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Fixed Point in Cone Banach Space

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Abstract: ***Let*** $(X, \|.\|_p)$ ***be a cone Banach space and*** T ***a self map on*** X. ***We say that*** T ***satisfy condition*** B ***if there exists*** $0<\delta<1$ ***and*** $L>0$ ***such that*** $\|T_x - T_y\| < \delta\|x-y\| + L.u$ ***where*** $u \in \{\|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\}$ ***in this paper we have the existences and uniqueness theorems for a fixed point to a map*** T ***that satisfy condition*** B. ***We also prove that under a certain condition every map*** T ***that satisfy condition*** B ***is continuous.***

1. Introduction

Many authors study fixed point theorems for a mapping is a cone metric space satisfying different contraction condition for example Sh.Rezapour, R.Hamlbarani[4] prove fixed point theorem for contractive mapping. Must these theorem extended to a cone Banach space after some modification , Edral karapiner[2] prove fixed point theorem in cone Banach space while R.krishnakumar and D.Dharnodharan[3] prove fixed point theorem in cone Banach space by using Φ operator

The following definition and results will be needed in the sequel Let E be a real Banach space .A subset p of E is called cone if

(1) p is closed , nonempty and $p \neq \{0\}$

(2) $a,b \in R, a,b \geq 0$ and $x,y \in p$ imply $ax+by \in p$

(3) $p \cap (-p) = \{0\}$

Definition 1.1 *[2] Let* X *be a vector space over* $\mathbb{R}$, *suppose the mapping* $\|.\|_p : X \to E$ *satisfies*

(1) $\|x\|_p > 0$ for all $x \in X$

(2) $\|x\|_p = 0$ if and only if $x = 0$

(3) $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ for all $x, y \in X$

(4) $\|kx\|_p = |k| \|x\|_p$ for all $k \in \mathbb{R}$

then $\|.\|_p$ is called a cone norm on X and the pair $(X, \|.\|_p)$ is called a cone normed space . (CNS)

Definition 1.2 *[2] Let* $(X, \|.\|_p)$ *be a cone normed space* $x \in X$ *and* $\{x_n\}_{n\geq1}$ *a sequence in* X , *then*

(1) $\{x_n\}_{n\geq1}$ converge to x whenever for each $c \in E$ with $0 = c$ there is natural number n such that $\|x_n \to x\|_p = c$ for all $n \in N$ it is denoted be $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$

(2) $\{x_n\}_{n\geq1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 = c$ there is a natural number n , such that $\|x_n - x_m\|_p = c$ for all $n, m \in N$

(3) $(X, \|.\|_p)$ is complete cone normd space i.e. every Cauchy sequence is convergent complete cone normed space will be called cone Banach space.

Lemma 1.3 *[1] Let* $(X.\|.\|)$ *be a cone Banach space then the following properties are often used*

h_1) if $u \leq v$ and $v = w$ then $u = w$

h_2) if $0 \leq u = c$ for each $c \in intp$ then $u = 0$

2. Main Result

Defintion 2.1 *Let* $(X, \|.\|)$ *be a cone Banach space a map* $T: X \to X$ *is said to be satisfy condition* B *if there exists* $0<\delta<1$ *and* $L>0$ *such that for all* $x, y \in X$ *we have*

$\|Tx - Ty\| < \delta\|x-y\| + L.u$

when

$u \in \{\|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\}$

Lemma 2.2 *Let* $(X, \|.\|)$ *be a cone Banach space* $T: X \to X$ *satisfy condition* B *if for some point* x *of the sequence* $\{T^n x\}$ *of picard iteration converge to point* $z \in X$ *then* z *is it's fixed point*

Proof. by definition of condition B and the triangle inequality we get

$$\|z - Tz\| \leq \|z - T^{n+1}x\| + \|T^{n+1}x - Tz\|$$
$$= \|z - T^{n+1}x\| + \|TT^n x - Tz\|$$
$$\leq \|z - T^{n+1}x\| + \delta\|T^n x - z\| + L.u$$

where

$$u \in \{\|T^n - TT^n x\| . \|z - Tz\|, \|T^n x - Tz\| . \|z - TT^n x\|\}$$

let $0 = c$, clearly at least one of the following four cases holds for infinitely many $n \in N$

