, Sarkar [36-38] have studied non-interacting holographic dark energy with linearly varying deceleration parameter in Bianchi type-I and V universe and interacting holographic dark energy in Bianchi type-II respectively. Besides, some interacting models are discussed in many works because these models can help to understand or alleviate the coincidence problem by considering the possible interaction between dark energy and cold dark matter due to the unknown nature of dark energy and dark matter. In addition, the proposal of interacting dark energy is compatible with the current observations such as the SNIa and CMB data . Currently, an interesting attempt for probing the nature of dark energy within the framework of quantum gravity is the so-called "holographic dark energy". This principle is enlightened by investigations of the quantum property of black holes. Roughly speaking, in a quantum gravity system, the conventional local quantum field theory will break down. The holographic dark energy model has been tested and constrained by various astronomical observations. We focus in this paper on the holographic dark energy in a non-flat universe. The anisotropy plays a significant role in the early stage of evolution of the universe and hence the study of anisotropic and homogeneous cosmological models becomes important. The Bianchi type universe models are spatially homogeneous cosmological models that are in general anisotropic. The Bianchi type-I space-time is the straight forward generalization of Robertson-Walkar (RW) metric. However, Bianchi type I, V, VII models isotropy at late times even for ordinary matter, and the possible anisotropy of the Bianchi metrics necessarily die away during the inflationary era. The Bianchi type-I space-time reduces to flat FRW soon after inflation.

According to the accelerating expansion of the universe, we work on three kinds of scenarios which are based on different eras in the evolutionary process of the universe which all of them consist of a kind of expanding exponential scale factor. Motivated by the above discussion, in the present paper, we consider spatially homogeneous and anisotropic Bianchi type-V universe filled with interacting Dark matter and Holographic dark energy. The geometrical and physical aspects of the models are also studied. The

physical parameters that are of cosmological importance for Bianchi type-V space-time are

- The mean Hubble parameter:

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\frac{\dot{V}}{V} \qquad (1.1)$$

- The deceleration parameter:

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 \qquad (1.2)$$

Where a is the average scale factor and

$H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$, $H_3 = \frac{\dot{C}}{C}$ are the directional Hubble

parameters in the directions of x, y, z axes respectively.

2. Metric and Field Equations

We consider the Bianchi type-V metric given by

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{-2mx} dy^2 + C^2 e^{-2mx} dz^2 \quad (2.1)$$

where A, B, and C are the metric functions of cosmic time t and m is constant.

The Einstein's field equations are $\left(8\pi G = 1 \ and \ c = 1\right)$

$$R_{ij} - \frac{1}{2} R g_{ij} = -T_{ij}^M + T_{ij}^\Lambda \qquad (2.2)$$

where

$$T_{ij}^M = \rho_m u_i u_j \quad T_{ij}^\Lambda = \left(\rho_\Lambda + \overline{p}_\Lambda\right) u_i u_j + g_{ij} \overline{p}_\Lambda - 2\eta \sigma_{ij} \qquad (2.3)$$

with $\overline{p}_\Lambda = \left(p_\Lambda - \xi u_{;i}^i\right)$

are matter tensor for dark matter (pressure less i.e. $\omega_m = 0$) and holographic dark energy. Here ρ_m is the energy density of dark matter and ρ_Λ and p_Λ are the energy density and pressure of holographic dark energy. $\eta \geq 0$, $\xi \geq 0$ are the coefficients of shear and bulk viscosity respectively, v_i, the four-velocity vector of the fluid satisfying, $v_i v^i = -1$, θ is the expansion scalar and σ_{ij} is the shear tensor.

The Einstein's field equations (2.2) for metric (2.1) with the help of Eqs. (2.3) can be written as

$$-\left[p_\Lambda - \left(\xi - \frac{2}{3}\eta\right)\theta\right] + 2\eta\frac{\dot{A}}{A} = \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{m^2}{A^2} \quad (2.4)$$

$$-\left[p_\Lambda - \left(\xi - \frac{2}{3}\eta\right)\theta\right] + 2\eta\frac{\dot{B}}{B} = \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m^2}{A^2} \quad (2.5)$$

$$-\left[p_\Lambda - \left(\xi - \frac{2}{3}\eta\right)\theta\right] + 2\eta\frac{\dot{C}}{C} = \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m^2}{A^2} \quad (2.6)$$

$$\rho_m + \rho_\Lambda = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3m^2}{A^2} \qquad (2.7)$$

$$0 = \frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \qquad (2.8)$$

where overhead dot (.) represents derivative with respect to time t. The volume scale factor V is given by

$$V = a^3 = ABC \qquad (2.9)$$

θ is the expansion scalar defined by

$$\theta = u_{;i}^i \qquad (2.10)$$

Subtracting (2.4) from (2.5), we get

$$2\eta\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) = \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC} \qquad (2.11)$$

On integrating (2.11), We obtain

$$\frac{A}{B} = d_1 \exp\left[x_1 \int a^{-3} e^{-2\int \eta\, dt} dt\right] \qquad (2.12)$$

Similarly, Subtracting (2.5) from (2.6) and (2.4) from (2.6), and integrating we get

$$\frac{A}{C} = d_2 \exp\left[x_2 \int a^{-3} e^{-2\int \eta\, dt} dt\right] \qquad (2.13)$$

$$\frac{A}{B} = d_3 \exp\left[x_3 \int a^{-3} e^{-2\int \eta\, dt} dt\right] \qquad (2.14)$$

Where the relations $d_1 d_2, d_3 = 1$ and $x_1 + x_2 + x_3 = 0$ are satisfied by the constants d_1, d_2, d_3, x_1, x_2, x_3.

