

# Fixed Point in Cone Banach Space

Buthainah A.A. Ahmed<sup>1</sup>, Essraa A. Hasan<sup>2</sup>

Department of Mathematics -College of Science -University of Baghdad

**Abstract:** Let  $(X, \|\cdot\|_p)$  be a cone Banach space and  $T$  a self map on  $X$ . We say that  $T$  satisfy condition  $B$  if there exists  $0 < \delta < 1$  and  $L > 0$  such that  $\|T_x - T_y\| < \delta \|x - y\| + L.u$  where  $u \in \{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}$  in this paper we have the existences and uniqueness theorems for a fixed point to a map  $T$  that satisfy condition  $B$ . We also prove that under a certain condition every map  $T$  that satisfy condition  $B$  is continuous.

## 1. Introduction

Many authors study fixed point theorems for a mapping is a cone metric space satisfying different contraction condition for example Sh.Rezapour, R.Hamlbarani[4] prove fixed point theorem for contractive mapping. Must these theorem extended to a cone Banach space after some modification, Edral karapiner[2] prove fixed point theorem in cone Banach space while R.krishnakumar and D.Dharnodharan[3] prove fixed point theorem in cone Banach space by using  $\Phi$  operator

The following definition and results will be needed in the sequel Let  $E$  be a real Banach space. A subset  $p$  of  $E$  is called cone if

- (1)  $p$  is closed, nonempty and  $p \neq \{0\}$
- (2)  $a, b \in R, a, b \geq 0$  and  $x, y \in p$  imply  $ax + by \in p$
- (3)  $p \cap (-p) = \{0\}$

**Definition 1.1** [2] Let  $X$  be a vector space over  $R$ , suppose the mapping  $\|\cdot\|_p: X \rightarrow E$  satisfies

- (1)  $\|x\|_p > 0$  for all  $x \in X$
- (2)  $\|x\|_p = 0$  if and only if  $x = 0$
- (3)  $\|x + y\|_p \leq \|x\|_p + \|y\|_p$  for all  $x, y \in X$
- (4)  $\|kx\|_p = |k| \|x\|_p$  for all  $k \in R$

then  $\|\cdot\|_p$  is called a cone norm on  $X$  and the pair  $(X, \|\cdot\|_p)$  is called a cone normed space. (CNS)

**Definition 1.2** [2] Let  $(X, \|\cdot\|_p)$  be a cone normed space  $x \in X$  and  $\{x_n\}_{n \geq 1}$  a sequence in  $X$ , then

- (1)  $\{x_n\}_{n \geq 1}$  converge to  $x$  whenever for each  $c \in E$  with  $0 = c$  there is natural number  $n$  such that  $\|x_n - x\|_p = c$  for all  $n \in N$  it is denoted be  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$
- (2)  $\{x_n\}_{n \geq 1}$  is a Cauchy sequence whenever for every  $c \in E$  with  $0 = c$  there is a natural number  $n$ , such that  $\|x_n - x_m\|_p = c$  for all  $n, m \in N$

(3)  $(X, \|\cdot\|_p)$  is complete cone normd space i.e. every Cauchy sequence is convergent complete cone normed space will be called cone Banach space.

**Lemma 1.3** [1] Let  $(X, \|\cdot\|_p)$  be a cone Banach space then the following properties are often used

- $h_1$ ) if  $u \leq v$  and  $v = w$  then  $u = w$
- $h_2$ ) if  $0 \leq u = c$  for each  $c \in \text{int}p$  then  $u = 0$

## 2. Main Result

**Defintion 2.1** Let  $(X, \|\cdot\|_p)$  be a cone Banach space a map  $T: X \rightarrow X$  is said to be satisfy condition  $B$  if there exists  $0 < \delta < 1$  and  $L > 0$  such that for all  $x, y \in X$  we have

$$\|Tx - Ty\| < \delta \|x - y\| + L.u$$

when  $u \in \{\|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}$

**Lemma 2.2** Let  $(X, \|\cdot\|_p)$  be a cone Banach space  $T: X \rightarrow X$  satisfy condition  $B$  if for some point  $x$  of the sequence  $\{T^n x\}$  of picard iteration converge to point  $z \in X$  then  $z$  is it's fixed point

