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Improved Cauchy-Schwarz Inequality for Matrices

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Abstract: ***In this paper, using Hermite-Hadamard Integral Inequality we give an improvement of the matrix Cauchy-Schwarz inequality which was due to Bhatia and Davis in [5], and also obtain a further improvement of a refined matrix Cauchy-Schwarz inequality which was due to Ali et al. in [2].***

Keywords: Unitarily invariant norms; Cauchy- Schwarz inequality; Hermite-Hadamard Integral Inequality; Positive semidefinite matrices

Mathematics subject classification (2010): 15A60, 15A45.

1. Introduction

Let $M_{m,n}$ be the space of $m\times n$ complex matrices and $M_n = M_{n,n}$. $\|\cdot\|$ denotes the unitarily norm.

For all $A, B, X \in M_n$, A, B are positive semidefinite , for all real number $r\ge 0$ and every v with $0\le v\le 1$, the well-known Cauchy-Schwarz inequality for matrix [5] (see also [6, p. 267]) says that

$$\phi(\frac{1}{2})\le\phi(0) \qquad (1.1)$$

where $\phi(v)=\left\|\left|A^{v}XB^{1-v}\right|^{r}\right\|\cdot\left\|\left|A^{1-v}XB^{v}\right|^{r}\right\|$.

Hiai and Zhan [1] showed the following inequality

$$\phi(\frac{1}{2})\le\phi(v)\le\phi(0), \qquad (1.2)$$

which is a refinement of (1.1).

In [2], Ali et al. gave another refinement of (1.1) as follows:

$$\phi(\frac{1}{2})+2(\int_0^1\phi(v)dv-\phi(\frac{1}{2}))\le\phi(0). \qquad (1.3)$$

In this paper, we present new refinements of (1.1) and (1.3).

2. Main Results

Lemma 2.1[3. p.21] Let f be a convex function defined on an interval I. If $x, y, z\in I$ such that $x\le y\le z (x\ne z)$ then

$$f(y)-f(x)\le\frac{f(z)-f(x)}{z-x}(y-x). \qquad (2.1)$$

Lemma 2.2 (Improved Hermite-Hadamard Integral Inequality) [4] Let f be a convex function defined on an interval $[a,b]$. Then

$$f(\frac{a+b}{2})\le\frac{1}{b-a}\int_a^b f(t)dt$$
$$\le\frac{1}{4}(f(a)+2f(\frac{a+b}{2})+f(b))$$
$$\le\frac{f(a)+f(b)}{2}.$$

Lemma 2.3 Let f be a convex function defined on an interval $[a,b]$.Then for positive integers m, n with $n\ge m$

$$f(\frac{a+b}{2})\le\frac{1}{b-a}\int_a^b f(t)dt$$
$$\le\frac{1}{4n}((2n-m)f(a)+2mf(\frac{a+b}{2})+(2n-m)f(b))$$
$$\le\frac{f(a)+f(b)}{2}.$$

Proof. Since f is convex on $[0,1]$, we have

$$2f(\frac{a+b}{2})\le f(a)+f(b)).$$

Thus

$$\frac{1}{4n}((2n-m)f(a)+2mf(\frac{a+b}{2})+(2n-m)f(b))$$
$$\le\frac{1}{4n}((2n-m)f(a)+mf(a)+mf(b)+(2n-m)f(b))$$
$$=\frac{f(a)+f(b)}{2},$$

which is the last inequality.

On the other hand, form Lemma 2.2, we have

$$\frac{1}{b-a}\int_a^b f(t)dt$$
$$\le\frac{1}{4}(f(a)+2f(\frac{a+b}{2})+f(b))$$
$$=\frac{1}{4n}(nf(a)+2nf(\frac{a+b}{2})+nf(b))$$
$$=\frac{1}{4n}(nf(a)+(2n-2m)f(\frac{a+b}{2})+2mf(\frac{a+b}{2})+nf(b))$$
$$\le\frac{1}{4n}(nf(a)+(n-m)(f(a)+f(b))+2mf(\frac{a+b}{2})+nf(b))$$
$$\le\frac{1}{4n}((2n-m)f(a)+2mf(\frac{a+b}{2})+(2n-m)f(b)),$$

which is the middle inequality.