From equations (2.12)-(2.14), the metric functions can be written explicitly as

$$A = aa_1 \exp\left[b_1 \int a^{-3} e^{-2\int \eta\, dt} dt\right] \quad (2.15)$$

$$B = aa_2 \exp\left[b_2 \int a^{-3} e^{-2\int \eta\, dt} dt\right] \quad (2.16)$$

$$C = aa_3 \exp\left[b_3 \int a^{-3} e^{-2\int \eta\, dt} dt\right] \quad (2.17)$$

In order to introduce an interaction between dark energy and dark matter, we assume that both components do not conserve separately but interact with each other in such a manner that the balance equations take the form

$$\dot{\rho}_m + \frac{\dot{V}}{V}\rho_m = Q \qquad (2.18)$$

$$\dot{\rho}_\Lambda + \frac{\dot{V}}{V}\left(\rho_\Lambda + \overline{p}_\Lambda\right) = -Q \qquad (2.19)$$

where $\omega_\Lambda = \frac{p_\Lambda}{\rho_\Lambda}$ is the equation of state parameter for holographic dark energy and $Q > 0$ measures the strength of the interaction. A vanishing Q implies that matter and dark energy remain separately conserved. In view of continuity equations, the interaction between dark energy and dark matter must be a function of the energy density multiplied by a quantity with units of inverse of time, which can be chosen as the Hubble factor H. Models featuring an interaction matter-dark energy were introduced by Wetterich [39-40] (see also Billyard and Coley [41]) and first used alongside the holographic dark energy by Horvat [42]. Further, there is no known symmetry that would suppress such interaction and arguments in favour of interacting models have been put forward recently (Farrar and Peebles [43]). There is freedom to choose the form of the energy density, which can be any combination of dark energy and dark matter. Thus, the interaction between dark energy and dark matter could be expressed phenomenologically in forms such as (Amendola et al. [44-45])

$$Q = b^2 \frac{\dot{V}}{V} \rho_m = 3b^2 H \rho_0 V^{(b^2-1)} \qquad (2.20)$$

Where b^2 is coupling constant.

From Eqs. (2.20) and (2.18), we get the energy density of dark matter as

$$\rho_m = \rho_0 V^{(b^2-1)} \qquad (2.21)$$

Where $\rho_0 > 0$ is constant of integration.

Using Eqs. (2.20) and (2.21), we get the interacting term as

$$Q = 3b^2 H \rho_0 V^{(b^2-1)} \qquad (2.22)$$

3. Cosmological Solution

To solve the cosmological coincidence problem, we consider the interaction between the components on phenomenological level. Generally, interaction could be considered as a function of energy densities and their derivatives: $Q(\rho_i, \dot{\rho}_i ...)$. According to the accelerating expansion of the universe, we consider three kinds of scenarios which are based on different eras in the evolutionary process of the universe which all of them consist of a kind of expanding exponential scale factor as the followings.

3.1. Emergent Scenario

The scale factor in this scenario is given by,

$$a(t) = a_0 \left(\lambda + e^{\mu t} \right)^n \qquad (3.1)$$

Where $a_0 > 0$, $\lambda > 0$, $\mu > 0$ $and\ n > 1$ (Fabris, Gonsalves and de Sonza [46]).

we assume that the coefficient of shear viscosity is proportional to the scale of expansion, i.e. $\eta \ \alpha \ \theta$
That is

$$\eta = \eta_0 \theta \qquad (3.2)$$

Integrating, we get

$$\int \eta \, dt = 3\eta_0 n \log\left(\lambda + e^{\mu t} \right) \qquad (3.3)$$

Using Eq. (3.1) and (3.3) in Eqs. (2.15)–(2.17), we obtain the exact value of scale factors as

$$A = a_0 a_1 \left(\lambda + e^{\mu t} \right)^n \exp\left[\frac{b_1 a_0^{-3} \left(\lambda + e^{\mu t} \right)^{-3n(1+2\eta_0)+1}}{(-3n(1+2\eta_0)+1)\mu e^{\mu t}} \right] \quad (3.4)$$

$$B = a_0 a_2 \left(\lambda + e^{\mu t} \right)^n \exp\left[\frac{b_2 a_0^{-3} \left(\lambda + e^{\mu t} \right)^{-3n(1+2\eta_0)+1}}{(-3n(1+2\eta_0)+1)\mu e^{\mu t}} \right] \quad (3.5)$$

$$C = a_0 a_3 \left(\lambda + e^{\mu t} \right)^n \exp\left[\frac{b_3 a_0^{-3} \left(\lambda + e^{\mu t} \right)^{-3n(1+2\eta_0)+1}}{(-3n(1+2\eta_0)+1)\mu e^{\mu t}} \right] \quad (3.6)$$

