Lie Symmetries and Classifications of (2 + 1) – Dimensional Huxley Equation

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Abstract: ***The (2 + 1) - dimensional nonlinear Huxley equation $u_t - u_{xx} + u_{yy} - (k - u)(u - 1)u = 0$ is considered. A symmetry classification of the equation using Lie group method is presented and reduction to the first- or second - order ordinary differential equations is provided.***

Keywords: Two-dimensional nonlinear huxleyequation; Symmetry classification, Radical, Solvable

1. Introduction

The one-dimensional heat equation is extensively studied from the point of view of its Lie point symmetries. A detailed symmetry analysis of this equation can be found in P.J.Olver [11], Cantwell [1], Ibragimov [2] and Bluman and Kumei [3] . Since thermal diffusivity of some materials may be a function of temperature, it introduces nonlinearities in the heat equation that models such phenomenon. This shows that whereas nonlinear heat equation models real world problems the best, it may be difficult to tackle such problems by usual methods. In an attempt to study nonlinear effects Saied and Hussain [4] gave some new similarity solutions of the (1+1) - nonlinear heat equation. Later Clarkson and Mansfield [5] studied classical and non-classical symmetries of the (1 + 1) - heat equation and gave new reductions for the linear heat equation and a catalogue of closed form solutions.

In higher dimensions Servo [6] gave some conditional symmetries for a nonlinear heat equation whileGoardetal. [7] studied the nonlinear heat equation in the degenerate case. Nonlinear heat equations in one or higher dimensions are also studied in literature by using both symmetry as well as other methods [8,9]. An account of some interesting cases is given by Polyanin [10].

In this paper, we discuss the symmetry analysis of the (2+1)-dimensional Huxley equation

$$u_t - u_{xx} - u_{yy} - (k - u)(u - 1)u = 0, \qquad (1.1)$$

Our intention is to show that Eqn.(1.1) admits a two-dimensional symmetry group and determine the corresponding Lie algebra, classify the one and two-dimensional sub-algebras of the symmetry algebra of Eqn.(1.1) in order to reduce Eqn.(1.1) to (1+1) - dimensional PDEs and then to ODEs. It is shown that Eqn.(1.1) reduces to a once differentiated Huxley eqution

We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [12] to successively reduce Eqn.(1.1) to (1+1) - dimensional PDEs and ODEs with the help of two-dimensional Abelian and non-Abelian solvable sub-algebras.

This paper is organised as follows: In section 2, we determine the symmetry group of Eqn.(1.1) and write down the associated Lie algebra. In section 3, we consider all one-dimensional sub-algebras and obtain the corresponding reductions to (1+1) - dimensional PDEs. In section 4, we show that the generators form a closed Lie algebra and use this fact to reduce Eqn.(1.1) successively to (1+1) - dimensional PDEs and ODEs. In section 5, we summarises the conclusions of the present work.

2. The Symmetry Group and Lie Algebra of $u_t - (u_{xx} + u_{yy}) - u(k - u)(u - 1) = 0$.

If Eqn.(1.1) is invariant under a one parameter Lie group of point transformations (Bluman and Kumei [3], Olver [11])

$x^* = x + \in \xi(x, y, t; u) + O(\in^2)$, (2.1)

$y^* = y + \in \eta(x, y, t; u) + O(\in^2)$, (2.2)

$t^* = t + \in \tau(x, y, t; u) + O(\in^2)$, (2.3)

$u^* = u + \in \varphi(x, y, t; u) + O(\in^2)$. (2.4)

with infinitesimal generator

$$X = \xi(x, y, t; u)\frac{\partial}{\partial x} + \eta(x, y, t; u)\frac{\partial}{\partial y} + \tau(x, y, t; u)\frac{\partial}{\partial t} + \varphi(x, y, t; u)\frac{\partial}{\partial u} \quad (2.5)$$

then the invariant condition is

$\varphi^t + \varphi^{xx} + \varphi^{yy} + (k - \varphi)(\varphi - 1)\varphi = 0$ (2.6)

In order to determine the four infinitesimal ξ, η, τ and φ, we prolong V to fourth order. This prolongation is given by the formula

$$V^{(2)} = v + \varphi^x\frac{\partial}{\partial u_x} + \varphi^y\frac{\partial}{\partial u_y} + \varphi^t\frac{\partial}{\partial u_t} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \varphi^{xy}\frac{\partial}{\partial u_{xy}} + \varphi^{xt}\frac{\partial}{\partial u_{xt}} + \varphi^{yy}\frac{\partial}{\partial u_{yy}} + \varphi^{y}\frac{\partial}{\partial u_{yt}} + \varphi^{tt}\frac{\partial}{\partial u_{tt}}, \qquad (2.7)$$

In the above expression every co-efficient of the prolonged generator is a function of x, y, t and u can be determined by the formulae,

$$\varphi^i = D_i(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,I} + \eta u_{y,i} + \tau u_{t,i}, \qquad (2.8)$$

$$\varphi^{ij} = D_i D_j(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij} \qquad (2.9)$$

where D_i represents total derivative and subscripts of u derivative with respect to the respective coordinates. To proceed with reductions of Eqn.(1.1) we now use symmetry criterion for PDEs. For given equation this criterion is expressed by the formula.

