Fixed Point in Cone Banach Space

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Abstract: Let \((X, || . ||_p)\) be a cone Banach space and \(T\) a self map on \(X\). We say that \(T\) satisfy condition \(B\) if there exists \(0 < \delta < 1\) and \(L > 0\) such that \(\|T_x - T_y\| < \delta \|x - y\| + Lu\) where \(u \in \{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|\}\) and \(Tz - Tx \in B\). We also prove that under a certain condition every map \(T\) that satisfy condition \(B\) is continuous.

1. Introduction

Many authors study fixed point theorems for a mapping is a cone metric space satisfying different contraction condition for example Sh.Rezapour, R.Hambarani[4] prove fixed point theorem for contractive mapping. Must these theorem extended to a cone Banach space after some modification. Edral karapiner[2] prove fixed point theorem in cone Banach space while R.krishnakumar and D.Dhamodharan[3] prove fixed point theorem in cone Banach space by using \(\Phi\) operator.

The following definition and results will be needed in the sequel Let \(E\) be a real Banach space. A subset \(p\) of \(E\) is called cone if

1. \(p\) is closed, nonempty and \(p \neq \{0\}\)
2. \(a, b \in R, a, b > 0\) and \(x, y \in p\) imply \(ax + by \in p\)
3. \(p \cap (-p) = \{0\}\)

Definition 1.1 [2] Let \(X\) be a vector space over \(R\), suppose the mapping \(\| . \|_p: X \rightarrow E\) satisfies

1. \(\|x\| > 0\) for all \(x \in X\)
2. \(\|x\| = 0\) if and only if \(x = 0\)
3. \(\|x + y\| \leq \|x\| + \|y\|\) for all \(x, y \in X\)
4. \(\|kx\| = |k|\|x\|\) for all \(k \in R\)

then \(\| . \|_p\) is called a cone norm on \(X\) and the pair \((X, \| . \|_p)\) is called a cone normed space. (CNS)

Definition 1.2 [2] Let \((X, || . ||_p)\) be a cone normed space \(x \in X\) and \(\{x_n\}_{n=1}^{\infty}\) a sequence in \(X\), then

1. \(\{x_n\}_{n=1}^{\infty}\) converge to \(x\) whenever for each \(c \in E\) with \(0 = c\) there is natural number \(n\) such that \(\|x_n - x\|_p = c\) for all \(n \in N\) it is denoted be \(\lim_{n \rightarrow \infty} x_n = x\) or \(x_n \rightarrow x\)
2. \(\{x_n\}_{n=1}^{\infty}\) is a Cauchy sequence whenever for every \(c \in E\) with \(0 = c\) there is a natural number \(n\), such that \(\|x_n - x_m\|_p = c\) for all \(n, m \in N\)

(3) \((X, || . ||_p)\) is complete cone normed space i.e. every Cauchy sequence is convergent complete cone normed space will be called cone Banach space.

Lemma 1.3 [1] Let \((X, || . ||_p)\) be a cone Banach space then the following properties are often used

\(h_1)\) if \(u \leq v\) and \(v = w\) then \(u = w\)

\(h_2)\) if \(0 \leq u = c\) for each \(c \in intp\) then \(u \geq 0\)

2. Main Result

Definition 2.1 Let \((X, || . ||_p)\) be a cone Banach space a map \(T: X \rightarrow X\) is said to be satisfy condition \(B\) if there exists \(0 < \delta < 1\) and \(L > 0\) such that for all \(x, y \in X\) we have

\(\|T_T - T_y\| < \delta \|x - y\| + Lu\)

when \(u \in \{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|\}\)

Lemma 2.2 Let \((X, || . ||_p)\) be a cone Banach space \(T: X \rightarrow X\) satisfy condition \(B\) if for some point \(x\) of the sequence \(\{T^n x\}\) of picard iteration converge to point \(z \in X\) then \(z\) is it's fixed point

Proof. by definition of condition \(B\) and the triangle inequality we get

\(\|z - T^n x\| = \|z - T_{n+1} x\| + \|T^n x - T_{n+1} x\|\)

\(\leq \|z - T^{n+1} x\| + \|T^n x - T_{n+1} x\|\)

\(\leq \|z - T^{n+1} x\| + \delta \|T^n x - z\| + Lu\)

where \(u \in \{\|T^n x - T^{n+1} x\|, \|z - T_{n+1} x\|, \|T^n x - T_{n+1} x\|, \|z - T^n x\|\}\)

let \(0 = c\), clearly at least one of the following four cases holds for infinitely many \(n \in N\)

(1) \(\|z - T^n x - z\| = \|z - T^{n+1} x\| + \|T^n x - z\| + Lu\)

(2) \(\|z - T^n x - z\| = \|z - T^{n+1} x\| + \|T^n x - z\| + Lu\)

(3) \(\|z - T^n x - z\| = \|z - T^{n+1} x\| + \|T^n x - z\| + Lu\)

(4) \(\|z - T^n x - z\| = \|z - T^{n+1} x\| + \|T^n x - z\| + Lu\)
Now we give the main theorem that gave the condition which granted existence and uniqueness of the fixed point for a map $T$ that satisfies condition $B$, but first we need the following lemma.