Using Eqs. (3.4)–(3.6) in Eqs. (2.21) and (2.22), we get

$$\rho_m = \rho_0 a_0^{3(b^2-1)} \left(\lambda + e^{\mu t} \right)^{3n(b^2-1)} \qquad (3.7)$$

$$Q = 3b^2 \mu n \rho_0 a_0^{3(b^2-1)} e^{\mu t} \left(\lambda + e^{\mu t} \right)^{3n(b^2-1)-1} \quad (3.8)$$

Using Eqs. (3.4)–(3.6) and (3.7) in Eq. (2.7), we obtain the energy density of holographic dark energy as

$$\rho_\Lambda = (k_1 k_2 + k_2 k_3 + k_1 k_3)\left(\lambda + e^{\mu t} \right)^{2h} \left(h e^{\mu t} - \lambda \right)^2 + 2n(k_1 + k_2 + k_3)\mu e^{\mu t} \left(\lambda + e^{\mu t} \right)^{h-1} \left(h e^{\mu t} - \lambda \right)$$

$$+ 3n^2 \mu^2 \ e^{2\mu t} \left(\lambda + e^{\mu t} \right)^{-2} - 3m^2 (a_0 a_1)^{-2} \left(\lambda + e^{\mu t} \right)^{-2n} \exp\left[\frac{-2k_1 \left(\lambda + e^{\mu t} \right)^{h+1}}{\mu e^{\mu t}} \right]$$

$$- \rho_0 a_0^{3(b^2-1)} \left(\lambda + e^{\mu t} \right)^{3n(b^2-1)} \quad (3.9)$$

where $h = -3n(1+2\eta_0)$, $k_1 = \dfrac{b_1 a_0^{-3}}{-3n(1+2\eta_0)+1}$, $k_2 = \dfrac{b_2 a_0^{-3}}{-3n(1+2\eta_0)+1}$, $k_3 = \dfrac{b_3 a_0^{-3}}{-3n(1+2\eta_0)+1}$.

Using Eqs. (3.3) and (3.6) Eq. (2.4), we obtain the pressure of holographic dark energy as

$$p_\Lambda = -(k_2^2 + k_3^2 + k_2 k_3)\left(\lambda + e^{\mu t} \right)^{2h} \left(h e^{\mu t} - \lambda \right)^2 - (k_2 + k_3)h\mu e^{\mu t} \left(\lambda + e^{\mu t} \right)^h$$

$$- \left[(k_2 + k_3)h\mu e^{\mu t} + 2(k_2 + k_3) + k_2 e^{\mu t} + n\mu k_3 - \sigma \eta_0 \mu n e^{\mu t} k_1 \right]\left(\lambda + e^{\mu t} \right)^{h-1} \left(h e^{\mu t} - \lambda \right)$$

$$- \left[-2n\mu^2 e^{2\mu t} + n^2 \mu^2 e^{2\mu t} + \sigma \eta_0 \mu^2 n^2 e^{2\mu t} \right]\left(\lambda + e^{\mu t} \right)^{-2}$$

$$- e^{\mu t}\left[\eta\mu^2 - 3\mu n\left[\xi - 2\eta_0 \mu n e^{\mu t} \left(\lambda + e^{\mu t} \right)^{-1} \right]\right]\left(\lambda + e^{\mu t} \right)^{-1} \quad (3.10)$$

The EoS parameter of holographic dark energy is given by

$$\omega_\Lambda = \frac{-(k_2^2 + k_3^2 + k_2 k_3)\left(\lambda + e^{\mu t} \right)^{2h} \left(h e^{\mu t} - \lambda \right)^2 - (k_2 + k_3)h\mu e^{\mu t} \left(\lambda + e^{\mu t} \right)^h}{(k_1 k_2 + k_2 k_3 + k_1 k_3)\left(\lambda + e^{\mu t} \right)^{2h} \left(h e^{\mu t} - \lambda \right)^2}$$

$$\frac{- \left[(k_2 + k_3)h\mu e^{\mu t} + 2(k_2 + k_3) + k_2 e^{\mu t} + n\mu k_3 - \sigma \eta_0 \mu n e^{\mu t} k_1 \right]\left(\lambda + e^{\mu t} \right)^{h-1} \left(h e^{\mu t} - \lambda \right)}{+ 2n(k_1 + k_2 + k_3)\mu e^{\mu t} \left(\lambda + e^{\mu t} \right)^{h-1} \left(h e^{\mu t} - \lambda \right)}$$