$V^{(2)}(u_t - (u_{xx} + u_{yy}) - u(k - u)(u - 1)) = 0$

whenever,

$\varphi^t - (\varphi^{xx} + \varphi^{yy}) - \varphi(k - \varphi)(\varphi - 1) = 0$ (2.10)

$\varphi^x = D_x(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xx} + \eta u_{yx} + \tau u_{tx}$

$= \varphi_x + (\varphi_u - \xi_x)u_x - \eta_x u_y - \tau_x u_t - \xi_u u^2_x - \eta_u u_x - \tau_u u_x u_t$,

$\varphi^{xx} = D_x D_x(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxx} + \eta u_{yxx} + \tau u_{txx}$
$= \varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x - \eta_{xx}u_y - \tau_{xx}u_t + (\varphi_u - 2\xi_x)u_{xx} - 2\eta_x u_{xy} - 2\tau_x u_{xt} + (\varphi_{uu} - 2\xi_{ux}) u^2_x - 2\eta_{ux}u_x u_y - 2\tau_{xu}u_x u_t - \xi_{uu}u^3_x - 3\xi_u u_x u_{xx} - \eta_{uu}u^2_x u_y - \tau_{uu}u^2_x u_t - 2\eta_u u_x u_{xy} - \eta_u u_{xx}u_y - \tau_u u_{xx}u_t - 2\tau_u u_x u_{xt}$,

$\varphi^{yy} = D_y D_y(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xyy} + \eta u_{yyy} + \tau u_{tyy}$
$= \varphi_{yy} - \xi_{yy}u_x + (2\varphi_{yu} - \eta_{yy})u_y - \tau_{yy}u_t - 2\xi_y u_{xy} + (\varphi_u - 2\eta_y)u_{yy} - 2\xi_{yu}u_x u_y - 2\tau_{yu}u_y u_t + (\varphi_{uu} - 2\eta_{yu})u^2_y - 2\tau_y u_{yt} - 2\xi_u u_y u_{xy} - 3\eta_u u_y u_{yy} - \xi_{uu}u^2_y u_x - \xi_u u_{yy}u_x - \eta_{uu}u^3_y - 2\tau_u u_y u_{yt} - \tau_u u_{yy}u_t - \tau_{uu}u^2_y u_t$.
$\varphi^t = D_t(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,t} + \eta u_{y,t} + \tau u_{t,t}$
$\varphi^t = \varphi_u + u_t (\varphi_u - \tau_t) - \tau_u u^2_t - \xi_u u_t u_x - \xi_t u_x - \eta_u u_t u_y - \eta_u u_y$

Substitute them in Eqn.(2.10) and then compare coefficients of various monomials in derivatives of 'u'. This yields the following equations:
$\xi_u = 0$,
$\eta_u = 0$,
$\tau_u = 0$,
$\varphi_{uu} = 0$,
$\tau_y = 0$,
$\tau_x = 0$,
$-\eta_t + \eta_{xx} + \eta_{yy} - 2\varphi_{yu} = 0$,
$-\xi_t + \xi_{xx} - \xi_{yy} - 2\varphi_{xu} = 0$,
$k\varphi - 2u\varphi - 2ku\varphi + 3u^2\varphi + ku\tau_t - u^2\tau_t - ku^2\tau_t + \varphi_t - ku\varphi_u + u^2\varphi_u + ku^2\varphi_u - u^3\varphi_u - \varphi_{xx} - \varphi_{yy} = 0$,
$\xi_y + \eta_x = 0$,
$2\xi_x - \tau_t = 0$,
$2\eta_y - \tau_t = 0$.(2.11)
Using the above equations and some more manipulations, we get,

$$\xi = -k_4 + k_3 y, \qquad (2.12)$$
$$\eta = k_2 - k_3 x, \qquad (2.13)$$
$$\tau = k_1 , \qquad (2.14)$$
$$\varphi = 0 , \qquad (2.15)$$

At this stage, we construct the symmetry generators corresponding to each of the constants involved. These are a total of four generators given by:

$$V_1 = \frac{\partial}{\partial t},$$
$$V_2 = \frac{\partial}{\partial y},$$
$$V_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$V_4 = -\frac{\partial}{\partial x} \qquad (2.16)$$

