

Fixed Point in Cone Banach Space

Buthainah A.A. Ahmed¹, Essraa A. Hasan²

Department of Mathematics -College of Science -University of Baghdad

Abstract: Let $(X, \|\cdot\|_p)$ be a cone Banach space and T a self map on X . We say that T satisfy condition B if there exists $0 < \delta < 1$ and $L > 0$ such that $\|T_x - T_y\| < \delta \|x - y\| + L.u$ where $u \in \{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}$ in this paper we have the existences and uniqueness theorems for a fixed point to a map T that satisfy condition B . We also prove that under a certain condition every map T that satisfy condition B is continuous.

1. Introduction

Many authors study fixed point theorems for a mapping is a cone metric space satisfying different contraction condition for example Sh.Rezapour, R.Hamlbarani[4] prove fixed point theorem for contractive mapping. Must these theorem extended to a cone Banach space after some modification, Edral karapiner[2] prove fixed point theorem in cone Banach space while R.krishnakumar and D.Dharnodharan[3] prove fixed point theorem in cone Banach space by using Φ operator

The following definition and results will be needed in the sequel Let E be a real Banach space. A subset p of E is called cone if

- (1) p is closed, nonempty and $p \neq \{0\}$
- (2) $a, b \in R, a, b \geq 0$ and $x, y \in p$ imply $ax + by \in p$
- (3) $p \cap (-p) = \{0\}$

Definition 1.1 [2] Let X be a vector space over R , suppose the mapping $\|\cdot\|_p: X \rightarrow E$ satisfies

- (1) $\|x\|_p > 0$ for all $x \in X$
- (2) $\|x\|_p = 0$ if and only if $x = 0$
- (3) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for all $x, y \in X$
- (4) $\|kx\|_p = |k| \|x\|_p$ for all $k \in R$

then $\|\cdot\|_p$ is called a cone norm on X and the pair $(X, \|\cdot\|_p)$ is called a cone normed space. (CNS)

Definition 1.2 [2] Let $(X, \|\cdot\|_p)$ be a cone normed space $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X , then

- (1) $\{x_n\}_{n \geq 1}$ converge to x whenever for each $c \in E$ with $0 = c$ there is natural number n such that $\|x_n - x\|_p = c$ for all $n \in N$ it is denoted be $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$
- (2) $\{x_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 = c$ there is a natural number n , such that $\|x_n - x_m\|_p = c$ for all $n, m \in N$

(3) $(X, \|\cdot\|_p)$ is complete cone normd space i.e. every Cauchy sequence is convergent complete cone normed space will be called cone Banach space.

Lemma 1.3 [1] Let $(X, \|\cdot\|_p)$ be a cone Banach space then the following properties are often used

- h_1) if $u \leq v$ and $v = w$ then $u = w$
- h_2) if $0 \leq u = c$ for each $c \in \text{int}p$ then $u = 0$

2. Main Result

Defintion 2.1 Let $(X, \|\cdot\|_p)$ be a cone Banach space a map $T: X \rightarrow X$ is said to be satisfy condition B if there exists $0 < \delta < 1$ and $L > 0$ such that for all $x, y \in X$ we have

$$\|Tx - Ty\| < \delta \|x - y\| + L.u$$

when

$$u \in \{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}$$

Lemma 2.2 Let $(X, \|\cdot\|_p)$ be a cone Banach space $T: X \rightarrow X$ satisfy condition B if for some point x of the sequence $\{T^n x\}$ of picard iteration converge to point $z \in X$ then z is it's fixed point