(1) $\|T^n x - z\| \leq \|z - T^{n+1}x\| + \delta\|T^n x - z\| + L\|T^n x - TT^n x\|$
$\leq \|z - T^{n+1}x\| + \delta\|T^n x - z\| + L\|T^n x - z\| + L\|z - T^{n+1}x\|$
$\leq \|z - T^{n+1}x\| + (\delta + L)\|T^n x - z\| + L\| - T^{n+1}x\|$

$\leq \| (1+L) \| z - T^{n+1}x \| + (\delta + L) \| T^n x - z \|$

$\| T^n x - z \| \leq \frac{(1+L)}{1-(\delta+L)} \| z - T^{n+1} \|$

$\leq \frac{(1+L)}{1-(\delta+L)} \cdot \frac{c}{\frac{1+l}{1-(\delta+L)}} = c$

(2) $\| T^n x - z \| \leq \| z - T^{n+1}x \| + \delta \| T^n x - z \| + l \| z - Tz \|$

$\| z - T^n x \| \leq \frac{1}{\delta - L} \| z - T^{n+1}x \| = \frac{1}{\delta - L} \cdot \frac{c}{\frac{1}{\delta - L}} = c$

(3) $\| T^n x - z \| \leq \| z - T^{n+1}x \| + \delta \| T^n x - x \| + L \| z - TT^n x \|$

$\leq (1+L) \| z - T^{n+1}x \| = \delta \| T^n x - z \| = \frac{(1+L)}{1-\delta} \cdot \frac{c}{\frac{(1+L)}{(1-\delta)}} = c$

(4) $\| T^n x - z \| \leq \| z - T^{n+1}x \| + \delta \| T^n x - z \| + L \| T^n - Tz \|$

$\leq \| z - T^{n+1}x \| + \delta \| T^n x - z \| + L \| T^n x - z \| + L \| z - Tz \|$

$\leq \| z - T^{n+1}x \| + (\delta + L) \| T^n x - z \| + L \| z - Tz \|$

$\leq \frac{1}{1-(\delta+L)} \| z - T^{n+1}x \| + \frac{L}{1-(\delta+L)} \| z - Tz \|$

$= \frac{1}{1-(\delta+L)} \cdot \frac{c}{2\frac{1}{1-(\delta+l)}} + \frac{L}{1-(\delta+L)} \cdot \frac{c}{2\frac{L}{1-(\delta+L)}} = c$

we have fined that all cases $\| T^n x - x \| = c$ for each interior point c of the cone P. By (h_2), it wollow that $\| T^n x - z \| = 0$ i.e $T^n x = z$

Now we give the main theorem that gave the condition which granted existence and uniqueness of the fixed point for a map T that satisfies condition B, but first we need the following lemma.

Lemma 2.3 *Let* $(X.\|.\|)$ *be a cone Banach space and* $T: X \to X$ *satisfying condition* B, *if* $x \in X$ *and let* $x \neq Tx$ *then* $\| T^2 x - Tx \| < \| Tx - x \|$

