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Some Constructions of BIB Designs with Quasi Symmetric Structure

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Abstract: ***Some construction methods of k-multiple resolvable balanced incomplete block designs with no repeated blocks with parameters $v = 2k$, $b^* = 4k^2 - 2k = kb$, $r^* = 2k^2 - k = kr$, k, $\lambda^* = k^2 - k = k\lambda$ are proposed with illustrations.***

Keywords: Balanced Incomplete Block Design, Resolvable Design, Affine Resolvable Design, Quasi Symmetric Designs, Galois Field.

1. Introduction

BIB design play important role in design of experiments especially in field of experiments. In statistical design of experiments the importance of BIB designs for varietal trial was realized when in 1936 Yates discussed these designs in the context of biological and agriculture experiments. Many constructional methods of BIB design were given by Prof. Fisher [7], Yates [22] and Bose [3]. These BIB designs ensure that treatments are compared with equal precision. Ray et al.[15] gave the solution of Kirkman's school girl problem which is the best example of balanced incomplete block design.

Patterson et al. [12] described an algorithm for constructing resolvable incomplete block designs where v is multiple of k. These designs are called α-designs. In 1853 Steiner proposed the problem of arranging n objects in triplets such that every pair of objects appears in precisely one triplet such type of arrangement is known as Steiner's triple system or BIB design. In United Kingdom resolvable incomplete block designs are mostly used in statutory trials of agricultural crop varieties. In the combinatorial context resolvable designs have many applications in field of experiments. Bose [4] defined the concept of resolvability for a BIB design. The concept of resolvability to α-resolvability where in each of the resolvable group, each treatment appears exactly α times was further elaborated by Shrikhande and Raghavrao [19].

A balanced incomplete block design is an arrangement of v treatments in b blocks, of size k where each treatment replicated r times, and every pair of treatment appears together in λ blocks. A BIB design is symmetric iff v = b and r = k.

Quasi Symmetric Design - Let S be a finite set of v objects (points), and γ be a finite family of distinct k subsets of S (blocks). Then the pair D = { S, γ } is called a block design (or 2-design) with parameters (v, b, r, k, λ). For $0 \le x < k$, where an intersection number of D is x, if there exist B, B' ∈ γ such that $|B \cap B'| = x$. A 2-design *D* is **quasi-symmetric design** with two numbers of intersection x and y and $0 \le x < y < k$ if every two distinct blocks intersect in either x or *y* points.

An incomplete block design with parameters v, b, r, k, λ is known as resolvable, if b blocks can be divides into r groups of b/r = n blocks each n being an integer, such that n blocks forming any of these groups give complete replication of all the v treatments. If b = v + r - 1 and any two blocks from different groups have k^2 / v treatments in common, where k^2 / v is an integer then it is called an affine resolvable design. Any two blocks from different groups intersect in a constant y (say) of points, the design is affine. Thus an affine design is a type of quasi symmetric design with two type of intersection numbers x=0 and y. An incomplete block design is said to be α - resolvable if the blocks can be grouped into s sets, each containing γ blocks, such that in each set every treatment occurs α (≥1) times. For α – resolvable BIB design, vα = kγ, b = sγ and r = sα. An α- resolvable incomplete block design is known as affine α- resolvable if any pair of blocks from the same α- resolution set has q_1 treatments in common and any pair of blocks from different sets has q_2 treatments in common where $q_1 = k(\alpha-1) / (\gamma-1) = k + \lambda - r$ and $q_2 = k\alpha / \gamma = k^2/v$.

In the present article we study k-multiple resolvable balanced incomplete block design with no repeated blocks.

2. Method of Construction

Let D_1 be an affine resolvable balanced incomplete block design with parameters v = 2k, b = 4k –2, r = 2k –1, k, λ =k–1 where $v-1 = p^n$ is a prime or prime power and x is a primitive element of $GF(p^n)$. Design D_1 have initial block sets $(\infty, x^0, x^2, \ldots, x^{2(k-2)})$, $(0, x^1, x^3, \ldots, x^{2(k-1)-1})$. Now we interchange first element of initial block sets (∞ and 0) and get another design D_2 with initial block sets $(0, x^0, x^2, \ldots, x^{2(k-2)})$, $(\infty, x^1, x^3, \ldots, x^{2(k-1)-1})$. Similarly, we interchange the element of each new obtained design one by one and obtained $D_2, D_3, \ldots, D_k$, (k-1) new designs, now arrange $\mathbf{D_1, D_2, \ldots, \ldots, D_k}$ these k designs in form of $D^* = \mathbf{[D_1: D_2: \ldots: D_k]}$ and then by developing D^* we get k-multiple solution of resolvable balanced incomplete block design with no repeated blocks with parameters $\mathbf{v = 2k, b^* = 4k^2 - 2k = kb, r^* = 2k^2 - k = kr, k, \lambda^* = k^2 - k = k\lambda}$.

