

# Improved Cauchy-Schwarz Inequality for Matrices

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**Abstract:** In this paper, using Hermite-Hadamard Integral Inequality we give an improvement of the matrix Cauchy-Schwarz inequality which was due to Bhatia and Davis in [5], and also obtain a further improvement of a refined matrix Cauchy-Schwarz inequality which was due to Ali et al. in [2].

**Keywords:** Unitarily invariant norms; Cauchy- Schwarz inequality; Hermite-Hadamard Integral Inequality; Positive semidefinite matrices

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## 1. Introduction

Let  $M_{m,n}$  be the space of  $m \times n$  complex matrices and  $M_n = M_{n,n}$ .  $\|\cdot\|$  denotes the unitarily norm.

For all  $A, B, X \in M_n$ ,  $A, B$  are positive semidefinite, for all real number  $r \geq 0$  and every  $\nu$  with  $0 \leq \nu \leq 1$ , the well-known Cauchy-Schwarz inequality for matrix [5] (see also [6, p. 267]) says that

$$\phi\left(\frac{1}{2}\right) \leq \phi(0) \tag{1.1}$$

where  $\phi(\nu) = \left\| A^\nu X B^{1-\nu} \right\| \cdot \left\| A^{1-\nu} X B^\nu \right\|$ .

Hiai and Zhan [1] showed the following inequality

$$\phi\left(\frac{1}{2}\right) \leq \phi(\nu) \leq \phi(0), \tag{1.2}$$

which is a refinement of (1.1).

In [2], Ali et al. gave another refinement of (1.1) as follows:

$$\phi\left(\frac{1}{2}\right) + 2\left(\int_0^1 \phi(\nu) d\nu - \phi\left(\frac{1}{2}\right)\right) \leq \phi(0). \tag{1.3}$$

In this paper, we present new refinements of (1.1) and (1.3).

## 2. Main Results

**Lemma 2.1**[3. p.21] Let  $f$  be a convex function defined on an interval  $I$ . If  $x, y, z \in I$  such that  $x \leq y \leq z (x \neq z)$  then

$$f(y) - f(x) \leq \frac{f(z) - f(x)}{z - x} (y - x). \tag{2.1}$$

**Lemma 2.2** (Improved Hermite-Hadamard Integral Inequality) [4] Let  $f$  be a convex function defined on an interval  $[a, b]$ . Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{4} \left( f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right) \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

**Lemma 2.3** Let  $f$  be a convex function defined on an interval  $[a, b]$ . Then for positive integers  $m, n$  with  $n \geq m$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{4n} \left( (2n-m)f(a) + 2mf\left(\frac{a+b}{2}\right) + (2n-m)f(b) \right) \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

**Proof.** Since  $f$  is convex on  $[0, 1]$ , we have

$$2f\left(\frac{a+b}{2}\right) \leq f(a) + f(b).$$

Thus

$$\begin{aligned} &\frac{1}{4n} \left( (2n-m)f(a) + 2mf\left(\frac{a+b}{2}\right) + (2n-m)f(b) \right) \\ &\leq \frac{1}{4n} \left( (2n-m)f(a) + mf(a) + mf(b) + (2n-m)f(b) \right) \\ &= \frac{f(a) + f(b)}{2}, \end{aligned}$$

which is the last inequality.

On the other hand, from Lemma 2.2, we have

$$\begin{aligned} &\frac{1}{b-a} \int_a^b f(t) dt \\ &\leq \frac{1}{4} \left( f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right) \\ &= \frac{1}{4n} \left( nf(a) + 2nf\left(\frac{a+b}{2}\right) + nf(b) \right) \\ &= \frac{1}{4n} \left( nf(a) + (2n-2m)f\left(\frac{a+b}{2}\right) + 2mf\left(\frac{a+b}{2}\right) + nf(b) \right) \\ &\leq \frac{1}{4n} \left( nf(a) + (n-m)(f(a) + f(b)) + 2mf\left(\frac{a+b}{2}\right) + nf(b) \right) \\ &\leq \frac{1}{4n} \left( (2n-m)f(a) + 2mf\left(\frac{a+b}{2}\right) + (2n-m)f(b) \right), \end{aligned}$$

which is the middle inequality.