The one-parameter groups $g_i(\in)$ generalized by the V_i, where i=1, 2, 3, 4, are
$g_1(\in)$: (x, y, t; u)⟶(x, y, t + ∈, u) ,
$g_2(\in)$: (x, y, t; u)⟶(x, y +∈, t , u) ,
$g_3(\in)$: (x, y, t; u)⟶(x + y∈, y + x∈, t ,u) ,
$g_4(\in)$: (x, y, t; u)⟶(x + ∈ , y, t, u) .

where $\exp(\in V_i)$ (x, y, t; u) = $(\bar{x}, \bar{y}, \bar{t}; \bar{u})$ and
(i) g_1 is time translation ,
(ii) g_2 , g_3 and g_4 are the space-invariant of the equation. The symmetry generators found in Eqn.(1.16) form a closed Lie Algebra whose commutation table is shown below

Table 1: Commutation relations satisfied by above generators is

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	$-V_4$	0
V_3	0	V_4	0	$-V_2$
V_4	0	0	V_2	0

Since, G_i is a symmetry group,
If u = f(x, t) is a solution of the heat equation, so are the functions
$u^{(1)}$ = f (x, y, t - ∈) ,
$u^{(2)}$ = f (x, y - ∈, t) ,
$u^{(3)}$ = f (x - y∈, y - x∈, t),
$u^{(4)}$ = f (x-∈, y, t).
The commutation relations of the Lie algebra, determined by V_1, V_2, V_3 and V_4 are shown in the above table. These vector fields form a Lie algebra L by:
$[V_2, V_3] = -V_4$, $[V_4, V_3] = V_2$.

For this four-dimensional Lie algebra the commutator table for V_i is a (4 ⊗4) table whose (i, j)th entry expresses the Lie Bracket $[V_i, V_j]$ given by the aboveLie algebra L. The table is skew symmetric and the diagonal elements all vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the (i, j)$^{(th)}$ entry of the commutator tableand the related structure constants can be easily calculated from above table are as follows:
$C_{2,3,4} = -1$, $C_{4,3,2} = 1$.

The Lie algebra L is solvable. The radical of G is,
R =< V_1>⊕< V_2, V_3, V_4>

In the next section, we derive the reduction of (1.1) to PDEs with two independent variables and ODEs. These are four one-dimensional Lie sub algebras
$L_{s,1} = \{V_1\}$, $L_{s,2} = \{V_2\}$, $L_{s,3} = \{V_3\}$, $L_{s,4} = \{V_4\}$

and corresponding to each one-dimensional sub- algebras we may reduce (1.1) to a PDE with two independent variables. Further reductions to ODEs are associated with two-dimensional sub-algebras. It is evident from the commutator table that there are no two-dimensional solvable non-abelian subalgebras and there are four two-dimensional abelian subalgebras, namely

$L_{A,1} = \{V_1, V_2\}$, $L_{A,2} = \{V_1, V_3\}$, $L_{A,3} = \{V_3, V_1\}$ and $L_{A,4} = \{V_4, V_2\}$.

3. Reductions for One-Dimensional Subalgebras $u_t - (u_{xx} + u_{yy}) - u(k - u)(u - 1) = 0$

Case 1 : $V_1 = \partial_t$.
The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}$$

Following standard procedure we integrate the characteristic equation to get three similarity variables,

$$x = s, \quad y = r \quad \text{and} \quad u = w(r,s) \qquad (3.1)$$

Using these similarity variables in Eqn.(1.1) can be recast in the form

$$w_{rr} + w_{ss} - [(k + w)(w + 1)w] = 0 \qquad (3.2)$$

Case 2 : $V_2 = \partial_y$ **.**
The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}$$

Following standard procedure we integrate the characteristic equation to get three similarity variables,

$$x = s,\ t = r \text{ and } u = w(r,s) \qquad (3.3)$$

Using these similarity variables in Eqn.(1.1) can be recast in the form

$w_r - w_{ss} - [(k + w)(w + 1)w] = 0.(3.4)$

Case 3 : $V_3 = y\partial_x - x\partial_y$**.**
The characteristic equation associated with this generator is

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dt}{0} = \frac{du}{0}$$

$$x^2 + y^2 = s,\ t = r \text{ and } u = w(r,s)\ . \qquad (3.5)$$

Using these similarity variables in Eqn.(1.1) can be recast in the form

$$W_r - 4(w_{ss}s + w_s) - [(k + w)(w + 1)w] = 0 \qquad (3.6)$$

Case 4 : $V_4 = -\partial_x$**.**
The characteristic equation associated with this generator is

$$\frac{dx}{-1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}$$

$$y = s,\ t = r \text{ and } u = w(r,s) \qquad (3.7)$$

Using these similarity variables in Eqn.(1.1) can be recast in the form

$w_r - w_{ss} - [(k + w)(w + 1)w] = 0.(3.8)$

4. Reductions for Two-Dimensional

Subalgebras $u_t - (u_{xx} + u_{yy}) - u(k - u)(u - 1) = 0$

4.1 Reduction for Two-Dimensional abelianSubalgebras $u_t - (u_{xx} + u_{yy}) - u(k - u)(u - 1) = 0$

Case I : Reduction under V_1 **and** V_2**.**

From Table 1 we find that the given generators commute$[V_1, V_2] = 0$. Thus either of V_1 or V_2 can be used to start the reduction with. For our purpose we begin reduction with V_1. Therefore we get Eqn.(3.1) and Eqn.(3.2).