Theorem 2.1 The existence of an affine resolvable balanced incomplete block design D_1 with v = 2k, b = 4k –2, r = 2k – 1, k, λ =k–1 where $v-1 = p^n$ is a prime or prime power and x is a primitive element of $GF(p^n)$, implies the existence of D^* a k-multiple resolvable balanced incomplete block design with no repeated blocks with parameters $\mathbf{v = 2k, b^* = 4k^2\text{-}2k = kb, r^* = 2k^2 - k = kr, k, \lambda^* = k^2\text{-} k = k\lambda}$.

Proof: Consider an affine resolvable BIB design D_1 with given parameters v = 2k, b = 4k –2, r = 2k –1, k, λ =k–1 where $v\text{-}1 = p^n$ is a prime or prime power and x is a primitive element of $GF(p^n)$. Then, the solution of the design D_1 with the help of initial block sets

$D_1 = (\infty, x^0, x^2, \ldots, x^{2(k-3)}, x^{2(k-2)}), (0, x^1, x^3, \ldots, x^{2(k-3)+1}, x^{2(k-1)-1})$ (1)

Now, get another solution of the design D_1 which is D_2 with initial block sets

$D_2 = (0, x^0, x^2, \ldots, x^{2(k-3)}, x^{2(k-2)}), (\infty, x^1, x^3, \ldots, x^{2(k-3)+1}, x^{2(k-1)-1})$ (2)

From design D_2 obtained design D_3 with initial block sets

$D_3 = (0, x^1, x^2, \ldots, x^{2(k-3)}, x^{2(k-2)}), (\infty, x^0, x^3, \ldots, x^{2(k-3)+1}, x^{2(k-1)-1})$.

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We will continue this process and from the design D_{k-2} we get design D_{k-1} with initial block sets

$D_{k-1} = (0, x^1, x^3, \ldots, x^{2(k-3)}, x^{2(k-2)}), (\infty, x^0, x^2, \ldots, x^{2(k-3)+1}, x^{2(k-1)-1})$ (3)

Similarly, from the design D_{k-1} we get design D_k have initial block sets

$D_k = (0, x^1, x^3, \ldots, x^{2(k-3)+1}, x^{2(k-2)}), (\infty, x^0, x^2, \ldots, x^{2(k-3)}, x^{2(k-1)-1})$ (4)

Thus, from (1) to (4) we get (k-1) new designs, arrange $\mathbf{D_1, D_2, \ldots, D_k}$ these k designs in form of $\mathbf{D^* = [D_1 : D_2 : \ldots : D_{k-1} : D_k]}$. Hence, by developing D* we get the complete solution of k-multiple resolvable balanced incomplete block design with no repeated blocks with parameters

$\mathbf{v = 2k, b^* = 4k^2\text{-}2k = kb, r^* = 2k^2 - k = kr, k, \lambda^* = k^2\text{-} k = k\lambda}$.

This complete the proof.

Example 2.2 We illustrate the theorem 2.1, if k = 6 then $\mathbf{D_1}$ an affine resolvable balanced incomplete block design with the parameters

v = 12, b = 22, r = 11, k=6, λ =5.

Since primitive element of GF (11) is x = **2** and the solution of the design is given by the initial block sets $(0, 2^0, 2^2, 2^4, 2^6, 2^8), (\infty, 2^1, 2^3, 2^5, 2^7, 2^9)$ that is (0, 1, 4, 5, 9, 3), (∞, **2,** 8, 10, 7, 6) mod 11 provides

(0, 1, 3, 4, 5, 9), (∞, 2, 6, 7, 8, 10); (1, 2, 4, 5, 6, 10), (∞, **3,** 7, 8, 9, 0);
(2, 3, 5, 6, 7, 0), (∞, **4,** 8, 9, 10, 1); (3, 4, 6, 7, 8, 1), (∞, 5, 9, 10, 0, 2);
D₁: (4, 5, 7, 8, 9, 2), (∞, 6, 10, 0, 1, 3); (5, 6, 8, 9, 10, 3), (∞, 7, 0, 1, 2, 4);
(6, 7, 9, 10, 0, 4), (∞, 8, 1, 2, 3, 5); (7, 8, 10, 0, 1, 5), (∞, 9, 2, 3, 4, 6);
(8, 9, 0, 1, 2, 6), (∞, 10, 3, 4, 5, 7); (9, 10, 1, 2, 3, 7), (∞, 0, 4, 5, 6, 8);
(10, 0, 2, 3, 4, 8), (∞, 1, 5, 6, 7, 9).

