Gluon Density Function in Leading Order by the Method of Characteristics

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Abstract: ***The application of method of characteristics in perturbative quantum chromodynamics (pQCD) is relatively new. In the present paper, we solve Dokshitzer, Gribov, Lipatov, Altarelli, Parisi (DGLAP) equations in Leading Order (LO) by using this method and obtain an analytical form of gluon density function at small-x. Comparison with exact results as well as with data are reported.***

Keywords: Gluon density function; method of characteristics; DGLAP equation; small-x physics.

1. Introduction

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi (DGLAP) evolution equations [1]–[4] have been playing very important role in understanding the dynamics of evolutions of quark and gluons. Several approximate and numerical solutions of DGLAP evolution equations are available in literature [5]–[8], but their exact analytical solutions are not known [9], [10]. Because these evolution equations are partial differential equations (PDE), their ordinary solutions are not unique solutions, rather a range of solutions. Moreover, they are based on an *ad-hoc* assumption of factorizability of *x* and *t* dependence of the gluon distribution *G(x,t)*. These limitations can be over come by the use of Method of Characteristics [16].

The application of method of characteristics in perturbative quantum chromo-dynamics (pQCD), specially in the solution of DGLAP equations is relatively new. Some of these applications are available in recent literatures [13] ,[14] with considerable phenomenological success. In this paper, we solve DGLAP equations in leading order (LO) by using method of characteristics and obtain an analytical form of gluon number density function at small-*x* which is free from the above mentioned limitations.

2. Formalism

The DGLAP equations for gluon distribution have the standard form [1]–[4] :

$$t\frac{\partial G(x,t)}{\partial t}=\frac{3\alpha_s(t)}{\pi}\times\left[\left\{\frac{11}{12}-\frac{n_f}{18}+\ln(1-x)\right\}G(x,t)+\int_x^1 dz\frac{zG\left(\frac{x}{z},t\right)-G(x,t)}{(1-x)}+\int_x^1 dz\left\{z(1-z)+\frac{(1-z)}{z}\right\}G\left(\frac{x}{z},t\right)+\frac{2}{9}\int_x^1\frac{1+(1-z)^2}{z}F_2^s\left(\frac{x}{z},t\right)\right] \quad (1)$$

where $t=\ln\left(\frac{Q^2}{\Lambda^2}\right)$, $\alpha_s(t)=\frac{4\pi}{\beta_0}$, $\beta_0=11\text{-}\frac{2}{3}\,n_f$, n_f being the number of flavours.

To evaluate the integrals of eq.(1), we introduce a variable u [8], [15] as $u=1-z$. Since $x<z<1$, so $0<u<1-x$, x/z can be approximated at small-*x* as $x/z=x(1-u)^{-1}\approx x(1+u)=x+xu$ and hence Taylor's expansion of $F_2^s(x/z,t)$ and $G(x/z,t)$ in approximated form [11], [12] at small-*x* can be given by :

$$F_2^s\left(\frac{x}{z},t\right)\approx F_2^s(x,t)+xu\frac{\partial F_2^s(x,t)}{\partial x}$$

and

$$G\left(\frac{x}{z},t\right)\approx G(x,t)+xu\frac{\partial G(x,t)}{\partial x} \quad (2)$$

Since *x* is small, terms containing x^2 and higher powers of *x* are neglected. Using eq.(2) in eq.(1) and performing the integrations w.r.t. *z* ,

$$t\frac{\partial G(x,t)}{\partial t}=P(x)G(x,t)+Q(x)\frac{\partial G(x,t)}{\partial x}+R(x)F_2^s(x,t)+S(x,t)\frac{\partial F_2^s}{\partial x} \quad (1)$$

where at small-*x*,

$$P(x)=\frac{12}{\beta_0}\left\{\frac{\beta_0}{12}+\ln(1/x)-\frac{1}{6}(11-12x)\right\}$$
$$Q(x)=\frac{11}{\beta_0}x \qquad (4)$$
$$R(x)=\frac{4}{3\beta_0}\{4\ln(1/x)+(4x-3)\}$$
$$S(x)=-\frac{68}{9\beta_0}x$$

A reasonable approximate relationship between $F_2^S(x,t)$ and $G(x,t)$, representing the relative strength of gluon to singlet distribution, can be taken as $F_2^S(x,t)=kG(x,t)$, where k is a suitable function of x or may be a constant [7], [14]. For simplicity and well adaptation to method of characteristics, k is considered here as a constant with $0<k<1$, since gluon distribution is always higher than singlet distributions at any Q^2. Using this relationship in eq.(3),

$$J(x)\frac{\partial G(x,t)}{\partial x}-t\frac{\partial G(x,t)}{\partial t}+H(x)G(x,t)=0 \qquad (5)$$

where,

$H(x)=P(x)+kR(x)$ and $J(x)=Q(x)+kS(x)$.

Eq.(5) is a first order PDE, which can be solved by Method of Characteristics.