*Proof.* by definition of condition  $B$  and the triangle inequality we get

$$\begin{aligned} \|z - Tz\| &\leq \|z - T^{n+1}x\| + \|T^{n+1}x - Tz\| \\ &= \|z - T^{n+1}x\| + \|TT^n x - Tz\| \\ &\leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L.u \end{aligned}$$

where  $u \in \{\|T^n - TT^n x\|, \|z - Tz\|, \|T^n x - Tz\|, \|z - TT^n x\|\}$

let  $0 = c$ , clearly at least one of the following four cases holds for infinitely many  $n \in N$

$$\begin{aligned} (1) \quad &\|T^n x - z\| \leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L \|T^n x - TT^n x\| \\ &\leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L \|T^n x - z\| + L \|z - T^{n+1}x\| \\ &\leq \|z - T^{n+1}x\| + (\delta + L) \|T^n x - z\| + L \|z - T^{n+1}x\| \end{aligned}$$

$$\begin{aligned} &\leq \|(1+L)\|z - T^{n+1}x\| + (\delta + L)\|T^n x - z\| \\ &\|T^n x - z\| \leq \frac{(1+L)}{1 - (\delta + L)} \|z - T^{n+1}x\| \\ &\leq \frac{(1+L)}{1 - (\delta + L)} \cdot \frac{c}{1+l} = c \\ (2) \quad &\|T^n x - z\| \leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + l \|z - Tz\| \\ &\|z - T^n x\| \leq \frac{1}{\delta - L} \|z - T^{n+1}x\| = \frac{1}{\delta - L} \cdot \frac{c}{1} = c \\ (3) \quad &\|T^n x - z\| \leq \|z - T^{n+1}x\| + \delta \|T^n x - x\| + L \|z - TT^n x\| \\ &\leq (1+L)\|z - T^{n+1}x\| = \delta \|T^n x - z\| = \frac{(1+L)}{1 - \delta} \cdot \frac{c}{(1+\delta)} = c \\ (4) \quad &\|T^n x - z\| \leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L \|T^n - Tz\| \\ &\leq \|z - T^{n+1}x\| + \delta \|T^n x - z\| + L \|T^n x - z\| + L \|z - Tz\| \\ &\leq \|z - T^{n+1}x\| + (\delta + L)\|T^n x - z\| + L \|z - Tz\| \\ &\leq \frac{1}{1 - (\delta + L)} \|z - T^{n+1}x\| + \frac{L}{1 - (\delta + L)} \|z - Tz\| \\ &= \frac{1}{1 - (\delta + L)} \cdot \frac{c}{2 \frac{1}{1 - (\delta + l)}} + \frac{L}{1 - (\delta + L)} \cdot \frac{c}{2 \frac{L}{1 - (\delta + L)}} = c \end{aligned}$$

we have found that all cases  $\|T^n x - x\| = c$  for each interior point  $c$  of the cone  $P$ . By  $(h_2)$ , it follows that  $\|T^n x - z\| = 0$  i.e.  $T^n x = z$ .

Now we give the main theorem that gave the condition which granted existence and uniqueness of the fixed point for a map

$$\|T^n x - T^{n-1}x\| \leq \frac{\delta + 2L}{1 - L} \|T^{n-1}x - T^{n-2}x\| \dots n \geq 2 \quad \dots (2.1)$$

indeed  $\|T^n x - T^{n-1}x\| \leq \delta \|T^{n-1}x - T^{n-2}x\| + Lu$

where

$$u \in \{\|T^{n-1}x - TT^{n-1}x\|, \|T^{n-2}x - TT^{n-2}x\|, \|T^{n-2}x - TT^{n-1}x\|, \|T^{n-1}x - TT^{n-2}x\|\}$$

$$\begin{aligned} (1) \quad &\|T^n x - TT^{n-1}x\| \leq \delta \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-1}x - TT^{n-1}x\| \\ &\leq \delta \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-1}x - T^n x\| \\ &\leq \delta \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - T^n x\| \\ &\leq (\delta + L)\|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - T^n x\| \\ &\leq (\delta + L)\|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - T^{n-1}x\| + L \|T^{n-1}x - T^n x\| \\ &\leq (\delta + 2L)\|T^{n-1}x - T^{n-2}x\| + L \|T^{n-1}x - T^n x\| \end{aligned}$$

$T$  that satisfies condition  $B$ , but first we need the following lemma.