$$\frac{-\left[-2n\mu^2 e^{2\mu t}+n^2\mu^2 e^{2\mu t}+\sigma\eta_0\mu^2\, n^2 e^{2\mu t}\right]\left(\lambda+e^{\mu t}\right)^{-2}}{+3n^2\mu^2\, e^{2\mu t}\left(\lambda+e^{\mu t}\right)^{-2}-3m^2\left(a_0 a_1\right)^{-2}\left(\lambda+e^{\mu t}\right)^{-2n}\exp\left[\dfrac{-2k_1\left(\lambda+e^{\mu t}\right)^{h+1}}{\mu e^{\mu t}}\right]}$$

$$\frac{-e^{\mu t}\left[\eta\mu^2-3\mu\,n\left[\xi-2\eta_0\mu\,ne^{\mu t}\left(\lambda+e^{\mu t}\right)^{-1}\right]\left(\lambda+e^{\mu t}\right)^{-1}}{-\rho_0 a_0^{3\left(b^2-1\right)}\left(\lambda+e^{\mu t}\right)^{3n\left(b^2-1\right)}} \quad (3.11)$$

Using Eqs. (3.2)–(3.4) in Eqs. (1.1) and (1.2), we get the mean Hubble parameter, deceleration parameter of expansion as

$$H=\frac{\mu\,n\,e^{\mu t}}{\left(\lambda+e^{\mu t}\right)} \quad (3.12)$$

$$q=-\left[1+\frac{\lambda}{ne^{\mu t}}\right] \quad (3.13)$$

The coincidence parameter $\bar{r}=\dfrac{\rho_m}{\rho_\Lambda}$ i.e. the ratio of dark matter energy density to the dark energy density is given by

$$\bar{r}=\frac{\rho_0 a_0^{3\left(b^2-1\right)}\left(\lambda+e^{\mu t}\right)^{3n\left(b^2-1\right)}}{\left(k_1 k_2+k_2 k_3+k_1 k_3\right)\left(\lambda+e^{\mu t}\right)^{2h}\left(he^{\mu t}-\lambda\right)^2+2n\left(k_1+k_2+k_3\right)\mu e^{\mu t}\left(\lambda+e^{\mu t}\right)^{h-1}\left(he^{\mu t}-\lambda\right)}$$

$$\frac{1}{+3n^2\mu^2\, e^{2\mu t}\left(\lambda+e^{\mu t}\right)^{-2}-3m^2\left(a_0 a_1\right)^{-2}\left(\lambda+e^{\mu t}\right)^{-2n}\exp\left[\dfrac{-2k_1\left(\lambda+e^{\mu t}\right)^{h+1}}{\mu e^{\mu t}}\right]}$$

$$\frac{1}{-\rho_0 a_0^{3\left(b^2-1\right)}\left(\lambda+e^{\mu t}\right)^{3n\left(b^2-1\right)}} \quad (3.14)$$

First, we consider the simplest case of modified holographic dark energy interacting with dark matter in emergent era and reconstruct the coincidence parameter and equation of state parameter numerically. By imposing the coefficient of shear viscosity is proportional to the scale of expansion, and variable parameter λ in the equation of state, we have the holographic dark energy, which is in general form. In this case the behavior of equation of state parameter is like minimal coupling case. The equation of state parameter $\omega_\Lambda<-1$, which means the phantom-like behavior. This shows that the Emergent scenario is useful for describing the universe because it is in agreement with the real data. The coincidence parameter has a faster decreasing time. The universe at large scale is isotropic and homogeneous and there is no time-like singularity. Cosmological solutions for phantom matter which violates the weak energy condition were found by Dabrowski et al.[47]. Caldwell [48], Srivastava [49], Yadav [50] have investigated phantom models with $\omega_\Lambda<-1$ and also suggested that at late time, phantom energy has appeared as a potential DE candidate which violets the weak as well as strong energy condition.

3.2. Logamediate Scenario

The Logamediate scenario of the universe is motivated by considering a class of possible cosmological solutions with indefinite expansion. In this model the scale factor showing the accelerating expansion of the universe is given by,

$$a(t)=e^{A_1(\log t)^\alpha} \quad (3.15)$$

where $A_1>0,\ \alpha>1$ (Bilic, Tupper and Violler [51]).

we assume that the coefficient of shear viscosity is proportional to the scale of expansion, i.e.