**Remark 2.4** Set  $m = 1$  in Lemma 2.2, then we get Lemma 1 of [7].

**Theorem 2.5** Let  $A, B, X \in M_n$  such that  $A, B$  are positive semidefinite, and let  $r \geq 0, 0 \leq \nu \leq 1$ , and  $0 \leq u \leq 1$ . Then for

positive integers  $m, n$  with  $n \geq m$

$$\begin{aligned} \phi\left(\frac{1}{2}\right) &\leq \frac{1}{|1-2u|} \left| \int_u^{1-u} \phi(v) dv \right| \\ &\leq \frac{1}{2n} ((2n-m)\phi(u) + m\phi\left(\frac{1}{2}\right)) \\ &\leq \phi(0), \end{aligned}$$

where  $\phi(v) = \left\| A^v XB^{1-v} \right\|^r \cdot \left\| A^{1-v} XB^v \right\|^r$ .

**Proof.** Since  $\phi(v) = \left\| A^v XB^{1-v} \right\|^r \cdot \left\| A^{1-v} XB^v \right\|^r$  is convex on  $[0, 1]$ , then for  $0 \leq u \leq \frac{1}{2}$ , by Lemma 2.3, we have

$$\begin{aligned} \phi\left(\frac{1}{2}\right) &= \phi\left(\frac{u+1-u}{2}\right) \\ &\leq \frac{1}{1-2u} \int_u^{1-u} \phi(v) dv \\ &\leq \frac{1}{4n} ((2n-m)\phi(u) + 2m\phi\left(\frac{1}{2}\right) + (2n-m)\phi(1-u)) \\ &= \frac{1}{2n} ((2n-m)\phi(u) + 2m\phi\left(\frac{1}{2}\right)) \\ &\leq \frac{\phi(u) + \phi(1-u)}{2} \leq \phi(0). \end{aligned} \tag{2.2}$$

If  $\frac{1}{2} \leq u \leq 1$ , then

$$\begin{aligned} \phi\left(\frac{1}{2}\right) &= \phi\left(\frac{u+1-u}{2}\right) \\ &\leq \frac{1}{2u-1} \int_{1-u}^u \phi(v) dv \\ &\leq \frac{1}{4n} ((2n-m)\phi(u) + 2m\phi\left(\frac{1}{2}\right) + (2n-m)\phi(1-u)) \\ &= \frac{1}{2n} ((2n-m)\phi(u) + 2m\phi\left(\frac{1}{2}\right)) \\ &\leq \frac{\phi(u) + \phi(1-u)}{2} \leq \phi(0). \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we get the required inequality.

**Theorem 2.6** Let  $A, B, X \in M_n$  such that  $A, B$  are positive semidefinite, and  $r \geq 0, 1 \geq v \geq 0$ . Then

$$\phi\left(\frac{1}{2}\right) + [4 \int_0^1 \phi(v) dv - 2\phi\left(\frac{1}{2}\right) - 2\phi\left(\frac{1}{4}\right)] \leq \phi(0) \tag{2.4}$$

where  $\phi(v) = \left\| A^v XB^{1-v} \right\|^r \cdot \left\| A^{1-v} XB^v \right\|^r$ .

**Proof.** Since  $\phi(v) = \left\| A^v XB^{1-v} \right\|^r \cdot \left\| A^{1-v} XB^v \right\|^r$  is convex on  $[0, 1]$ , then for  $0 \leq v \leq 1$ , by Lemma 2.2, we have

$$\phi\left(\frac{1}{2}\right) \leq \phi(v) \leq \phi(0).$$

If  $0 \leq v \leq \frac{1}{4}$ , by inequality (2.1), then we have

$$\phi(v) - \phi(0) \leq 4v[\phi\left(\frac{1}{4}\right) - \phi(0)].$$

Thus

$$\int_0^{\frac{1}{4}} \phi(v) dv \leq \int_0^{\frac{1}{4}} 4v[\phi\left(\frac{1}{4}\right) - \phi(0)] dv + \frac{1}{4} \phi(0),$$

which is

$$\int_0^{\frac{1}{4}} \phi(v) dv \leq \frac{1}{8} [\phi\left(\frac{1}{4}\right) + \phi(0)]. \tag{2.5}$$