**Lemma 2.3** Let  $(X, \|\cdot\|)$  be a cone Banach space and  $T: X \rightarrow X$  satisfying condition  $B$ , if  $x \in X$  and let  $x \neq Tx$  then  $\|T^2x - Tx\| < \|Tx - x\|$

*Proof.* Let  $x \in X$  and let  $x \neq Tx$  then

$$\|T^2x - Tx\| = \|TTx - Tx\| \leq \delta \|Tx - x\| + Lu$$

$$u \in \{\|Tx - T^2x\|, \|x - Tx\|, \|Tx - Tx\|, \|x - Tx\|\}$$

$$u \in \{\|Tx - T^2x\|, \|z - T^2x\|, 0\}$$

we will get the following possibilities

$$(1) \quad \|T^2x - Tx\| \leq \delta \|Tx - x\| + L \cdot 0 = 0 < \|Tx - x\| \text{ because } x \neq Tx$$

$$(2) \quad \|T^2x - Tx\| \leq \delta \|Tx - x\| + L \|Tx - T^2x\| = 0 < \|Tx - x\| \text{ because } x \neq Tx$$

$$(3) \quad \|T^2x - Tx\| \leq \delta \|Tx - x\| + L \|x - T^2x\| \leq \delta \|Tx - x\| + L \|x - Tx\| + L \|Tx - T^2x\|$$

$$\|T^2x - Tx\| \leq \frac{\delta + L}{1_L} \|x - Tx\| < \|Tx - x\|$$

because  $\frac{\delta + L}{1 - L} < 1 \Leftarrow L \in (0, \frac{1}{3})$  and  $\delta \in (0, \frac{1}{3})$  and  $x \neq Tx$

**Theorem 2.4** Suppose that  $(X, \|\cdot\|)$  is a cone Banach space with a cone  $P$  such that  $\text{int}P \neq \emptyset$  and  $T: X \rightarrow X$  that satisfy condition  $B$  then  $T$  has a unique fixed point

*Proof.* Let  $x \in X$  we shall show that  $\{Tx\}$  is a Cauchy sequence, but first we prove that

$$\Rightarrow \|T^{n-1}x - T^n x\| \leq \frac{(\delta + 2L)}{(1 - L)} \|T^{n-1}x - T^{n-2}x\|$$

$$\leq w \|T^{n-1}x - T^{n-2}x\|, w = \frac{\delta + 2L}{1 - L} \in (0, 1)$$

$$(2) \quad \|T^n x - T^{n-1}x\| \leq \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - TT^{n-2}x\|$$

$$\leq \delta \|T^{n-1}x - T^{n-2}x\| + L \|T^{n-2}x - T^{n-1}x\|$$

$$\leq (\delta + L)\|T^{n-1}x - T^{n-2}x\|$$

$$\text{then } (\delta + L) \leq \frac{\delta + L}{1 - L} < \frac{\delta + 2L}{1 - L}$$

$$\begin{aligned} (3) \quad & \|T^n x - T^{n-1} x\| \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-2} x - T^{n-1} x\| \\ & \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-2} x - T^{n-1} x\| \\ & \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-2} x - T^{n-1} x\| + L \|T^{n-1} x - T^n x\| \\ & \leq (\delta + L) \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-1} x - T^n x\| \\ & \|T^{n-1} x - T^n x\| \leq \frac{(\delta + L)}{1 - L} \|T^{n-1} x - T^{n-2} x\| \\ & \leq w \|T^{n-1} x - T^{n-2} x\|, w \in \frac{\delta + L}{1 - L} \in (0, 1) \end{aligned}$$

$$\begin{aligned} (4) \quad & \|T^n x - T^{n-1} x\| \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \|T^{n-1} x - T^n x\| \\ & \leq \delta \|T^{n-1} x - T^{n-2} x\| + L \cdot 0 \end{aligned}$$

then (2.1) is hold

Hence, in all the possible cases it follows that (2.1) holds, and so let

$$\Lambda = \max\left\{\delta + L, \frac{\delta + 2L}{1 - L}\right\}$$

$$\|T^n x - T^{n-1} x\| \leq \Lambda \|T^{n-1} x - T^{n-2} x\| = \frac{\delta + 2L}{1 - L} \|T^{n-1} x - T^{n-2} x\| \dots (2.2)$$

By induction and using (2.1) and (2.2)

$$\|T^n x - T^{n-1} x\| \leq (\delta + L) \|T^{n-1} x - T^{n-2} x\| \leq \dots \leq (\delta + L)^{n-1} \|T x - x\| \dots (2.3)$$

