Fixed Point in Cone Banach Space

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Abstract: Let \((X,\|\|_{p})\) be a cone Banach space and \(T\) a self map on \(X\). We say that \(T\) satisfy condition \(B\) if there exists \(0 < \delta < 1\) and \(L > 0\) such that \(\|T x - T y\| < \delta \|x - y\| + Lu\) where \(u \in \{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}\) in this paper we have the existences and uniqueness theorems for a fixed point to a map \(T\) that satisfy condition \(B\). We also prove that under a certain condition every map \(T\) that satisfy condition \(B\) is continuous.

1. Introduction

Many authors study fixed point theorems for a mapping is a cone metric space satisfying different contraction condition for example Sh.Rezapour, R.Hamlbarani[4] prove fixed point theorem for contractive mapping. Most theses theorem extended to a cone Banach space after some modification. Edral karapiner[2] prove fixed point theorem in cone Banach space while R.krishnakumar and D.Dhamodharan[3] prove fixed point theorem in cone Banach space by using \(\Phi\) operator.

The following definition and results will be needed in the sequel Let \(E\) be a real Banach space. A subset \(p\) of \(E\) is called cone if

1. \(p\) is closed , nonempty and \(p \neq \{0\}\)
2. \(a, b \in \mathbb{R}, a, b \geq 0\) and \(x, y \in p\) imply \(ax + by \in p\)
3. \( p \cap (-p) = \{0\} \)

Definition 1.1 [2] Let \(X\) be a vector space over \(\mathbb{R}\), suppose the mapping \(\|\|_{p} : X \to E\) satisfies

1. \(\|x\| > 0\) for all \(x \in X\)
2. \(\|x\| = 0\) if and only if \(x = 0\)
3. \(\|x + y\| \leq \|x\| + \|y\|\) for all \(x, y \in X\)
4. \(\|kx\| = k\|x\|\) for all \(k \in \mathbb{R}\)

then \(\|\|_{p}\) is called a cone norm on \(X\) and the pair \((X,\|\|_{p})\) is called a cone normed space. (CNS)

Definition 1.2 [2] Let \((X,\|\|_{p})\) be a cone normed space \(x \in X\) and \(\{x_{n}\}_{n \in \mathbb{N}}\) a sequence in \(X\), then

1. \(\{x_{n}\}_{n \in \mathbb{N}}\) converge to \(x\) whenever for each \(c \in \mathbb{E}\) with \(0 = c\) there is natural number \(n\) such that \(\|x_{n} - x\|_{p} = c\) for all \(n \in \mathbb{N}\) it is denoted be \(\lim_{n \to \infty} x_{n} = x\) or \(x_{n} \to x\)
2. \(\{x_{n}\}_{n \in \mathbb{N}}\) is a Cauchy sequence whenever for every \(c \in \mathbb{E}\) with \(0 = c\) there is a natural number \(n\), such that \(\|x_{n} - x_{m}\|_{p} = c\) for all \(n, m \in \mathbb{N}\)

(3) \((X,\|\|_{p})\) is complete cone norm space i.e. every Cauchy sequence is convergent complete cone normed space will be called cone Banach space.

Lemma 1.3 [1] Let \((X,\|\|_{p})\) be a cone Banach space then the following properties are often used

1. \(h_{i}\) if \(u \leq v\) and \(v = w\) then \(u = w\)
2. \(h_{j}\) if \(0 \leq u = c\) for each \(c \in \text{int}p\) then \(u = 0\)

2. Main Result

Definition 2.1 Let \((X,\|\|_{p})\) be a cone Banach space a map \(T : X \to X\) is said to satisfy condition \(B\) if there exists \(0 < \delta < 1\) and \(L > 0\) such that for all \(x, y \in X\) we have

\[
\|Tx - Ty\| < \delta \|x - y\| + Lu
\]

when \(u \in \{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}\)

Lemma 2.2 Let \((X,\|\|_{p})\) be a cone Banach space \(T : X \to X\) satisfy condition \(B\) if for some point \(x\) of the sequence \(\{T^{n}x\}\) of picard iteration converge to point \(z \in X\) then \(z\) is it’s fixed point

Proof. by definition of condition \(B\) and the triangle inequality we get

\[
\|z - Tz\| \leq \|z - T^{n+1}x\| + \|T^{n+1}x - Tz\|
\]

\[
= \|z - T^{n+1}x\| + \|TT^{n}x - Tz\|
\]

\[
\leq \|z - T^{n+1}x\| + \delta \|T^{n}x - z\| + Lu
\]

where

\(u \in \{\|T^{n}x - TT^{n}x\|, \|z - Tz\|, \|T^{n}x - Tz\|, \|z - TT^{n}x\|\}\)

let \(0 = c\), clearly at least one of the following four cases holds for infinitely many \(n \in \mathbb{N}\)

1. \(\|T^{n}x - z\| \leq \|z - T^{n+1}x\| + \delta \|T^{n}x - z\| + Lu\|T^{n}x - TT^{n}x\|
\]

\[
\leq \|z - T^{n+1}x\| + \delta \|T^{n}x - z\| + Lu\|T^{n}x - Tz\| + Lu\|z - T^{n+1}x\|
\]

\[
\leq \|z - T^{n+1}x\| + \delta \|T^{n}x - z\| + Lu\|T^{n}x - z\| + Lu\|T^{n}x - T^{n+1}x\|
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\leq \|z - T^{n+1}x\| + \delta \|T^{n}x - z\| + Lu\|T^{n}x - T^{n+1}x\|
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\[
\leq \|z - T^{n+1}x\| + \delta \|T^{n}x - z\| + Lu\|T^{n}x - T^{n+1}x\|
\]
Suppose that and let \( T : X \to X \) that satisfies condition \( B \), but first we need the following lemma.