Second initial block sets are $(\infty, 2^0, 2^2, 2^4, 2^6, 2^8), (0, 2^1, 2^3, 2^5, 2^7, 2^9)$ that is (∞, 1, 4, 5, 9, 3), (**0**, 2, 8, 10, 7, 6) mod 11 provides

(∞, 1, 3, 4, 5, 9), (0, 2, 6, 7, 8, 10); (∞, 2, 4, 5, 6, 10), (1, 3, 7, 8, 9, 0);
(∞, 3, 5, 6, 7, 0), (2, 4, 8, 9, 10, 1); (∞, 4, 6, 7, 8, 1), (3, 5, 9, 10, 0, 2);
D₂: (∞, 5, 7, 8, 9, 2), (4, 6, 10, 0, 1, 3); (∞, 6, 8, 9, 10, 3), (5, 7, 0, 1, 2, 4);
(∞, 7, 9, 10, 0, 4), (6, 8, 1, 2, 3, 5); (∞, 8, 10, 0, 1, 5), (7, 9, 2, 3, 4, 6);
(∞, 9, 0, 1, 2, 6), (8, 10, 3, 4, 5, 7); (∞, 10, 1, 2, 3, 7), (9, 0, 4, 5, 6, 8);
(∞, 0, 2, 3, 4, 8), (10, 1, 5, 6, 7, 9).

Third initial block sets are $(\infty, \mathbf{2^1}, 2^2, 2^4, 2^6, 2^8), (0, \mathbf{2^0}, 2^3, 2^5, 2^7, 2^9)$ that is (∞, **2**, 4, 5, 9, 3), (0, **1,** 8, 10, 7, 6) mod 11 provides

(∞, 2, 3, 4, 5, 9), (0, 1, 6, 7, 8, 10); (∞, 3, 4, 5, 6, 10), (1, 2, 7, 8, 9, 0);
(∞, 4, 5, 6, 7, 0), (2, 3, 8, 9, 10, 1); (∞, 5, 6, 7, 8, 1), (3, 4, 9, 10, 0, 2);
D₃: (∞, 6, 7, 8, 9, 2), (4, **5**, 10, 0, 1, 3); (∞, 7, 8, 9, 10, 3), (5, 6, 0, 1, 2, 4);
(∞, 8, 9, 10, 0, 4), (6, 7, 1, 2, 3, 5); (∞, 9, 10, 0, 1, 5), (7, 8, 2, 3, 4, 6);
(∞, 10, 0, 1, 2, 6), (8, 9, 3, 4, 5, 7); (∞, 0, 1, 2, 3, 7), (9, 10, 4, 5, 6, 8);
(∞, 1, 2, 3, 4, 8), (10, 0, 5, 6, 7, 9).

Fourth initial block sets are $(\infty, 2^1, \mathbf{2^3}, 2^4, 2^6, 2^8), (0, 2^0, \mathbf{2^2}, 2^5, 2^7, 2^9)$ that is (∞, 2, **8**, 5, 9, 3), (0, 1, **4**, 10, 7, 6) mod 11 provides

(∞, 2, 3, 5, 8, 9), (0, 1, 4, 6, 7, 10); (∞, 3, 4, 6, 9, 10), (1, 2, 5, 7, 8, 0);
(∞, 4, 5, 7, 10, 0), (2, 3, 6, 8, 9, 1); (∞, 5, 6, 8, 0, 1), (3, 4, 7, 9, 10, 2);
D₄: (∞, 6, 7, 9, 1, 2), (4, **5**, 8, 10, 0, 3); (∞, 7, 8, 10, 2, 3), (5, 6, 9, 0, 1, 4);
(∞, 8, 9, 0, 3, 4), (6, 7, 10, 1, 2, 5); (∞, 9, 10, 1, 4, 5), (7, 8, 0, 2, 3, 6);
(∞, 10, 0, 2, 5, 6), (8, 9, 1, 3, 4, 7); (∞, 0, 1, 3, 6, 7), (9, 10, 2, 4, 5, 8);
(∞, 1, 2, 4, 7, 8), (10, 0, 3, 5, 6, 9).