$$\eta\,\alpha\,\theta \quad (3.16)$$

That is

$$\eta=\eta_0\theta \quad (3.17)$$

Integrating, we get

$$\int\eta\,dt=3A_1\eta_0\left(\log t\right)^\alpha \quad (3.18)$$

Using Eq. (3.15) and (3.18) in Eqs. (2.15)–(2.17), we obtain the exact value of scale factors as

$$A=a_1 e^{A_1(\log t)^\alpha}\exp\left[\frac{b_1 e^{-3A_1(1+2\eta_0)(\log t)^\alpha t}}{-3A_1\left(1+2\eta_0\right)\alpha\left(\log t\right)^{\alpha-1}}\right] \quad (3.19)$$

$$B=a_2 e^{A_1(\log t)^\alpha}\exp\left[\frac{b_2 e^{-3A_1(1+2\eta_0)(\log t)^\alpha t}}{-3A_1\left(1+2\eta_0\right)\alpha\left(\log t\right)^{\alpha-1}}\right] \quad (3.20)$$

$$C=a_3 e^{A_1(\log t)^\alpha}\exp\left[\frac{b_3 e^{-3A_1(1+2\eta_0)(\log t)^\alpha t}}{-3A_1\left(1+2\eta_0\right)\alpha\left(\log t\right)^{\alpha-1}}\right] \quad (3.21)$$

Using Eqs. (3.19)-(3.21) in (1.1), (1.2), the Hubble parameter, its derivative, deceleration parameter and expansion scalar are given by

$$H=\frac{A_1\alpha\left(\log t\right)^{\alpha-1}}{t} \quad (3.22)$$

$$\dot{H}=\frac{A_1\alpha\left(\log t\right)^{\alpha-2}\left(\alpha-1-\log t\right)}{t^2} \quad (3.23)$$

$$q=-\left[\frac{A_1\alpha+\left(\log t\right)^{3\alpha-4}\left(\alpha-1-\log t\right)}{A_1\alpha}\right] \quad (3.24)$$

$$\theta = \frac{3A_1\alpha(\log t)^{\alpha-1}}{t} \qquad (3.25)$$

$$Q = 3b^2 A_1\alpha(\log t)^{\alpha-1}\rho_0 e^{3A_1(b^2-1)(\log t)^\alpha}t^{-1} \quad (3.27)$$

Using Eqs. (3.19)–(3.21) and (3.26) in Eq. (2.7), we obtain the energy density of holographic dark energy as

Using Eqs. (3.19)–(3.21) in Eqs. (2.21) and (2.22), we get

$$\rho_m = \rho_0 e^{3A_1(b^2-1)(\log t)^\alpha} \qquad (3.26)$$

$$\rho_\Lambda = 3\left(\frac{A_1\alpha(\log t)^{\alpha-1}}{t}\right)^2 - 2(b_1+b_2+b_3)\frac{e^{-3A_1(1+2\eta_0)(\log t)^\alpha}}{3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}}\left(1-3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}-\frac{\alpha-1}{\log t}\right)$$

$$-\frac{3m^2}{a_1^2\exp 2\left[A_1(\log t)^\alpha - \frac{b_1 e^{-3A_1(1+2\eta_0)(\log t)^\alpha}t}{3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}}\right]} - \rho_0 e^{3A_1(b^2-1)(\log t)^\alpha} \quad (3.28)$$

Using Eqs. (3.18)-(3.20) in Eq. (2.4), we obtain the pressure of holographic dark energy as

$$p_\Lambda = \left(\xi - \frac{2\eta_0 A_1\alpha(\log t)^{\alpha-1}}{t}\right)\frac{3A_1\alpha(\log t)^{\alpha-1}}{t} + 3(2\eta_0-1)\left(\frac{A_1\alpha(\log t)^{\alpha-1}}{t}\right)^2 + (b_2+b_3-2\eta_0 b_1)e^{-3A_1(1+2\eta_0)(\log t)^\alpha}$$

$$\left(1-\frac{3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}}{t}-\frac{(\alpha-1)}{\log t}\right)-\frac{2A_1\alpha}{t^2}\left((\alpha-1)t(\log t)^{\alpha-2}-(\log t)^{\alpha-1}\right)$$

$$+\frac{(b_2+b_3)}{3A_1\alpha(1+2\eta_0)}e^{-3A_1(1+2\eta_0)(\log t)^\alpha}\left\{\frac{1}{t}-\frac{3A_1\alpha(1+2\eta_0)(\log t)^\alpha}{t}\right.$$

$$\left.-\frac{3A_1\alpha(1+2\eta_0)}{t^2}\left[-3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}-1\right]-\frac{(\alpha-1)}{(\log t)^{2\alpha}t}\left[-3A_1\alpha(1+2\eta_0)(\log t)^{2\alpha-1}-\alpha(\log t)^{\alpha-1}\right]\right\}$$

$$-\frac{(b_2^2+b_3^2+b_2 b_3)}{(3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1})^2}e^{-6A_1(1+2\eta_0)(\log t)^\alpha}\left(1-\frac{3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}}{t}-\frac{(\alpha-1)}{\log t}\right)^2 \quad (3.29)$$