Remark 2.4 Set $m=1$ in Lemma 2.2, then we get Lemma 1 of [7].

Theorem 2.5 Let $A, B, X\in M_n$ such that A, B are positive semidefinite, and let $r\ge 0, 0\le v\le 1$, and $0\le u\le 1$. Then for positive integers m,n with $n \ge m$

$$\begin{aligned}\phi(\frac{1}{2}) &\le \frac{1}{|1-2u|}\left|\int_u^{1-u}\phi(v)dv\right| \\ &\le \frac{1}{2n}((2n-m)\phi(u)+m\phi(\frac{1}{2})) \\ &\le \phi(0),\end{aligned}$$

where $\phi(v)=\left\|\left|A^v XB^{1-v}\right|^r\right\|\cdot\left\|\left|A^{1-v}XB^v\right|^r\right\|$.

Proof. Since $\phi(v)=\left\|\left|A^v XB^{1-v}\right|^r\right\|\cdot\left\|\left|A^{1-v}XB^v\right|^r\right\|$ is convex on $[0,1]$, then for $0\le u\le\frac{1}{2}$, by Lemma 2.3, we have

$$\begin{aligned}\phi(\frac{1}{2}) &= \phi(\frac{u+1-u}{2}) \\ &\le \frac{1}{1-2u}\int_u^{1-u}\phi(v)dv \\ &\le \frac{1}{4n}((2n-m)\phi(u)+2m\phi(\frac{1}{2})+(2n-m)\phi(1-u)) \\ &= \frac{1}{2n}((2n-m)\phi(u)+2m\phi(\frac{1}{2})) \\ &\le \frac{\phi(u)+\phi(1-u)}{2}\le\phi(0).\end{aligned}\tag{2.2}$$

If $\frac{1}{2}\le u\le 1$, then

$$\begin{aligned}\phi(\frac{1}{2}) &= \phi(\frac{u+1-u}{2}) \\ &\le \frac{1}{2u-1}\int_{1-u}^{u}\phi(v)dv \\ &\le \frac{1}{4n}((2n-m)\phi(u)+2m\phi(\frac{1}{2})+(2n-m)\phi(1-u)) \\ &= \frac{1}{2n}((2n-m)\phi(u)+2m\phi(\frac{1}{2})) \\ &\le \frac{\phi(u)+\phi(1-u)}{2}\le\phi(0).\end{aligned}\tag{2.3}$$

Combining (2.2) and (2.3), we get the required inequality.

Theorem 2.6 Let $A,B,X\in M_n$ such that A,B are positive semidefinite , and $r\ge 0, 1\ge v\ge 0$. Then

$$\phi(\frac{1}{2})+[4\int_0^1\phi(v)dv-2\phi(\frac{1}{2})-2\phi(\frac{1}{4})]\le\phi(0)\tag{2.4}$$

where $\phi(v)=\left\|\left|A^v XB^{1-v}\right|^r\right\|\cdot\left\|\left|A^{1-v}XB^v\right|^r\right\|$.

Proof. Since $\phi(v)=\left\|\left|A^v XB^{1-v}\right|^r\right\|\cdot\left\|\left|A^{1-v}XB^v\right|^r\right\|$ is convex on $[0,1]$, then for $0\le v\le 1$, by Lemma 2.2, we have

$$\phi(\frac{1}{2})\le\phi(v)\le\phi(0).$$

If $0\le v\le\frac{1}{4}$, by inequality (2.1), then we have

$$\phi(v)-\phi(0)\le 4v[\phi(\frac{1}{4})-\phi(0)].$$

Thus

$$\int_0^{\frac{1}{4}}\phi(v)dv\le\int_0^{\frac{1}{4}}4v[\phi(\frac{1}{4})-\phi(0)]dv+\frac{1}{4}\phi(0),$$

which is

$$\int_0^{\frac{1}{4}}\phi(v)dv\le\frac{1}{8}[\phi(\frac{1}{4})+\phi(0)].\tag{2.5}$$

