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S-nearly Semiprime Submodules and Some Related Concept

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Abstract: ***Let R be a commutative ring with unity and N be a submodule of anon zero left R-moduleM ,N is called S −nearly semiprime submodule if whenever $f^2(m) \in N$, for some $f \in End(M)$ and m ∈ M, then $f(m) \in N$, we mean f^2 is $f \circ f$.We will give many results of this type of submodules. In this search, we introduce and study the concepts of S-nearly semiprime submodules and some other related notions .We will study the relationship between the properties of S-nearly semiprime submodules and classes of the other module***

Keywords: semiprime submodule, nearly semiprime submodule $S-$ semiprime submodule,$S-$nearly semiprime submodule, projective and multiplication module.

1. Introduction

A submodule of an R- module M which Dauns was named semiprime submodules that they are generalized of semiprime ideals ,which get big importance at last year, many studies and searches are published about semiprime submodules by many people who care with the subject of commutative algebra.Let N be a proper submodule of an R-module M, then N is an S-semiprime submodule of M if and only if whenever f^n (m) ∈ N, for some f ∈ End(M), m ∈ M and for n ≥ 2, then f(m) ∈ N.

2. Preliminaries

Let R be a commutative ring with identity and M be a non-zero unitary left –module.N is called $S-$ semiprime submodule of M if whenever $f^2(m) \in N$, for some $f \in End(M)$, $m \in M$, then$f(m) \in N$, we mean f^2 is $f \circ f$.We know that Jacobson radical of$M$ (for short $J(M)$)is defined by the intersection of all maximal submodules of M

3. S-Nearly Semiprime Submodules

Recall that a proper submodule N of an $R-module\ M$ is said to be$S-$ semiprime submodule of M if whenever $f^2(m) \in N$, for some $f \in End(M)$ and $m \in N$, then $f(m) \in N$, we mean f^2 is $f \circ f$, [1].

Remark(2.1):Let N be a proper submodule N of an $R-module\ M$, then N is an $S-$ semiprime submodule of M iff whenever $f^n(m) \in N$, for some $f \in End(M)$, $m \in N$ and $n \geq 2$, then $f(m) \in N$, [1].

In this section we introduce the following:

Definition(2.2): A proper submodule N of an $R-module\ M$ is said to be $S-$ Nearly semiprime submodule of M if whenever $f^n(m) \in N$, for some $f \in End(M)$ and $m \in N$, $n \in Z^+$, then $f(m) \in N + J(M)$ where $J(M)$ is the Jacobson radical of M.

Remarks and examples(2.3):

(1) Every $S-$ Nearly semiprime submodule of an $R-module\ M$ isNearly semiprime submodule of M.

Proof: Let N be an $S-$ Nearly semiprime submodule of M and$r^k \in N$, $r \in R, m \in M, k \in Z^+$, we want to show that $r\,m \in N + J(M)$.Define $f: M \longrightarrow M$ by $f(m) = r\,m$, for some $m \in M, r \in R, f \in End(M)$

Since $f^k(m) = r^k\ m \in N$, $k \in Z^+$, but N is $S-$ Nearly semiprimesubmodule of M, then$f(m) = r\,m \in N + J(M)$ where $J(M)$ is theJacobson radical of M, but the converse is not true ingeneral for example

Let $M = Z \oplus Z$ as $Z-$module and $N = 6Z \oplus Z$, it is clearly N is Nearly semiprime submodule of M

Now , define $f: M \longrightarrow M$ by $f(n,m) = (m, 2n)$, for all $n, m \in Z$. clearly $f \in End(M)$.Now $f(0,3) = (3,0) \notin N + J(M)$, but $f^k(0,3) \in N$where $J(M) = (0,0)$.

(2) Every $S-$ semiprime submodule of an $R-$module M is an$S-$Nearly semiprime submodule of M, but the converse is not true ingeneral for example. Let $M = Z_8$ as $Z-$module and $N = \langle\bar{4}\rangle$

Define $f: M \longrightarrow M$ by $f(m) = 2m$, for all $m \in M$, clearly $f \in End(M)$.

Now, $f^2(m) \in N, for\ all\ m \in M$, but $f(m) \notin N$. Then N is not $S-$ semiprime submodule, but$f(m) \in N + J(M)$, then N is $S-$Nearlysemiprime submodule of M.