Using the triangle inequality and (2.3) we get that for  $m > n$

$$\begin{aligned} \|T^m x - T^n x\| & \leq \|T^m x - T^{m-1} x\| + \|T^{m-1} x - T^{m-2} x\| + \dots + \|T^n x - T^{n-1} x\| \\ & \leq [(\delta + L)^{m-1} + (\delta + L)^{m-2} + \dots + (\delta + L)^n] \|T x - x\| \\ & = (\delta + L)^m \frac{1 - (\delta + L)^{n-m+1}}{1 - L} \|T x - x\| \leq \frac{(\delta + 2L)^m}{1 - L} \|T x - x\| \rightarrow 0 \end{aligned}$$

as  $m \rightarrow \infty$

$$\text{since } \frac{(\delta + 2L)^m}{1 - L} \|T x - x\| \rightarrow 0 \text{ as } m, n \rightarrow \infty \text{ is}$$

follow that

$$\frac{(\delta + 2L)^m}{1 - L} \|T x - x\| = c \text{ where } \|T^n x - T^m x\| = c \text{ for each}$$

$c \in \text{int}P$

we have proved that the sequence  $\{T^n x\}$  is Cauchy sequence for the fixed  $x \in X$  since the  $(X, \|\cdot\|)$  is complete, there exists a point  $p \in X$  which is the limit of this sequence by lemma (2.2) we conclude that  $p$  is fixed point

**Theorem 2.5** Let  $T : X \rightarrow X$  satisfy condition  $B$  on a cone Banach space. If  $\text{int}P \neq \emptyset$  then  $T$  is continuous at its fixed point

*Proof.* Let  $z$  the unique fixed point and let  $X_n \rightarrow z$  where  $\{x_n\}$  is a sequence of point from the given cone

Banach space to show that  $T$  is continuous we must prove that  $T x_n \rightarrow T z = z$  we have

$$\|T x_n - T z\| \leq \delta \|x_n - z\| + L u$$

where

$$\begin{aligned} u \in & \{\|z_n - T x_n\|, \|z - T z\|, \|x_n - T z\|, \|z - T x_n\|\} \\ & = \{0, \|x_n - T_n\|, \|x_n - z\|, \|z - T x_n\|\} \end{aligned}$$

similarly as the previous proofs of this kind, must be considered the following cases

$$(1) \quad \|T x_n - T z\| \leq \delta \|x_n - z\| + L \cdot 0$$

$$\Rightarrow T x_n = T z \Rightarrow T x_n \rightarrow T z = z$$

$$(2) \quad \|T x_n - T z\| \leq \delta \|x_n - z\| + L \|x_n - z\|$$

$$= (\delta + L) \frac{c}{(\delta + L)} = c$$

$$(3) \quad \|T x_n - T z\| \leq \delta \|x_n - z\| + L \|x_n - T x_n\|$$

$$\leq \delta \|x_n - z\| + L \|x_n - z\| + L \|T x_n - z\|$$

$$\leq (\delta + L) \|x_n - z\| + L \|x - T x_n\|$$

$$\leq \frac{\delta + L}{1 - L} \|x_n - z\|$$

$$\leq \frac{\delta + L}{1 - L} \cdot \frac{c}{\delta + L} = c$$

$$(4) \quad \|T x_n - T z\| \leq \delta \|x_n - z\| + L \|T x_n - z\|$$

$$= \delta \|x_n - z\| + L \|T x_n - T z\|$$

$$\Rightarrow \|T x_n - T z\| = 0$$

$$\Rightarrow T x_n \rightarrow T z = z$$

denote as usual by  $F(T)$  the set of fixed point of the mapping  $T : X \rightarrow X$  it is said that the map  $T$  has property  $P$  if  $F(T) = F(T^n)$  for each  $n \in N$

## References

- [1] Zoran Kadelburg, Stojan Radenović, and Vladimir Rakoćević, *Remarks on  $\alpha$ -quasi-contraction on a cone metric space*, Applied Mathematics Letters **22** (2009), no. 11, 1674–1679.
- [2] Erdal Karapınar, *Fixed point theorems in cone Banach spaces*, Hindawi Publishing Corporation Fixed Point Theory and Applications **2009**, 9.
- [3] R Krishnakumar and D Dhamodharan, *Some fixed point theorems in cone Banach spaces using  $\phi$  operator*, **55** (2016), 7.
- [4] Sh Rezapour and R Hamlbarani, *Some notes on the paper  $\alpha$ -cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications **345** (2008), no. 2, 719–724.