**Lemma 2.3** Let $(X, ||.||)$ be a cone Banach space and $T : X \to X$ satisfying condition $B$, if $x \in X$ and let $x \neqTx$ then $\|T^2x-Tx\| \leq \|Tx-x\|$.

**Proof.** Let $x \in X$ and let $x \neqTx$ then

$$\|T^2x-Tx\| \leq \|TTx-Tx\| \leq \|Tx-x\| + L\mu$$

when

$$u \in \{ ||Tx-T^2x||, ||x-Tx||, ||Tx-Tx||, ||x-Tx|| \}$$

we will get the following possibilities

(1) $\|T^2x-Tx\| \leq \|Tx-x\| + L.0$

$$=0 < ||Tx-x|| \text{ because } x \neqTx$$

(2) $\|T^2x-Tx\| \leq \|Tx-x\| + L \|Tx-T^2x\|$

$$=0 < ||Tx-x|| \text{ because } x \neqTx$$

(3) $\|T^2x-Tx\| \leq \|Tx-x\| + L \|x-T^2x\|$

$$\leq \|Tx-x\| + L \|x-Tx\| + L \|T^2x-x\|$$

$$\|T^2x-Tx\| \leq \frac{\delta + L}{1-L} \|x-Tx\| < ||Tx-x||$$

because $\frac{\delta + L}{1-L} < 1 \leq L \in (0, \frac{1}{3})$ and $\delta \in (0, \frac{1}{3})$ and $x \neqTz$

**Theorem 2.4** Suppose that $(X, ||.||)$ is a cone Banach space with a cone $P$ such that $\text{int}P \neq \phi$ and $T : X \to X$ that satisfy condition $B$ then $T$ has a unique fixed point.

**Proof.** Let $x \in X$ we shall show that $\{Tx\}$ is a Cauchy sequence, but first we prove that

Now we give the main theorem that gave the condition which granted existence and uniqueness of the fixed point for a map $T$ that satisfies condition $B$, but first we need the following lemma.

$$\Rightarrow ||T^2x-T^3x|| \leq \frac{(\delta + 2L)}{(1L-L)} ||T^2x-T^3x||$$

where

$$u \in \{ ||T^{-1}x-TT^{-1}x||, ||T^{-2}x-TT^{-2}x||, ||T^{-3}x-TT^{-3}x||, ||T^{-1}x-TT^{-3}x|| \}$$

$$\Rightarrow ||T^{-1}x-T^{-2}x|| \leq \frac{(\delta + 2L)}{(1-L)} ||T^{-1}x-T^{-2}x||$$

$$\leq w ||T^{-2}x-T^3x||, w = \frac{\delta + 2L}{1-L} \in (0,1)$$

(2) $||T^{-1}x-T^{-2}x|| \leq ||T^{-1}x-T^{-2}x|| + L ||T^{-2}x-TT^{-2}x||$

$$\leq \|T^{-1}x-T^{-2}x\| + L \|T^{-2}x-T^{-3}x\|$$

$$\leq (\delta + L) \|T^{-3}x-T^{-2}x\| + L \|T^{-2}x-T^{-3}x\|$$

$$\leq (\delta + L) \|T^{-3}x-T^{-2}x\| + L \|T^{-3}x-T^{-2}x\|$$

$$\leq (\delta + L) \|T^{-1}x-x-T^{-2}x\| + L \|T^{-1}x-x-T^{-2}x\|$$

$$\leq (\delta + L) \|T^{-1}x-x-T^{-2}x\| + L \|T^{-1}x-x-T^{-2}x\|$$
then \((\delta + L) \leq \frac{\delta + L}{1 - L} \times \frac{\delta + 2L}{1 - L}\)

\[
\|T^nx - T^n-x\| \leq \delta \|T^{n-1}x-T^{n-2}x\| + L\|T^{n-2}x-T^{n-1}x\|
\]

\[
\leq \delta \|T^{n-1}x-T^{n-2}x\| + L\|T^{n-2}x-T^{n-1}x\|
\]

\[
\leq \delta \|T^{n-1}x-T^{n-2}x\| + L\|T^{n-2}x-T^{n-1}x\|
\]