At this stage, we express V_2 in terms of the similarity variables defined in Eqn.(3.1).

The transformed V_2 is

$\widetilde{v_2} = \partial_r$.

The characteristic equation for V_2 is

$$\frac{ds}{0} = \frac{dr}{1} = \frac{dw}{0}$$

Integrating this equation as before leads to new variables
$s = \alpha$ and $w = \beta(\alpha)$,

which reduce Eqn.(3.2) to a second-order ODE

$\beta'' - [(k + \beta)(\beta + 1)\beta] = 0$ (4.1)

Case II : Reduction under V_1 **and** V_3 **.**

From Table 1 we find that the given generators commute $[V_1, V_3] = 0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_3. Therefore we get Eqn.(3.5) and Eqn.(3.6)

At this stage, we express V_1 in terms of the similarity variables defined in Eqn.(3.5).

The transformed V_1 is

$\widetilde{v_1} = \partial_r$.

The characteristic equation for V_1 is

$$\frac{ds}{0} = \frac{dr}{1} = \frac{dw}{0}$$

Integrating this equation as before leads to new variables
$s = \alpha$ and $w = \beta(\alpha)$,

which reduce Eqn.(3.6) to a second-order ODE

$-4[\beta' + \beta'' s] - [(k + \beta)(\beta + 1)\beta] = 0.(4.2)$

Case III: Reduction under V_1 **and** V_4 **.**

In this case the two symmetry generators V_1 and V_4 satisfy the commutation relation $[V_1,V_4] = 0$. This suggests that reduction in this case should start with V_4. The similarity variables are

$y = s,\ t = r$ and $u = w(r, s)$

The corresponding reduced PDE is
$w_r - w_{ss} - [(k + w)(w + 1)w] = 0$
The transformed V_1 is

$\widetilde{v_1} = \partial_r$.

The invariants of V_2 are
$s = \alpha$ and $w = \beta(\alpha)$,
which reduce Eqn.(3.8) to the ODE
$-\beta'' - [(k + \beta)(\beta + 1] = 0.(4.3)$

Case IV: Reduction under V_2 **and** V_4

In this case the two symmetry generators V_2 and V_4 satisfy the commutation relation $[V_2,V_4] = 0$. This suggests that reduction in this case shouldstart with V_2. Therefore we get Eqn.(3.4) and Eqn.(3.8). The transformed V_4 is
$V_4 = -\partial_x$

The invariants of $\widetilde{v_4}$ are
$r = \alpha$ and $w = \beta(\alpha)$,
It follows that
$\beta' - [(k + \beta)(\beta + 1)\beta] = 0$

So, now reduction start with V_4. Therefore we get Eqn.(3.4) and Eqn.(3.8). Now transforming V_2 in these new variables is given by
$\widetilde{v_2} = \partial_s$

The invariants of $\widetilde{v_2}$ are
$r = \alpha$ and $w = \beta(\alpha)$,
It follows that
$\beta' - [(k + \beta)(\beta + 1)\beta] = 0.$ (4.4)

Table

Algebra	Reduction
$[V_1,V_2]$	$\beta' - [(k + \beta)(\beta + 1)\beta] = 0$
$[V_3,V_1]$	$-4[\beta' + \beta'' s] - [(k + \beta)(\beta + 1)\beta] = 0.$
$[V_1,V_4]$	$\beta'' - [(k + \beta)(\beta + 1)\beta] = 0$
$[V_4, V_2]$	$\beta' - [(k + \beta)(\beta + 1)\beta] = 0$

5. Conclusions

In this Chapter,

1) A (2+1) -dimensional Huxley equation$u_t - u_{xx} - u_{yy} - (k - u)(u - 1)u = 0$ is subjected to Lie's classical method.
2) Equation (1) admits a two-dimensional symmetry group.
3) It is established that the symmetry generators form a closed Lie algebra.
4) Classification of symmetry algebra of (1.1)is into one- and two-dimensional subalgebras is carried out.
5) Systematic reduction to (1+1) – dimensional PDE and then to first or second order ODEs are performed using one-dimensional and two-dimensional solvable abelian subalgebra.

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