**Lemma 2.3** Let \( (X, \| \cdot \|) \) be a cone Banach space and \( T : X \to X \) satisfying condition \( B \), if \( x \in X \) and let \( x \neq T x \) then \( \| T^2x - Tx \| < \| Tx - x \| \)

*Proof.* Let \( x \in X \) and let \( x \neq T x \) then

\[
\| T^2x - Tx \| = \| TTx - Tx \| \leq \delta \| Tx - x \| + Lu
\]

where \( u \in \{ \| Tx - T^2x \|, \| x - T^2x \|, \| Tx - Tx \|, \| x - Tx \| \} \)

we will get the following possibilities

(1) \( \| T^2x - Tx \| \leq \delta \| Tx - x \| + L \delta \)

\( = 0 < \| Tx - x \| \) because \( x \neq Tx \)

(2) \( \| T^2x - Tx \| \leq \delta \| Tx - x \| + L \| Tx - T^2x \| \)

\( = 0 < \| Tx - x \| \) because \( x \neq Tx \)

(3) \( \| T^2x - Tx \| \leq \delta \| Tx - x \| + L \| x - T^2x \| \)

\( \leq \delta \| Tx - x \| + L \| x - T^2x \| + L \| Tx - x \|^2 \)

\( \leq \frac{\delta + L}{1 - \delta} \| x - Tx \| \leq \| Tx - x \| \) because \( \delta + L < 1 < \in L \) and \( \frac{\delta + L}{1 - \delta} \) and \( x \neq Tx \)

**Theorem 2.4** Suppose that \( (X, \| \cdot \|) \) is a cone Banach space with a cone \( P \) such that \( \text{int} P \neq \phi \) and \( T : X \to X \) that satisfy condition \( B \) then \( T \) has a unique fixed point

*Proof.* Let \( x \in X \) we shall show that \( \{ Tx \} \) is a Cauchy sequence, but first we prove that

\[
\| T^nx - T^{n-1}x \| \leq \frac{\delta + 2L}{1 - L} \| T^{n-1}x - T^{n-2}x \| \quad \text{......(2.1)}
\]

Indeed \( \| T^nx - T^{n-1}x \| \leq \delta \| T^{n-1}x - T^{n-2}x \| + Lu \)

where \( u \in \{ \| T^{n-1}x - TT^{n-1}x \| , \| T^{n-2}x - TT^{n-2}x \| , \| T^{n-2}x - TT^{n-1}x \| , \| T^{n-1}x - TT^{n-2}x \| \} \)

(1) \( \| T^nx - TT^{n-1}x \| \leq \delta \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-1}x - T^{n-2}x \| \)

\( \leq \delta \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-1}x - T^{n-2}x \| \)

\( \leq \delta \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - T^{n-1}x \| + L \| T^{n-1}x - T^{n-2}x \| \)

\( \leq (\delta + L) \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - T^{n-1}x \| \)

\( \leq (\delta + L) \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - T^{n-1}x \| + L \| T^{n-1}x - T^{n-2}x \| \)

\( \leq (\delta + 2L) \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-1}x - T^{n-2}x \| \)

\( \leq w \| T^{n-1}x - T^{n-2}x \| , \quad w = \frac{\delta + 2L}{1 - L} \in (0, 1) \)

(2) \( \| T^nx - T^{n-1}x \| \leq \delta \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - TT^{n-2}x \| \)

\( \leq \delta \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - T^{n-1}x \| + L \| T^{n-2}x - T^{n-1}x \| \)

\( \leq (\delta + L) \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - T^{n-1}x \| \)

\( \leq (\delta + 2L) \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-1}x - T^{n-2}x \| \)
then \((\delta + L)\frac{\delta + 2L}{1 - L} \leq 1 - L\)

(3) \[\left\| T^n x - T^{n-1} x \right\| \leq \delta \left( \left\| T^n x - T^{n-2} x \right\| + L \left\| T^{n-1} x - TT^n x \right\| \right)\]

\[\leq \delta \left( \left\| T^n x - T^{n-1} x \right\| + L \left\| T^{n-2} x - T^n x \right\| \right)\]

\[\leq \delta \left( \left\| T^n x - T^{n-1} x \right\| + L \right) \left\| T^{n-1} x - T^n x \right\|\]

\[\leq \left( \delta + L \right) \frac{\delta + 2L}{1 - L} \left\| T^n x - T^{n-1} x \right\|\]

\[\leq w \left( \delta + L \right) \frac{\delta + 2L}{1 - L}, w \in (0, 1)\]