\[
\|T^{n-1}x-T^{n-2}x\| \leq \frac{(\delta + L)}{1 - L} \|T^{n-1}x-T^{n-2}x\|
\]

\[
\leq w\|T^{n-1}x-T^{n-2}x\|, \quad w \in \left(0, \frac{\delta + L}{1 - L}\right) \subseteq (0,1)
\]

\[
\|T^nx - T^n-x\| \leq \frac{\delta + L}{1 - L} \|T^{n-1}x-T^{n-2}x\| + L\|T^{n-2}x-T^{n-1}x\|
\]

\[
\leq \delta \|T^{n-1}x-T^{n-2}x\| + L0.0
\]

then (2.1) is hold

Hence, in all the possible cases it follows that (2.1) holds, and so let

\[
\Lambda = \max\{\delta + L, \frac{\delta + 2L}{1 - L}\}
\]

\[
\|T^nx - T^n-x\| \leq \Lambda \|T^{n-1}x-T^{n-2}x\| + \frac{\delta + 2L}{1 - L} \|T^{n-1}x-T^{n-2}x\| \quad ...(2.2)
\]

By induction and using (2.1) and (2.2)

\[
\|T^nx - T^n-x\| \leq \frac{\delta + L}{1 - L} \|T^{n-1}x-T^{n-2}x\| \leq \frac{\delta + L}{1 - L} \|T^{n-2}x-T^{n-3}x\| \quad ...(2.3)
\]

Using the triangle inequality and (2.3) we get that for some

\[
\|T^nx - T^n-x\| \leq \frac{\delta + L}{1 - L} \|T^{n-1}x-T^{n-2}x\| + \frac{\delta + 2L}{1 - L} \|T^{n-2}x-T^{n-1}x\| = (\delta + L)^n \frac{1}{1 - L} \|T^nx - x\| \leq \frac{\delta + L}{1 - L} \|T^nx - x\| \to 0
\]

as \(m \to \infty\)

so that

\[
\frac{(\delta + 2L)^m}{1 - L} \|T^nx - x\| \to 0 \quad \text{as} \quad m, n \to \infty
\]

where each \(\frac{(\delta + 2L)^m}{1 - L} \|T^nx - x\| = c\) for each

we have proved that the sequence \(\{T^nx\}\) is Cauchy sequence for the fixed \(x \in X\) since the \((X, \|\cdot\|)\) is complete, there exists a point \(p \in X\) which is the limit of this sequence be lemma(2.2) we conclude that \(p\) is fixed point

Theorem 2.5 Let \(T: X \to X\) satisfies condition \(B\) on a cone Banach space. If \(\text{int} p \neq \phi\) then \(T\) is continuous at its fixed point

Proof. Let \(z\) the unique fixed point and let \(X_n \to z\)

\[
\{x_n\}\quad \text{is a sequence of point form the given cone Banach space to show that} \quad T \quad \text{is continues we must prove that} \quad TX_n \to Tz = z \quad \text{we have}
\]

\[
\|TX_n - Tz\| \leq \delta \|x_n - z\| + Lu
\]

where

\[
u \in \{|z_n - TX_n|, |z - Tz|, |x_n - Tz|, |z - TX_n|\}
\]

\[
= \{0, |x_n - Tz_n|, |x_n - z|, |z - TX_n|\}
\]

similarly as the previous proofs of this kind, must be considered the following cases

\[
(1) \quad \|TX_n - Tz\| \leq \delta \|x_n - z\| + L0.0
\]

\[
\Rightarrow TX_n = Tz \Rightarrow TX_n \to Tz = z
\]

\[
(2) \quad \|TX_n - Tz\| \leq \delta \|x_n - z\| + L\|x_n - z\|
\]

\[
= (\delta + L) \frac{c}{(\delta + L)} = c
\]

\[
(3) \quad \|TX_n - Tz\| \leq \delta \|x_n - z\| + L\|x_n - Tx_n\|
\]

\[
\leq \delta \|x_n - z\| + L\|x_n - z\| + \|Tx_n - z\|
\]

\[
\leq (\delta + L) \|x_n - z\| + L\|Tx_n - z\|
\]

\[
\leq \frac{\delta + L}{1 - L} \|x_n - z\| \quad \text{...(2.3)}
\]

\[
\Rightarrow \|Tz_n - Tz\| = 0
\]

\[
\Rightarrow TX_n \to Tz = z
\]

denote as usual by \(F(T)\) the set of fixed point of the mapping \(T: X \to X\) it is said that the map \(T\) has property \(P\) if \(F(T) = F(T^n)\) for each \(n \in N\)

**References**