The EoS parameter of holographic dark energy is given by

$$w_\Lambda = \frac{\left(\xi - \frac{2\eta_0 A_1\alpha(\log t)^{\alpha-1}}{t}\right)\frac{3A_1\alpha(\log t)^{\alpha-1}}{t} + 3(2\eta_0-1)\left(\frac{A_1\alpha(\log t)^{\alpha-1}}{t}\right)^2 + (b_2+b_3-2\eta_0 b_1)e^{-3A_1(1+2\eta_0)(\log t)^\alpha}}{3\left(\frac{A_1\alpha(\log t)^{\alpha-1}}{t}\right)^2 - 2(b_1+b_2+b_3)\frac{e^{-3A_1(1+2\eta_0)(\log t)^\alpha}}{3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}}\left(1-3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}-\frac{\alpha-1}{\log t}\right)}$$

$$\frac{\left(1-\frac{3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}}{t}-\frac{(\alpha-1)}{\log t}\right)-\frac{2A_1\alpha}{t^2}\left((\alpha-1)t(\log t)^{\alpha-2}-(\log t)^{\alpha-1}\right)}{-\frac{3m^2}{a_1^2\exp 2\left[A_1(\log t)^\alpha - \frac{b_1 e^{-3A_1(1+2\eta_0)(\log t)^\alpha}t}{3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}}\right]} - \rho_0 e^{3A_1(b^2-1)(\log t)^\alpha}}$$

$$\frac{+\frac{(b_2+b_3)}{3A_1\alpha(1+2\eta_0)}e^{-3A_1(1+2\eta_0)(\log t)^\alpha}\left\{\frac{1}{t}-\frac{3A_1\alpha(1+2\eta_0)(\log t)^\alpha}{t}\right.}{}$$

$$\frac{\left.-\frac{3A_1\alpha(1+2\eta_0)}{t^2}\left[-3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}-1\right]-\frac{(\alpha-1)}{(\log t)^{2\alpha}t}\left[-3A_1\alpha(1+2\eta_0)(\log t)^{2\alpha-1}-\alpha(\log t)^{\alpha-1}\right]\right\}}{} \quad (3.30)$$

The coincidence parameter is given by

$$\overline{r} = \frac{\rho_0 e^{3A_1(b^2-1)(\log t)^\alpha}}{3\left(\dfrac{A_1\alpha(\log t)^{\alpha-1}}{t}\right)^2 - 2(b_1+b_2+b_3)\dfrac{e^{-3A_1(1+2\eta_0)(\log t)^\alpha}}{3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}}\left(1-3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1} - \dfrac{\alpha-1}{\log t}\right)}$$

$$-\frac{3m^2}{a_1^2 \exp 2\left[A_1(\log t)^\alpha - \dfrac{b_1 e^{-3A_1(1+2\eta_0)(\log t)^\alpha}t}{3A_1\alpha(1+2\eta_0)(\log t)^{\alpha-1}}\right]} - \rho_0 e^{3A_1(b^2-1)(\log t)^\alpha} \quad (3.31)$$

The Coincidence parameter and total equation of state parameter in this era with the intermediate scale factor reveals the quintessence-like behavior which is based on the $\omega_\Lambda > -1$. In this case Equation of state parameter is vanishes slower than coincidence parameter. the quintessence model is consistent with present and expected future evolution of the universe. The quintessence model approaches to isotropy at late time.

3.3. Intermediate Scenario

For the scale factor corresponding to intermediate scenario we have,

$$a(t) = e^{B_1 t^\beta} \quad (3.32)$$

Where $B_1 > 0,\ 0 < \beta < 1$ (Bento, Bertolami and Sen [52]).

we assume that the coefficient of shear viscosity is proportional to the scale of expansion, i.e. $\eta \propto \theta$

$$\eta = \eta_0 \theta \quad (3.33)$$

$$\int \eta\, dt = 3\eta_0 B_1 t^\beta \quad (3.34)$$

Using Eq. (3.32) and (3.34) in Eqs. (2.15)–(2.17), we obtain the exact value of scale factors as

$$A = a_1 \exp\left[B_1 t^\beta - \frac{b_1 e^{-3B_1(1+2\eta_0)t^\beta}}{3B_1(1+2\eta_0)\beta t^{\beta-1}}\right] \quad (3.35)$$

$$B = a_2 \exp\left[B_1 t^\beta - \frac{b_2 e^{-3B_1(1+2\eta_0)t^\beta}}{3B_1(1+2\eta_0)\beta t^{\beta-1}}\right] \quad (3.36)$$

$$C = a_3 \exp\left[B_1 t^\beta - \frac{b_3 e^{-3B_1(1+2\eta_0)t^\beta}}{3B_1(1+2\eta_0)\beta t^{\beta-1}}\right] \quad (3.37)$$