Proof. by definition of condition B and the triangle inequality we get

$$\begin{aligned} \|z - Tz\| &\leq \|z - T^{n+1}x\| + \|T^{n+1}x - Tz\| \\ &= \|z - T^{n+1}x\| + \|TT^n x - Tz\| \\ &\leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L.u \end{aligned}$$

where

$$u \in \{\|T^n - TT^n x\|, \|z - Tz\|, \|T^n x - Tz\|, \|z - TT^n x\|\}$$

let $0 = c$, clearly at least one of the following four cases holds for infinitely many $n \in N$

$$\begin{aligned} (1) \quad &\|T^n x - z\| \leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L \|T^n x - TT^n x\| \\ &\leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L \|T^n x - z\| + L \|z - T^{n+1}x\| \\ &\leq \|z - T^{n+1}x\| + (\delta + L) \|T^n x - z\| + L \|z - T^{n+1}x\| \end{aligned}$$

$$\begin{aligned} &\leq \|(1+L)\|z - T^{n+1}x\| + (\delta + L)\|T^n x - z\| \\ &\|T^n x - z\| \leq \frac{(1+L)}{1 - (\delta + L)} \|z - T^{n+1}x\| \\ &\leq \frac{(1+L)}{1 - (\delta + L)} \cdot \frac{c}{1+l} = c \\ (2) \quad &\|T^n x - z\| \leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + l \|z - Tz\| \\ &\|z - T^n x\| \leq \frac{1}{\delta - L} \|z - T^{n+1}x\| = \frac{1}{\delta - L} \cdot \frac{c}{1} = c \\ (3) \quad &\|T^n x - z\| \leq \|z - T^{n+1}x\| + \delta \|T^n x - x\| + L \|z - TT^n x\| \\ &\leq (1+L)\|z - T^{n+1}x\| = \delta \|T^n x - z\| = \frac{(1+L)}{1 - \delta} \cdot \frac{c}{(1+l)} = c \\ (4) \quad &\|T^n x - z\| \leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L \|T^n - Tz\| \\ &\leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L \|T^n x - z\| + L \|z - Tz\| \\ &\leq \|z - T^{n+1}x\| + (\delta + L)\|T^n x - z\| + L \|z - Tz\| \\ &\leq \frac{1}{1 - (\delta + L)} \|z - T^{n+1}x\| + \frac{L}{1 - (\delta + L)} \|z - Tz\| \\ &= \frac{1}{1 - (\delta + L)} \cdot \frac{c}{2 \frac{1}{1 - (\delta + l)}} + \frac{L}{1 - (\delta + L)} \cdot \frac{c}{2 \frac{L}{1 - (\delta + L)}} = c \end{aligned}$$

we have found that all cases $\|T^n x - x\| = c$ for each interior point c of the cone P . By (h_2) , it follows that $\|T^n x - z\| = 0$ i.e. $T^n x = z$.

Now we give the main theorem that gave the condition which granted existence and uniqueness of the fixed point for a map

$$\|T^n x - T^{n-1}x\| \leq \frac{\delta + 2L}{1 - L} \|T^{n-1}x - T^{n-2}x\| \dots n \geq 2 \quad \dots (2.1)$$

indeed $\|T^n x - T^{n-1}x\| \leq \delta \|T^{n-1}x - T^{n-2}x\| + Lu$

where

$$u \in \{\|T^{n-1}x - TT^{n-1}x\|, \|T^{n-2}x - TT^{n-2}x\|, \|T^{n-2}x - TT^{n-1}x\|, \|T^{n-1}x - TT^{n-2}x\|\}$$

$$\begin{aligned} (1) \quad &\|T^n x - TT^{n-1}x\| \leq \delta \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-1}x - TT^{n-1}x\| \\ &\leq \delta \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-1}x - T^n x\| \\ &\leq \delta \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - T^n x\| \\ &\leq (\delta + L)\|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - T^n x\| \\ &\leq (\delta + L)\|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - T^{n-1}x\| + L \|T^{n-1}x - T^n x\| \\ &\leq (\delta + 2L)\|T^{n-1}x - T^{n-2}x\| + L \|T^{n-1}x - T^n x\| \end{aligned}$$

T that satisfies condition B , but first we need the following lemma.