(4) \[\left\| T^n x - T^{n-1} x \right\| \leq \delta \left( \left\| T^n x - T^{n-1} x \right\| + L \right) \left\| T^{n-1} x - T^n x \right\|\]

\[\delta \left( \left\| T^n x - T^{n-1} x \right\| + L \right) \left\| T^{n-1} x - T^n x \right\|\]

then (2.1) is hold

Hence, in all the possible cases it follows that (2.1) holds, and so let

\[\Lambda = \max\left\{ \delta + L, \frac{\delta + 2L}{1 - L} \right\}\]

\[\left\| T^n x - T^{n-1} x \right\| \leq \Lambda \left\| T^n x - T^{n-2} x \right\| \leq \frac{\delta + 2L}{1 - L} \left\| T^{n-1} x - T^n x \right\|\]

By induction and (2.1) and (2.2)

\[\left\| T^n x - T^{n-1} x \right\| \leq (\delta + L)^{n-1} \frac{\delta + 2L}{1 - L} \left\| T x - x \right\|\]

Using the triangle inequality and (2.3) we get that for \(m \geq n\)

\[\left\| T^n x - T^m x \right\| \leq \left\| T^n x - T^{n-1} x \right\| + \left\| T^{n-1} x - T^{n-2} x \right\| + \ldots + \left\| T x - x \right\|\]

\[\leq \left( \delta + L \right)^{n-1} + \left( \delta + L \right)^{n-2} + \ldots + \left( \delta + L \right)^{m} \left\| T x - x \right\|\]

\[= \left( \delta + L \right)^{m-1} \frac{\delta + 2L}{1 - L} \left\| T x - x \right\| \rightarrow 0\]

as \(m \rightarrow \infty\)

since \(\frac{\delta + 2L}{1 - L} \left\| T x - x \right\| \rightarrow 0\) as \(m, n \rightarrow \infty\)

is that \(\frac{\delta + 2L}{1 - L} \left\| T x - x \right\| \rightarrow 0\) for each \(c \in \text{int} \ P\)

we have proved that the sequence \(\{T^n x\}\) is Cauchy sequence for the fixed \(x \in X\) since the \((X, \left\| \cdot \right\|)\) is complete, there exists a point \(p \in X\) which is the limit of this sequence be lemma (2.2) we conclude that \(p\) is fixed point

**Theorem 2.5** Let \(T : X \rightarrow X\) satisfy condition \(B\) on a cone Banach space. If \(\text{int} \ P \neq \emptyset\) then \(T\) is continues at its fixed point

**Proof.** Let \(z\) the unique fixed point and let \(X_n \rightarrow z\) where \(\{X_n\}\) is a sequence of point form the given cone Banach space to show that \(T\) is continues we must prove that \(T x_n \rightarrow T z = z\) we have

\[\left\| T x_n - T z \right\| \leq \delta \left\| x_n - z \right\| + Lu\]

where

\[u \in \left\{ \left\| T x_n - T z \right\|, \left\| z - T z \right\|, \left\| x_n - T z \right\|, \left\| z - T x_n \right\| \right\}\]

\(= \{0, \left\| x_n - T z \right\|, \left\| x_n - z \right\|, \left\| z - T x_n \right\| \}\)

similarly as the previous proofs of this kind, must be considered the following cases

(1) \[\left\| T x_n - T z \right\| \leq \delta \left\| x_n - z \right\| + L 0\]

\[\Rightarrow T x_n = T z \Rightarrow T x_n \rightarrow T z = z\]

(2) \[\left\| T x_n - T z \right\| \leq \delta \left\| x_n - z \right\| + L \left\| x_n - z \right\|\]

\[= \delta + L \left( \frac{c}{\delta + L} = c\right)\]

(3) \[\left\| T x_n - T z \right\| \leq \delta \left\| x_n - z \right\| + L \left\| x_n - T x_n \right\|\]

\[\leq \delta \left\| x_n - z \right\| + L \left\| x_n - z \right\| + L \left\| T x_n - z \right\|\]

\[\leq \left( \delta + L \right) \left\| x_n - z \right\| + L \left\| x - T x_n \right\|\]

\[\leq \frac{\delta + L}{1 - L} \left\| x_n - z \right\|\]

\[\leq \frac{\delta + L}{1 - L} \frac{c}{\delta + L} = c\]

(4) \[\left\| T x_n - T z \right\| \leq \delta \left\| x_n - z \right\| + L \left\| x_n - T z \right\|\]

\[= \delta \left\| x_n - z \right\| + L \left\| x_n - T z \right\|\]

\[\Rightarrow \left\| T x_n - T z \right\| = 0\]

\[\Rightarrow T x_n \rightarrow T z = z\]

denote as usual by \(F(T)\) the set of fixed point of the mapping \(T : X \rightarrow X\) it is said that the map \(T\) has property \(P\) if \(F(T) = F(T^n)\) for each \(n \in N\)

**References**