If $\frac{1}{4}\le v\le\frac{1}{2}$, by inequality (2.1), then we have

$$\phi(v)-\phi(\frac{1}{4})\le 4(v-\frac{1}{4})[\phi(\frac{1}{2})-\phi(\frac{1}{4})].$$

Thus

$$\int_{\frac{1}{4}}^{\frac{1}{2}}\phi(v)dv\le\int_{\frac{1}{4}}^{\frac{1}{2}}4(v-\frac{1}{4})[\phi(\frac{1}{2})-\phi(\frac{1}{4})]dv+\frac{1}{4}\phi(\frac{1}{4}),$$

which is

$$\int_{\frac{1}{4}}^{\frac{1}{2}}\phi(v)dv\le\frac{1}{8}[\phi(\frac{1}{4})+\phi(\frac{1}{2})].\tag{2.6}$$

If $\frac{1}{2}\le v\le\frac{3}{4}$, by inequality (2.1), then we have

$$\phi(v)-\phi(\frac{1}{2})\le 4(v-\frac{1}{2})[\phi(\frac{3}{4})-\phi(\frac{1}{2})].$$

Thus

$$\int_{\frac{1}{2}}^{\frac{3}{4}}\phi(v)dv\le\int_{\frac{1}{2}}^{\frac{3}{4}}4(v-\frac{1}{2})[\phi(\frac{3}{4})-\phi(\frac{1}{2})]dv+\frac{1}{4}\phi(\frac{1}{2}),$$

which is

$$\int_{\frac{1}{2}}^{\frac{3}{4}}\phi(v)dv\le\frac{1}{8}[\phi(\frac{1}{2})+\phi(\frac{3}{4})].\tag{2.7}$$

If $\frac{3}{4}\le v\le 1$, by inequality (2.1), then we have

$$\phi(v)-\phi(\frac{3}{4})\le 4(v-\frac{1}{4})[\phi(1)-\phi(\frac{1}{4})].$$

Thus

$$\int_{\frac{3}{4}}^{1}\phi(v)dv\le\int_{\frac{3}{4}}^{1}4(v-\frac{1}{4})[\phi(1)-\phi(\frac{3}{4})]dv+\frac{1}{4}\phi(\frac{3}{4}),$$

which is

$$\int_{\frac{3}{4}}^{1}\phi(v)dv\le\frac{1}{8}[\phi(\frac{3}{4})+\phi(1)].\tag{2.8}$$

Form inequalities (2.5)—(2.8) and $\phi(\frac{1}{4})=\phi(\frac{3}{4}),\phi(1)=\phi(0)$, we have

$$4\int_0^1\phi(v)dv\le\phi(0)+\phi(\frac{1}{2})+2\phi(\frac{1}{4}),$$

which is equivalent to

$$\phi(\frac{1}{2})+[4\int_0^1\phi(v)dv-2\phi(\frac{1}{2})-2\phi(\frac{1}{4})]\le\phi(0).$$

Remark 2.5 Inequality (2.4) is better than (1.3). In fact, by the convexity of ϕ, we have

$$\phi(\frac{1}{4})\le 2\int_0^{\frac{1}{2}}\phi(v)dv$$

and

$$\phi(\frac{1}{4})=\phi(\frac{3}{4})\le 2\int_{\frac{1}{2}}^{1}\phi(v)dv.$$

Thus

$$\int_0^1 \phi(v)dv - \phi(\frac{1}{4}) \geq 0$$

and

$$4\int_0^1 \phi(v)dv - 2\phi(\frac{1}{2}) - 2\phi(\frac{1}{4}) - 2[\int_0^1 \phi(v)dv - \phi(\frac{1}{2})]$$

$$= 2\int_0^1 \phi(v)dv - 2\phi(\frac{1}{4}) \geq 0.$$

Thus inequality (2.4) is an improvement of (1.3).

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