Proposition(2.4)
Let R be a good ring and N be an $S-$Nearlysemiprime submodule of an $R-$module M , K be any proper submoduleof M such that $K \not\subseteq N$ and $f(K) \subseteq K$, then $N \cap K$ is an $S-$Nearly semiprime submodule of K.

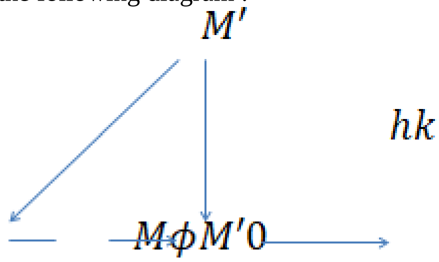
Proof: Let $m \in K$ and $f^n(m) \in N \cap K$, for some $n \in Z^+$. We want toshow that $f(m) \in (N \cap K) + J(K)$. Since $f^n(m) \in N \cap K \subseteq N$, but $N$is $S-$Nearly semiprime submodule of M, then $f(m) \in N + J(M)$ and $m \in K$, then $f(m) \in f(K) \subseteq K$, then $f(m) \in (N + J(M)) \cap K$, thus $f(m) \in (N \cap K) + J(K)$. Which implies that $N \cap K$ is an S −Nearly semiprime submodule of K.

Compare the following with Proposition(2,2,12) [1]

Proposition(2.5): Let $\phi: M \longrightarrow M'$ be an epimorphism ; M and M' be two R −modules. If N is an S −Nearly semiprime submodule of M and $ker\phi \subseteq N$, then $\phi(N)$ is S −Nearly semiprime submodule of M', whenever M' is a projective module.

Proof: We must prove that $\phi(N)$ is a proper submodule of M' .Suppose that $\phi(N) = M'$, but ϕ is an epimorphism, then $\phi(N) = \phi(M)$, then $M = ker\phi + N$, but $ker\phi \subseteq N$, thus $N = M$ (contradiction).

Now, let $h^n(m') \in \phi(N)$; $h \in End(M')$, $m \in M'$, $n \in Z^+$.We must prove that $h(m') \in \phi(N) + J(M')$. Since ϕ is an epimorphism and $m' \in M'$, then there exists $m \in M$ such that $\phi(m) = m'$.
Consider the following diagram :



Since M' is a projective module , then there exists a homorrphism k such that $\phi \circ k = h$

Now, $h^n(m) = (h \circ h \circ h \ldots \ldots h)(m') \in \phi(N)$, but $h = \phi \circ k$, then $(\phi \circ k \circ \phi \circ k \ldots \ldots \phi \circ k)(m') = (\phi \circ k \circ \phi \circ k \ldots \ldots \phi \circ k)\left(\phi(m)\right)$, then $\phi\left((k \circ \phi \ldots \ldots k \circ \phi m \in \phi N\right.$ where $ker\phi \subseteq N$, thus

$(k \circ \phi)^n(m) \in N$, but N is an S −Nearly semiprime submodule of M, then $(k \circ \phi)(m) \in N + J(M)$, then $(\phi \circ k)(m') \in \phi(N) + \phi\,(J(M))$,then $h(m') \in \phi(N) + \phi\,(J(M)) \subseteq (N) + J(M')$. Which implies that
$\phi(N)$ is S −Nearly semiprime submodule of M' .

Corollary(2.6): Let N and K be two submodules of an R −module M such that $K \subseteq N$. If N is an S −Nearly semiprime submodule of M, then $\frac{N}{K}$ an S −Nearly semiprime submodule of $\frac{M}{K}$, whenever $\frac{M}{K}$ is a projective module.

Corollary(2.7): Let $\phi: M \longrightarrow M'$ be an isomorphism ; M and M' be two R −modules. If N is an S −Nearly semiprime submodule of M, then $\phi(N)$ is S −Nearly semiprime submodule of M', whenever M' is
a projective module .

Compare the following with Proposition (2,2,8) [1]

Proposition(2.8): Let M be a non –zero multiplication module . If $\{0\}$ is a semiprime submodule of M, then it is an S −Nearly semiprime submodule of M.

Proof: Since $\{0\}$ is a semiprime submodule of M, then by proposition(2,2,8) [1]$\{0\}$ is a $S-$ semiprime submodule , then by remark(2)$\{0\}$ is S −Nearly semiprime submodule of M.