Proof. Let $x \in X$ and let $x \neq Tx$ then

$\| T^2 x - Tx \| = \| TTx - Tx \| \leq \delta \| Tx - x \| + L.u$

when

$u \in \{ \| Tx - T^2 x \|, \| x - Tx \|, \| Tx - Tx \|, \| x - Tx \| \}$

$u \in \{ \| Tx - T^2 x \|, \| z - T^2 x \|, 0 \}$

we will get the following possibilities

(1) $\| T^2 x - Tx \| \leq \delta \| Tx - x \| + L.0$

$= 0 < \| Tx - x \|$ *because* $x \neq Tx$

(2) $\| T^2 x - Tx \| \leq \delta \| Tx - x \| + L \| Tx - T^2 x \|$

$= 0 < \| Tx - x \|$ *because* $x \neq Tx$

(3) $\| T^2 x - Tx \| \leq \delta \| Tx - x \| + L \| x - T^2 x \|$

$\leq \delta \| Tx - x \| + L \| x - Tx \| + L \| Tx_T^2 x \|$

$\| T^2 x - Tx \| \leq \frac{\delta + L}{1_L} \| x - Tx \| < \| Tx - x \|$

because $\frac{\delta + L}{1 - L} < 1 \Leftarrow L \in (0, \frac{1}{3})$ *and* $\delta \in (0, \frac{1}{3})$ *and* $x \neq Tx$

Theorem 2.4 *Suppose that* $(X, \|.\|)$ *is a cone Banach space with a cone* P *such that* $intP \neq \phi$ *and* $T: X \to X$ *that satisfy condition* B *then* T *has a unique fixed point*

Proof. Let $x \in X$ we shall show that $\{Tx\}$ is a Cauchy sequence, but first we prove that

$$\| T^n x - T^{n-1}x \| \leq \frac{\delta + 2L}{1 - L} \| T^{n-1}x - T^{n-2}x \| .....n \geq 2 \qquad ....(2.1)$$

$$indeed \quad \| T^n x - T^{n-1}x \| \leq \delta \| T^{n-1}x - T^{n-2}x \| + L.u$$

where

$$u \in \{ \| T^{n-1}x - TT^{n-1}x \|, \| T^{n-2}x - TT^{n-2}x \|, \| T^{n-2}x - TT^{n-1}x \|, \| T^{n-1}x - TT^{n-2}x \| \}$$

(1) $\| T^n x - TT^{n-1}x \| \leq \delta \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-1}x - TT^{n-1}x \|$

$\leq \delta \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-1}x - T^n x \|$

$\leq \delta \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - T^n x \|$

$\leq (\delta + L) \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - T^n \|$

$\leq (\delta + L) \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - T^{n-1}x \| + L \| T^{n-1}x - T^n x \|$

$\leq (\delta + 2L) \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-1}x - T^n \|$

$\Rightarrow \| T^{n-1}x - T^n x \| \leq \frac{(\delta + 2L)}{(1-L)} \| T^{n-1}x - T^{n-2}x \|$

$\leq w \| T^{n-1}xT^{n-2} \|, w = \frac{\delta + 2L}{1 - L} \in (0,1)$

(2) $\| T^n x - T^{n-1}x \| \leq \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - TT^{n-2}x \|$

$\leq \delta \| T^{n-1}x - T^{n-2}x \| + L \| T^{n-2}x - T^{n-1}x \|$

$\leq (\delta + L) \| T^{n-1}x - T^{n-2}x \|$

then $(\delta+L)\le\frac{\delta+L}{1-L}<\frac{\delta+2L}{1-L}$

(3) $\|T^n x-T^{n-1}x\|\le\delta\|T^{n-1}x-T^{n-2}x\|+L\|T^{n-2}x-TT^{n-1}x\|$

$\le\delta\|T^{n-1}-T^{n-2}x\|+L\|T^{n-2}x-T^n x\|$

$\le\delta\|T^{n-1}x-T^{n-2}x\|+L\|T^{n-2}x-T^{n-1}x\|+L\|T^{n-1}x-T^n x\|$

$\le(\delta+L)\|T^{n-1}x-T^{n-2}\|+L\|T^{n-1}x-T^n x\|$

$\|T^{n-1}x-T^n x\|\le\frac{(\delta+L)}{1-L}\|T^{n-1}x-T^{n-2}x\|$

$\le w\|T^{n-1}x-T^{n-2}\|, w\in\frac{\delta+L}{1-L}\in(0,1)$

(4) $\|T^n x-T^{n-1}x\|\le\delta\|T^{n-1}x-T^{n-2}x\|+L\|T^{n-1}x-T^{n-1}x\|$

$\le\delta\|T^{n-1}x-T^{n-2}x\|+L.0$

then (2.1) *is hold*

Hence, in all the possible cases it follows that (2.1) holds, and so let

$\Lambda=\max\{\delta+L,\frac{\delta+2L}{1-L}\}$

$\|T^n x-T^{n-1}x\|\le\Lambda\|T^{n-1}x-T^{n-2}\|=\frac{\delta+2L}{1-L}\|T^{n-1}x-T^{n-2}x\|$.....(2.2)