Lemma 2.3 Let $(X, \|\cdot\|)$ be a cone Banach space and $T: X \rightarrow X$ satisfying condition B , if $x \in X$ and let $x \neq Tx$ then $\|T^2x - Tx\| < \|Tx - x\|$

Proof. Let $x \in X$ and let $x \neq Tx$ then

$$\|T^2x - Tx\| = \|TTx - Tx\| \leq \delta \|Tx - x\| + Lu$$

$$u \in \{\|Tx - T^2x\|, \|x - Tx\|, \|Tx - Tx\|, \|x - Tx\|\}$$

$$u \in \{\|Tx - T^2x\|, \|z - T^2x\|, 0\}$$

we will get the following possibilities

$$(1) \quad \|T^2x - Tx\| \leq \delta \|Tx - x\| + L \cdot 0 = 0 < \|Tx - x\| \text{ because } x \neq Tx$$

$$(2) \quad \|T^2x - Tx\| \leq \delta \|Tx - x\| + L \|Tx - T^2x\| = 0 < \|Tx - x\| \text{ because } x \neq Tx$$

$$(3) \quad \|T^2x - Tx\| \leq \delta \|Tx - x\| + L \|x - T^2x\|$$

$$\leq \delta \|Tx - x\| + L \|x - Tx\| + L \|Tx - T^2x\|$$

$$\|T^2x - Tx\| \leq \frac{\delta + L}{1_L} \|x - Tx\| < \|Tx - x\|$$

because $\frac{\delta + L}{1 - L} < 1 \Leftarrow L \in (0, \frac{1}{3})$ and $\delta \in (0, \frac{1}{3})$ and $x \neq Tx$

Theorem 2.4 Suppose that $(X, \|\cdot\|)$ is a cone Banach space with a cone P such that $\text{int}P \neq \emptyset$ and $T: X \rightarrow X$ that satisfy condition B then T has a unique fixed point

Proof. Let $x \in X$ we shall show that $\{Tx\}$ is a Cauchy sequence, but first we prove that

$$\Rightarrow \|T^{n-1}x - T^n x\| \leq \frac{(\delta + 2L)}{(1 - L)} \|T^{n-1}x - T^{n-2}x\|$$

$$\leq w \|T^{n-1}x - T^{n-2}x\|, w = \frac{\delta + 2L}{1 - L} \in (0, 1)$$

$$(2) \quad \|T^n x - T^{n-1}x\| \leq \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - TT^{n-2}x\|$$

$$\leq \delta \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - T^{n-1}x\|$$

$$\leq (\delta + L)\|T^{n-1}x - T^{n-2}x\|$$

$$\text{then } (\delta + L) \leq \frac{\delta + L}{1 - L} < \frac{\delta + 2L}{1 - L}$$

$$\begin{aligned} (3) \quad & \|T^n x - T^{n-1} x\| \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-2} x - T^{n-1} x\| \\ & \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-2} x - T^{n-1} x\| \\ & \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-2} x - T^{n-1} x\| + L \|T^{n-1} x - T^n x\| \\ & \leq (\delta + L) \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-1} x - T^n x\| \\ & \|T^{n-1} x - T^n x\| \leq \frac{(\delta + L)}{1 - L} \|T^{n-1} x - T^{n-2} x\| \\ & \leq w \|T^{n-1} x - T^{n-2} x\|, w \in \frac{\delta + L}{1 - L} \in (0, 1) \end{aligned}$$

$$\begin{aligned} (4) \quad & \|T^n x - T^{n-1} x\| \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-1} x - T^n x\| \\ & \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \cdot 0 \end{aligned}$$

then (2.1) is hold

Hence, in all the possible cases it follows that (2.1) holds, and so let

$$\Lambda = \max\left\{\delta + L, \frac{\delta + 2L}{1 - L}\right\}$$

$$\|T^n x - T^{n-1} x\| \leq \Lambda \|T^{n-1} x - T^{n-2} x\| = \frac{\delta + 2L}{1 - L} \|T^{n-1} x - T^{n-2} x\| \dots (2.2)$$

By induction and using (2.1) and (2.2)

$$\|T^n x - T^{n-1} x\| \leq (\delta + L) \|T^{n-1} x - T^{n-2} x\| \leq \dots \leq (\delta + L)^{n-1} \|T x - x\| \dots (2.3)$$