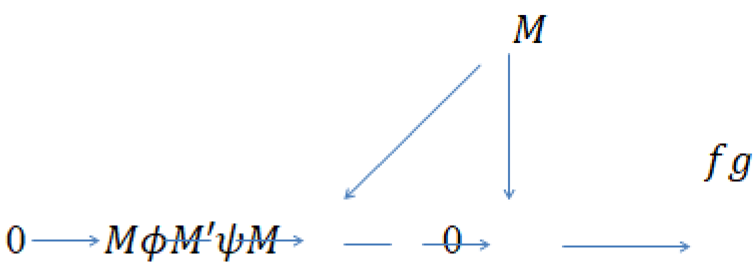
Corollary(2.9): Let M be a non –zero multiplication module . If $\{0\}$ is a Nearly semiprime submodule of M, then it is an S −Nearly semiprime submodule of M.
Compare the following with Proposition(2,2,10) [1]

Theorem (2.10): Let M be a non –zero multiplication module . If N is a Nearly semiprime submodule of M , then N is an S −Nearly semiprime submodule.

Proof: Since M is a multiplication module , then by[2,coro(3.22)] $\frac{M}{N}$ is also a multiplication module. By corollary(2.9)N is an S −Nearly semiprime submodule of M.

Proposition(2.11): If $0 \longrightarrow M \xrightarrow{\phi} M' \xrightarrow{\psi} M \longrightarrow 0$ is splits and K is S −Nearly semiprime submodule in M' such that $\ker\psi \subseteq K$, then $\psi(K)$is S −Nearly semiprime submodule in M.

Proof: Consider the following



Since exact sequins is splits, then M is projective module and by prop.(2.5) , then $\psi(K)$is S −Nearly semiprime submodule in M.

Recall that a submodule N of an R −module M is said to be fully invariant if $f(N) \subseteq N$, for each $R-endomorphism$ of M.[3]

Proposition(2.12): Let R be a good ring , N be an S −Nearly semiprime submodule of an R −module M. K be any a proper fully invariant submodule of M such that $K \nsubseteq N$, then $N \cap K$ is an S −Nearly semiprime submodule in K.

Proof: by prop.(2.4).

Recall that an R −module M is Z −nearly regular module if for each $m \in M, \exists f \in M^* = \mathrm{Hom}(\mathrm{M}, \mathrm{R})$ such that $m - f(m).m \in J(M)$.[4]

Proposition(2.13): If $0 \neq M$ is Z −nearly regular module , then every submodule of M is an S −Nearly semiprime submodule in M.

Proof: Let N be a proper submodule of an R −module M and $f^n(m) \in N, m \in M, f \in End(M), n \in Z^+$. We want to show that $f(m) \in N + J(M)$. Since M is Z −nearly regular module , then $\exists f \in M^*$ such that $m - f(m).m \in J(M)$, then $m = f(m).m + s\, ; s \in J(M)$, then $r\,m = r\,f(m).\ m + s_1$; $s_1 = r\,s$, then $r\,m = r^2\ m + s_1$, then $f(m) = f^2(m) + s_1 \in N + J(M)$. Which implies that every submodule of Z −nearly regular module is an S −Nearly semiprime submodule.

Proposition(2.14): Let $N_1$and N_2be two S −Nearly semiprime submodules of M_1 and M_2 respectively , then $N_1 \oplus N_2$ is an S −Nearly semiprime submodule of an R −module $M = M_1 \oplus M_2$.

Proof: Let $f^n(m) \in N_1 \oplus N_2$, $m = (a_1, a_2) \in M, f \in End(M), n \in Z^+$, $a_1 \in M_1$,$a_2 \in M_2$. We want to show that $f(m) \in N_1 \oplus N_2 + J(M)$. Since $f^n(m) = f^n(a_1, a_2) = (f^n(a_1), f^n(a_2)) \in N_1 \oplus N_2$, then $f^n(a_1) \in N_1$, $f^n(a_2) \in N_2$, but N_1 and N_2 are S −Nearly semiprime submodules of M_1 and M_2 respectively, then $f(a_1) \in N_1 + J(M_1)$ and $f(a_2) \in N_2 + J(M_2)$, then $((f(a_1), f(a_2)) \in (N_1 + J(M_1)) \oplus (N_2 + J(M_2))$, then $f(a_1, a_2) \in (N_1 + J(M_1)) \oplus (N_2 + J(M_2))$, then $f(m) \in N_1 \oplus N_2 + J(M)$; $J(M) = J(M_1) \oplus J(M_2)$.

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