Fifth initial block sets are $(\infty, 2^1, 2^3, \mathbf{2^5}, 2^6, 2^8), (0, 2^0, 2^2, \mathbf{2^4}, 2^7, 2^9)$ that is (∞, 2, 8, **10**, 9, 3), (0, 1, 4, **5**, 7, 6) mod 11 provides

(∞, 2, 3, 8, 9, 10), (0, 1, 4, 5, 6, 7); (∞, 3, 4, 9, 10, 0), (1, 2, 5, 6, 7, 8);
(∞, 4, 5, 10, 0, 1), (2, 3, 6, 7, 8, 9); (∞, 5, 6, 0, 1, 2), (3, 4, 7, 8, 9, 10);
D₅: (∞, 6, 7, 1, 2, 3), (4, **5**, 8, 9, 10, 0); (∞, 7, 8, 2, 3, 4), (5, 6, 9, 10, 0, 1);
(∞, 8, 9, 3, 4, 5), (6, 7, 10, 0, 1, 2); (∞, 9, 10, 4, 5, 6), (7, 8, 0, 1, 2, 3);
(∞, 10, 0, 5, 6, 7), (8, 9, 1, 2, 3, 4); (∞, 0, 1, 6, 7, 8), (9, 10, 2, 3, 4, 5);
(∞, 1, 2, 7, 8, 9), (10, 0, 3, 4, 5, 6).

Sixth initial block sets are $(\infty, 2^1, 2^3, 2^5, \mathbf{2^7}, 2^8), (0, 2^0, 2^2, 2^4, \mathbf{2^6}, 2^9)$ that is (∞, 2, 8, 10, **7**, 3), (0, 1, 4, 5, **9**, 6) mod 11 provides

(∞, 2, 3, 7, 8, 10), (0, 1, 4, 5, 6, 9); (∞, 3, 4, 8, 9, 0), (1, 2, 5, 6, 7, 10);
(∞, 4, 5, 9, 10, 1), (2, 3, 6, 7, 8, 0); (∞, 5, 6, 10, 0, 2), (3, 4, 7, 8, 9, 1);
D₆: (∞, 6, 7, 0, 1, 3), (4, **5**, 8, 9, 10, 2); (∞, 7, 8, 1, 2, 4), (5, 6, 9, 10, 0, 3);

(∞, 8, 9, 2, 3, 5), (6, 7, 10, 0, 1, 4); (∞, 9, 10, 3, 4, 6), (7, 8, 0, 1, 2, 5);
(∞, 10, 0, 4, 5, 7), (8, 9, 1, 2, 3, 6); (∞, 0, 1, 5, 6, 8), (9, 10, 2, 3, 4, 7);
(∞, 1, 2, 6, 7, 9), (10, 0, 3, 4, 5, 8).

Thus, from affine resolvable balanced incomplete block design D_1 with the parameters **v = 12, b = 22, r = 11, k=6, λ =5,** we obtained 5 new designs $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{D_5}$, $\mathbf{D_6}$ arrange these 6 designs in form of $\mathbf{D^* = [D_1: D_2: D_3: D_4: D_5: D_6]}$ by developing D^* we get 6-multiple solution of resolvable balanced incomplete block design with no repeated blocks with parameters **v =12, b*=132, r*= 66, k = 6, λ* =30.**

3. Conclusion

This k-multiple resolvable balanced incomplete block design with no repeated blocks with parameters $v = 2k$, $b^* = 4k^2-2k = kb$, $r^* = 2k^2 - k = kr$, k, $\lambda^* = k^2 - k = k\lambda$ have quasi symmetric structure with different intersection numbers corresponding to a block. The following table-1 provides a list of parameters which can be obtained by using theorem 2.1.

Table 1

S.No.	$v = 2k$	$b^* = 4k^2-2k = kb$	$r^* = 2k^2 - k = kr$	k	$\lambda^* = k^2 - k = k\lambda$
1	8	56	28	4	12
2	12	132	66	6	30
3	20	380	190	10	90
4	24	552	276	12	132
5	28	756	378	14	182
6	32	992	496	16	240

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