If  $\frac{1}{4} \leq v \leq \frac{1}{2}$ , by inequality (2.1), then we have

$$\phi(v) - \phi\left(\frac{1}{4}\right) \leq 4\left(v - \frac{1}{4}\right) [\phi\left(\frac{1}{2}\right) - \phi\left(\frac{1}{4}\right)].$$

Thus

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \phi(v) dv \leq \int_{\frac{1}{4}}^{\frac{1}{2}} 4\left(v - \frac{1}{4}\right) [\phi\left(\frac{1}{2}\right) - \phi\left(\frac{1}{4}\right)] dv + \frac{1}{4} \phi\left(\frac{1}{4}\right),$$

which is

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \phi(v) dv \leq \frac{1}{8} [\phi\left(\frac{1}{4}\right) + \phi\left(\frac{1}{2}\right)]. \tag{2.6}$$

If  $\frac{1}{2} \leq v \leq \frac{3}{4}$ , by inequality (2.1), then we have

$$\phi(v) - \phi\left(\frac{1}{2}\right) \leq 4\left(v - \frac{1}{2}\right) [\phi\left(\frac{3}{4}\right) - \phi\left(\frac{1}{2}\right)].$$

Thus

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \phi(v) dv \leq \int_{\frac{1}{2}}^{\frac{3}{4}} 4\left(v - \frac{1}{2}\right) [\phi\left(\frac{3}{4}\right) - \phi\left(\frac{1}{2}\right)] dv + \frac{1}{4} \phi\left(\frac{1}{2}\right),$$

which is

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \phi(v) dv \leq \frac{1}{8} [\phi\left(\frac{1}{2}\right) + \phi\left(\frac{3}{4}\right)]. \tag{2.7}$$

If  $\frac{3}{4} \leq v \leq 1$ , by inequality (2.1), then we have

$$\phi(v) - \phi\left(\frac{3}{4}\right) \leq 4\left(v - \frac{1}{4}\right) [\phi(1) - \phi\left(\frac{3}{4}\right)].$$

Thus

$$\int_{\frac{3}{4}}^1 \phi(v) dv \leq \int_{\frac{3}{4}}^1 4\left(v - \frac{1}{4}\right) [\phi(1) - \phi\left(\frac{3}{4}\right)] dv + \frac{1}{4} \phi\left(\frac{3}{4}\right),$$

which is

$$\int_{\frac{3}{4}}^1 \phi(v) dv \leq \frac{1}{8} [\phi\left(\frac{3}{4}\right) + \phi(1)]. \tag{2.8}$$

Form inequalities (2.5)–(2.8) and  $\phi\left(\frac{1}{4}\right) = \phi\left(\frac{3}{4}\right), \phi(1) = \phi(0)$ , we have

$$4 \int_0^1 \phi(v) dv \leq \phi(0) + \phi\left(\frac{1}{2}\right) + 2\phi\left(\frac{1}{4}\right),$$

which is equivalent to

$$\phi\left(\frac{1}{2}\right) + [4 \int_0^1 \phi(v) dv - 2\phi\left(\frac{1}{2}\right) - 2\phi\left(\frac{1}{4}\right)] \leq \phi(0).$$

**Remark 2.5** Inequality (2.4) is better than (1.3). In fact, by the convexity of  $\phi$ , we have

$$\phi\left(\frac{1}{4}\right) \leq 2 \int_0^{\frac{1}{2}} \phi(v) dv$$

and

$$\phi\left(\frac{1}{4}\right) = \phi\left(\frac{3}{4}\right) \leq 2 \int_{\frac{1}{2}}^1 \phi(v) dv.$$

Thus

$$\int_0^1 \phi(v)dv - \phi\left(\frac{1}{4}\right) \geq 0$$

and

$$\begin{aligned} & 4\int_0^1 \phi(v)dv - 2\phi\left(\frac{1}{2}\right) - 2\phi\left(\frac{1}{4}\right) - 2\left[\int_0^1 \phi(v)dv - \phi\left(\frac{1}{2}\right)\right] \\ & = 2\int_0^1 \phi(v)dv - 2\phi\left(\frac{1}{4}\right) \geq 0. \end{aligned}$$

Thus inequality (2.4) is an improvement of (1.3).

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