Using the triangle inequality and (2.3) we get that for $m > n$

$$\begin{aligned} \|T^m x - T^n x\| & \leq \|T^m x - T^{m-1} x\| + \|T^{m-1} x - T^{m-2} x\| + \dots + \|T^n x - T^{n-1} x\| \\ & \leq [(\delta + L)^{m-1} + (\delta + L)^{m-2} + \dots + (\delta + L)^n] \|T x - x\| \\ & = (\delta + L)^m \frac{1 - (\delta + L)^{n-m+1}}{1 - L} \|T x - x\| \leq \frac{(\delta + 2L)^m}{1 - L} \|T x - x\| \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$

$$\text{since } \frac{(\delta + 2L)^m}{1 - L} \|T x - x\| \rightarrow 0 \text{ as } m, n \rightarrow \infty \text{ is}$$

follow that

$$\frac{(\delta + 2L)^m}{1 - L} \|T x - x\| = c \text{ where } \|T^n x - T^m x\| = c \text{ for each}$$

$c \in \text{int}P$

we have proved that the sequence $\{T^n x\}$ is Cauchy sequence for the fixed $x \in X$ since the $(X, \|\cdot\|)$ is complete, there exists a point $p \in X$ which is the limit of this sequence by lemma (2.2) we conclude that p is fixed point

Theorem 2.5 Let $T : X \rightarrow X$ satisfy condition B on a cone Banach space. If $\text{int}P \neq \emptyset$ then T is continuous at its fixed point

Proof. Let z the unique fixed point and let $X_n \rightarrow z$ where $\{x_n\}$ is a sequence of point from the given cone

Banach space to show that T is continuous we must prove that $Tx_n \rightarrow Tz = z$ we have

$$\|Tx_n - Tz\| \leq \delta \|x_n - z\| + L u$$

where

$$\begin{aligned} u \in & \{\|z_n - Tx_n\|, \|z - Tz\|, \|x_n - Tz\|, \|z - Tx_n\|\} \\ & = \{0, \|x_n - Tz\|, \|x_n - z\|, \|z - Tx_n\|\} \end{aligned}$$

similarly as the previous proofs of this kind, must be considered the following cases

$$(1) \quad \|Tx_n - Tz\| \leq \delta \|x_n - z\| + L \cdot 0$$

$$\Rightarrow Tx_n = Tz \Rightarrow Tx_n \rightarrow Tz = z$$

$$(2) \quad \|Tx_n - Tz\| \leq \delta \|x_n - z\| + L \|x_n - z\|$$

$$= (\delta + L) \frac{c}{(\delta + L)} = c$$

$$(3) \quad \|Tx_n - Tz\| \leq \delta \|x_n - z\| + L \|x_n - Tx_n\|$$

$$\leq \delta \|x_n - z\| + L \|x_n - z\| + L \|Tx_n - z\|$$

$$\leq (\delta + L) \|x_n - z\| + L \|x - Tx_n\|$$

$$\leq \frac{\delta + L}{1 - L} \|x_n - z\|$$

$$\leq \frac{\delta + L}{1 - L} \cdot \frac{c}{\delta + L} = c$$

$$(4) \quad \|Tx_n - Tz\| \leq \delta \|x_n - z\| + L \|Tx_n - z\|$$

$$= \delta \|x_n - z\| + L \|Tx_n - Tz\|$$

$$\Rightarrow \|Tx_n - Tz\| = 0$$

$$\Rightarrow Tx_n \rightarrow Tz = z$$

denote as usual by $F(T)$ the set of fixed point of the mapping $T : X \rightarrow X$ it is said that the map T has property P if $F(T) = F(T^n)$ for each $n \in N$

References

- [1] Zoran Kadelburg, Stojan Radenović, and Vladimir Rakoćević, *Remarks on α -quasi-contraction on a cone metric space*, Applied Mathematics Letters **22** (2009), no. 11, 1674–1679.
- [2] Erdal Karapınar, *Fixed point theorems in cone Banach spaces*, Hindawi Publishing Corporation Fixed Point Theory and Applications **2009**, 9.
- [3] R Krishnakumar and D Dhamodharan, *Some fixed point theorems in cone Banach spaces using ϕ operator*, **55** (2016), 7.
- [4] Sh Rezapour and R Hamlbarani, *Some notes on the paper α -cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications **345** (2008), no. 2, 719–724.