Using Eqs. (3.35)-(3.37) in (1.1), (1.2), the Hubble parameter, its derivative, deceleration parameter and expansion scalar are given by

$$H = B_1 \beta t^{\beta-1} \quad (3.38)$$

$$\dot{H} = B_1 \beta(\beta-1) t^{\beta-2} \quad (3.39)$$

$$q = -\left[1 + \frac{(\beta-1)}{t e^{B_1 t^\beta}}\right] \quad (3.40)$$

$$\theta = 3B_1 \beta t^{\beta-1} \quad (3.41)$$

Using Eqs. (3.35)–(3.37) in Eqs. (2.21) and (2.22), we get

$$\rho_m = \rho_0 e^{3B_1(b^2-1)t^\beta} \quad (3.42)$$

$$Q = 3b^2 B_1 \rho_0 \beta t^{\beta-1} e^{3B_1(b^2-1)t^\beta} \quad (3.43)$$

Using Eqs. (3.35)–(3.37) and (3.41) in Eq. (2.7), we obtain the energy density of holographic dark energy as

$$\rho_\Lambda = 3\left(B_1 \beta t^{\beta-1}\right)^2 + \frac{2(b_1+b_2+b_3)}{3(1+2\eta_0)t^{\beta-1}}e^{-3B_1(1+2\eta_0)t^\beta}\left[3B_1\beta(1+2\eta_0)t^{2(\beta-1)} + (\beta-1)t^{\beta-2}\right]$$

$$\frac{2(b_1 b_2 + b_2 b_3 + b_1 b_3)}{3B_1\beta(1+2\eta_0)t^{2(\beta-1)}}e^{-3B_1(1+2\eta_0)t^\beta}\left[3B_1\beta(1+2\eta_0)t^{2(\beta-1)} + (\beta-1)t^{\beta-2}\right]$$

$$-\frac{3m^2}{a_1^2 \exp 2\left[B_1 t^\beta - \dfrac{b_1 e^{-3B_1(1+2\eta_0)t^\beta}}{3B_1\beta(1+2\eta_0)t^{\beta-1}}\right]} - \rho_0 e^{3(b^2-1)B_1 t^\beta} \quad (3.44)$$

Using Eqs. (3.35)-(3.37) in Eq. (2.4), we obtain the pressure of holographic dark energy as

$$p_\Lambda = 3B_1\beta\left(\xi - 2\eta_0 B_1 \beta t^{\beta-1}\right)t^{\beta-1} + 6\eta_0\left[\left(B_1\beta t^{\beta-1}\right)^2 + \frac{b_1 e^{-3B_1(1+2\eta_0)t^\beta}}{3(1+2\eta_0)t^{\beta-1}}\left(3B_1\beta(1+2\eta_0)t^{2(\beta-1)} + (\beta-1)t^{\beta-2}\right)\right]$$

$$-\Big[2B_1\beta(\beta-1)t^{\beta-2} - 3(b_2+b_3)B_1(1+2\eta_0)\beta t^{\beta-1}e^{-3B_1(1+2\eta_0)t^\beta}$$

$$-\frac{(\beta-1)(b_2+b_3)}{3(1+2\eta_0)B_1}e^{-3B_1(1+2\eta_0)t^\beta}t^{-2\beta}\left(3B_1(1+2\eta_0)t^{2\beta-1} + t^{\beta-1}\right)\Big]$$

$$-\left[3(B_1\beta\,t^{\beta-1})^2 + 3(b_2+b_3)B_1\frac{t^{\beta-1}e^{-3B_1(1+2\eta_0)t^\beta}}{3(1+2\eta_0)t^{2(\beta-1)}}\Big(3B_1\beta(1+2\eta_0)t^{2(\beta-1)}+(\beta-1)t^{(\beta-2)}\Big)\right]$$

$$+(b_2^2+b_3^2+b_2b_3)\frac{e^{-6B_1(1+2\eta_0)t^\beta}}{\big(3B_1\beta(1+2\eta_0)t^{2(\beta-1)}\big)^2}\Big(3B_1\beta(1+2\eta_0)t^{2(\beta-1)}+(\beta-1)t^{\beta-2}\Big)^2$$

$$+\frac{m^2}{a_1^2\exp 2\left[B_1 t^\beta - \dfrac{b_1 e^{-3B_1(1+2\eta_0)t^\beta}}{3B_1\beta(1+2\eta_0)t^{\beta-1}}\right]}\qquad(3.45)$$

The EoS parameter of holographic dark energy is given by

$$w_\Lambda = \frac{3B_1\beta(\xi-2\eta_0 B_1\beta t^{\beta-1})t^{\beta-1}+6\eta_0\left[(B_1\beta t^{\beta-1})^2+\dfrac{b_1 e^{-3B_1(1+2\eta_0)t^\beta}}{3(1+2\eta_0)t^{\beta-1}}\Big(3B_1\beta(1+2\eta_0)t^{2(\beta-1)}+(\beta-1)t^{\beta-2}\Big)\right]}{3(B_1\beta t^{\beta-1})^2+\dfrac{2(b_1+b_2+b_3)}{3(1+2\eta_0)t^{\beta-1}}e^{-3B_1(1+2\eta_0)t^\beta}\Big[3B_1\beta(1+2\eta_0)t^{2(\beta-1)}+(\beta-1)t^{\beta-2}\Big]}$$