Byinductionandusing(2.1) and (2.2)

$\|T^n x-T^{n-1}x\|\le(\delta+L)\|T^{n-1}x-T^{n-2}\|\le\ldots\le(\delta+L)^{n-1}\|Tx-x\|$...(2.3)

Usingthetriangleinequalityand(2.3)wegetthatforn > m

$\|T^n x-T^m x\|\le\|T^n x-T^{n-1}x\|+\|T^{n-1}x-T^{n-2}x\|+\ldots+\|T^{m+1}x-T^m x\|$

$\le[(\delta+L)^{n-1}+(\delta+L)^{n-2}+\ldots+(\delta+L)^m]\|Tx-x\|$

$=(\delta+L)^m\frac{1-(\delta+L)^{n-m-1}}{1-L}\|Tx-x\|\le\frac{(\delta+2L)^m}{1-L}\|Tx-x\|\to 0$

as $m\to\infty$

since $\frac{(\delta+2L)^m}{1-L}\|Tx-x\|\to 0$ *as* $m,n\to\infty$ is follow that

$\frac{(\delta+2L)^m}{1-L}\|Tx-x\|=$ where $\|T^n x-T^m x\|=c$ for each $c\in intP$

we have proved that the sequence $\{T^n x\}$ is Cauchy sequence for the fixed $x\in X$ since the $(X,\|.\|)$ is complete, there exists a point $p\in X$ which is the limit of this sequence be lemma(2.2) we conclude that p is fixed point

Theorem 2.5 *Let* $T:X\to X$ *satisfy condition* B *on a cone Banach space. If* $intp\ne\phi$ *then* T *is continues at its fixed point*

Proof. Let z the unique fixed point and let $X_n\to z$ where $\{x_n\}$ is a sequence of point form the given cone Banach space to show that T is continues we must prove that $Tx_n\to Tz=z$ we have

$$\|Tx_n-Tz\|\le\delta\|x_n-z\|+L.u$$

where

$u\in\{\|z_n-Tx_n\|,\|z-Tz\|,\|x_n-Tz\|,\|z-Tx_n\|\}$

$=\{0,\|x_n-T_n\|,\|x_n-z\|,\|z-Tx_n\|\}$

similarly as the previous proofs of this kind, must be considered the following cases

(1) $\|Tx_n-Tz\|\le\delta\|x_n-z\|+L.0$

$\Rightarrow Tx_n=Tz\Rightarrow Tx_n\to Tz=z$

(2) $\|Tx_n-Tz\|\le\delta\|x_n-z\|+L\|x_n-z\|$

$=(\delta+l)\frac{c}{(\delta+L)}=c$

(3) $\|Tx_n-Tz\|\le\delta\|x_n-z\|+L\|x_n-Tx_n\|$

$\le\delta\|x_n-z\|+L\|x_n-z\|+L\|Tx_n-z\|$

$\le(\delta+L)\|x_n-z\|+L\|x-Tx_n\|$

$\le\frac{\delta+L}{1-L}\|x_n-z\|$

$\le\frac{\delta+L}{1-L}.\frac{c}{\frac{\delta+L}{1-L}}=c$

(4) $\|Tx_n-Tz\|\le\delta\|x_n-z\|+L\|Tx_n-z\|$

$=\delta\|x_n-z\|+L\|Tx_n-Tz\|$

$\Rightarrow\|Tx_n-Tz\|=0$

$\Rightarrow Tx_n\to Tz=z$

denote as usual by $F(T)$ the set of fixed point of the mapping $T:X\to X$ it is said that the map T has property P if $F(T)=F(T^n)$ for each $n\in N$

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