To use method of characteristics, let us introduce two variables S and τ as :

$$\frac{dx}{dS}=J(x) \qquad (6)$$

and

$$\frac{dt}{dS}=-t \qquad (7)$$

Use of eq.(6) and (7) in eq.(5) gives,

$$\frac{dG}{dS}+U(S,\tau)G(S,\tau)=0 \qquad (8)$$

which is an ODE in new coordinates *(S,τ)*. Here, $U(S,\tau)=H(x)$. Solving eq.(6) and (7) with the boundary condition, at *S=0*, $t=t_0$ and $x=\tau$, we get the transformation equations as:

$$S=\ln\left(\frac{t_0}{t}\right) \qquad (9)$$

and

$$\tau=x\left(\frac{t}{t_0}\right)^{\frac{1}{\beta_0}\left(11-\frac{68}{9}k\right)} \qquad (10)$$

Expressing ODE (8) in terms of S and τ, integrating and transforming the resultant equation back to the original variables *(x,t)* with the help of transformation eq.(9) and (10),

$$G(x,t)=G(x,t_0)\times \exp\left[\begin{array}{l}\frac{4\left(\frac{4k}{3}+6\right)}{\left(11-\frac{68}{9}k\right)}x\left(\frac{t}{t_0}\right)^{\frac{1}{\beta_0}\left(11-\frac{68}{9}k\right)}\times\left\{1-\left(\frac{t}{t_0}\right)^{\frac{1}{\beta_0}\left(11-\frac{68}{9}k\right)}\right\}\\ -\frac{4}{\beta_0}\left(\frac{\beta_0}{4}+k-\frac{11}{2}\right)\ln\left(\frac{t_0}{t}\right)-\\ \frac{1}{2\beta_0^2}\left(\frac{4k}{3}+3\right)\left(\frac{68}{9}k-11\right)\times\left\{\ln\left(\frac{t_0}{t}\right)\right\}^2\\ +\frac{4}{\beta_0}\ln\left\{x\left(\frac{t_0}{t}\right)^{\frac{1}{\beta_0}\left(11-\frac{68}{9}k\right)}\right\}\left(\frac{4k}{3}+3\right)\ln\left(\frac{t_0}{t}\right)\end{array}\right] \qquad (11)$$

where $G(x,t_0)=G(S,\tau)=G(\tau)$ is the input function obtained from the boundary condition, at *S=0*, $t=t_0$ and $x=\tau$.

The gluon number density function is given by,

$$G_N(x,t)=\frac{G(x,t_0)}{\frac{4}{3}\pi R_N^3}\times \exp\left[\begin{array}{l}\frac{4\left(\frac{4k}{3}+6\right)}{\left(11-\frac{68}{9}k\right)}x\left(\frac{t}{t_0}\right)^{\frac{1}{\beta_0}\left(11-\frac{68}{9}k\right)}\times\left\{1-\left(\frac{t}{t_0}\right)^{\frac{1}{\beta_0}\left(11-\frac{68}{9}k\right)}\right\}\\ -\frac{4}{\beta_0}\left(\frac{\beta_0}{4}+k-\frac{11}{2}\right)\ln\left(\frac{t_0}{t}\right)\\ -\frac{1}{2\beta_0^2}\left(\frac{4k}{3}+3\right)\left(\frac{68}{9}k-11\right)\times\left\{\ln\left(\frac{t_0}{t}\right)\right\}^2\\ +\frac{4}{\beta_0}\ln\left\{x\left(\frac{t_0}{t}\right)^{\frac{1}{\beta_0}\left(11-\frac{68}{9}k\right)}\right\}\left(\frac{4k}{3}+3\right)\ln\left(\frac{t_0}{t}\right)\end{array}\right] \qquad (12)$$

where R_N is the target nucleon/nuclear radius.

3. Results and Discussions

For Quantitative analysis, we use MRST 2001LO input [8] given by the formula $xg=3.08x^{0.1}(1-x)^{6.49}(1-2.96x^{0.5}+9.26x)$ and considered Q^2_0=4 Gev^2, QCD cutoff parameter Λ=220 Mev and n_f=4 [17]. The best fit value of k (0.01) obtained through least-square method of curve fitting is considered. Also, we have used R_N=5 GeV^{-2} [18].

The numerical analysis shows that the gluon number density in LO increases with decrease in x at a representative Q^2=100 GeV^2 as shown in fig.1. This increase is in accordance with the MRST exact results.

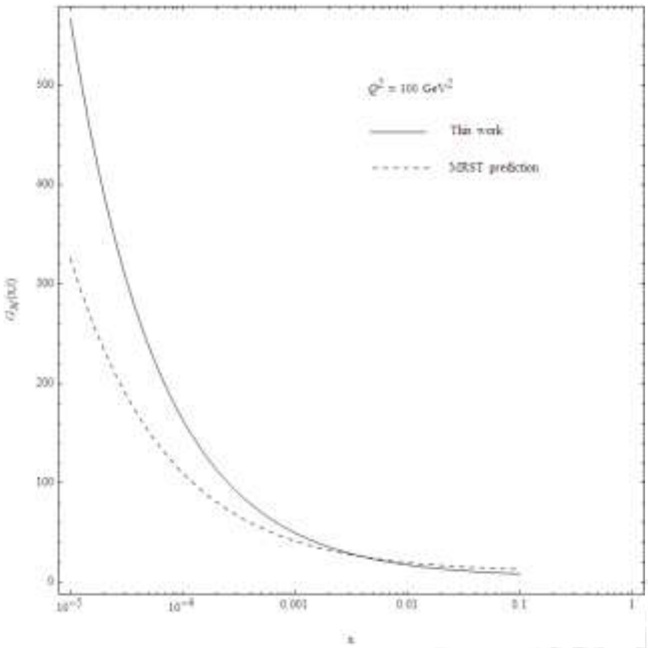

Figure 1: Predicted gluon density function at a representative value of Q^2, i.e., at Q^2=100 GeV^2 and its comparison with MRST 2001LO exact results.

4. Conclusion

In this paper, we have applied the method of characteristics to solve DGLAP equations in LO for gluon distribution functions without any *ad-hoc* assumption of factorizability of x and t dependence and obtained a form of gluon number density function $G_N(x,t)$. Predicted result shows that gluon density increase with decreasing Bjorken-x which is in accordance with the MRST exact results.

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