$$\frac{-\big[2B_1\beta(\beta-1)t^{\beta-2}-3(b_2+b_3)B_1(1+2\eta_0)\beta t^{\beta-1}e^{-3B_1(1+2\eta_0)t^\beta}}{\dfrac{2(b_1b_2+b_2b_3+b_1b_3)}{3B_1\beta(1+2\eta_0)t^{2(\beta-1)}}e^{-3B_1(1+2\eta_0)t^\beta}\Big[3B_1\beta(1+2\eta_0)t^{2(\beta-1)}+(\beta-1)t^{\beta-2}\Big]}$$

$$-\frac{\dfrac{(\beta-1)(b_2+b_3)}{3(1+2\eta_0)B_1}e^{-3B_1(1+2\eta_0)t^\beta}t^{-2\beta}\Big(3B_1(1+2\eta_0)t^{2\beta-1}+t^{\beta-1}\Big)\Big]}{-\dfrac{3m^2}{a_1^2\exp 2\left[B_1 t^\beta-\dfrac{b_1 e^{-3B_1(1+2\eta_0)t^\beta}}{3B_1\beta(1+2\eta_0)t^{\beta-1}}\right]}-\rho_0 e^{3(b^2-1)B_1 t^\beta}}$$

$$-\left[3(B_1\beta\,t^{\beta-1})^2+3(b_2+b_3)B_1\frac{t^{\beta-1}e^{-3B_1(1+2\eta_0)t^\beta}}{3(1+2\eta_0)t^{2(\beta-1)}}\Big(3B_1\beta(1+2\eta_0)t^{2(\beta-1)}+(\beta-1)t^{(\beta-2)}\Big)\right]$$

$$+(b_2^2+b_3^2+b_2b_3)\frac{e^{-6B_1(1+2\eta_0)t^\beta}}{\big(3B_1\beta(1+2\eta_0)t^{2(\beta-1)}\big)^2}\Big(3B_1\beta(1+2\eta_0)t^{2(\beta-1)}+(\beta-1)t^{\beta-2}\Big)^2$$

$$+\frac{m^2}{a_1^2\exp 2\left[B_1 t^\beta-\dfrac{b_1 e^{-3B_1(1+2\eta_0)t^\beta}}{3B_1\beta(1+2\eta_0)t^{\beta-1}}\right]}\qquad(3.46)$$

The coincidence parameter is given by

$$\bar{r}=\frac{\rho_0 e^{3B_1(b^2-1)t^\beta}}{3(B_1\beta t^{\beta-1})^2+\dfrac{2(b_1+b_2+b_3)}{3(1+2\eta_0)t^{\beta-1}}e^{-3B_1(1+2\eta_0)t^\beta}\Big[3B_1\beta(1+2\eta_0)t^{2(\beta-1)}+(\beta-1)t^{\beta-2}\Big]}$$

$$\frac{}{\dfrac{2(b_1b_2+b_2b_3+b_1b_3)}{3B_1\beta(1+2\eta_0)t^{2(\beta-1)}}e^{-3B_1(1+2\eta_0)t^\beta}\Big[3B_1\beta(1+2\eta_0)t^{2(\beta-1)}+(\beta-1)t^{\beta-2}\Big]}$$

$$-\frac{3m^2}{a_1^2 \exp 2\left[B_1 t^\beta - \frac{b_1 e^{-3B_1(1+2\eta_0)t^\beta}}{3B_1\beta(1+2\eta_0)t^{\beta-1}}\right]} - \rho_0 e^{3(b^2-1)B_1 t^\beta} \qquad (3.47)$$

It is observed that at t = 0, the parameters θ, H diverge. Hence the model starts with a big bang singularity at t = 0. This is a Point Type singularity (MacCallum [53]) since directional scale factor A, B and C vanish at initial time. The Coincidence parameter and total equation of state parameter in this era with the intermediate scale factor reveals the quintessence-like behavior which is based on the $\omega_\Lambda > -1$. In this case Equation of state parameter is vanishes slower than coincidence parameter.

4. Conclusion

In this paper we have studied the anisotropic and homogeneous Bianchi type-V universe filled with interacting Dark matter and Holographic dark energy. Here we discussed three different scenario. $w_\Lambda < 0$ is necessarily accompanied by the decay of the dark energy component into pressureless dark matter $(b^2 > 0)$. It is shown that for suitable choice of interaction between dark matter and holographic dark energy i.e. Eq. (2.20) with $(b^2 = 1)$, there is no coincidence problem in case of exponential volumetric expansion model and special form of deceleration parameter which matches with the observations as we now live in the stationary coincidence state of the universe, whereas there is coincidence problem in case of power-law volumetric expansion model. In the Emergent case, we found the phantom-like behavior due to the w_Λ, leads to uncommon cosmological scenarios and quintessence-like behaviors in the Intermediate and Logamediate era is obvious. Since the Phantom-like behavior is more consistent with the observational data, Then the Emergent scenario is better than the Logamediate and Intermediate ones. It is observed that such DE models are also in good harmony with current observations. Thus, the solutions demonstrated in this paper may be useful for better understanding of the characteristic of anisotropic DE in the evolution of the universe within the framework of Bianchi type-V space-time.

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