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The Linear Delay Fourth Order Eigen-Value Problems Solved By the Collocation Method

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Abstract: ***The main objective of this paper is to generalize the study of the linear fourth order eigen-value problems to be hold for the linear delay fourth order eigen-value problems, also to find the nontrivial solution of the introduced type of problems by using the collocation method.***

Keywords: Eventuality trait, fuzzy eventuality trait, firm casual stringy differential dissimilarity, fuzzy firm casual stringy differential dissimilarity

1. Introduction

This paper is devoted to study a special type of the linear delay fourth order eigen-value problems, namely the Sturm-Liouville problems to be considered as the linear delay fourth order eigen-value problems. One of the most important weighted residual methods namely the collocation method has been used to solve the considerable type of problems.

2. Basic Concepts

In this section, the basic concepts needed in this work are introduced, we start with the following definition that recalled by Baker, Paul and Wille, [1].

Definition 2.1
The delay differential equation is an equation that the unknown function or its derivative evaluated at arguments that differ by any fixed number of values.

While the following definition is given by Rama and Rao, [2].

Definition 2.2
The order of the delay differential equations is the order of the highest derivative involved in them, and the degree of delay differential equations is the degree of the highest order derivative that appeared in them.

Also, the following definition introduced by Brauer and Nohel, [3].

Definition 2.3
The delay differential equations are said to be linear in case they are linear with respect to the unknown functions that enter with different arguments and their derivatives that appeared in them.

Moreover, the following definition is given by Bellman and Cooke, [4].

Definition 2.4
The boundary value problems for the linear delay differential equations consists of linear delay differential equation together with two or more boundary conditions that must be satisfied by the nontrivial solution of them.

Consider the linear delay fourth order boundary Sturm-Liouville eigen-value problems:

$$-(p(x)y''(x))''+(q(x)-\lambda r(x))y(x-\tau)=0, \qquad (1.1)$$

Subject to the boundary conditions:

$$\begin{aligned}
&c_1 y(a)-c_2(py'')'(a)=0, \quad d_1 y(b)-d_2(py''')'(b)=0, \quad x\in[a-\tau,a] \\
&e_1 y'(a)-e_2 p(a)y''(a)=0, \quad f_1\, y'(b)-f_2 p(b)y''(b)=0, \quad x\in[b-\tau,b] \qquad (1.2)\\
&y(x-\tau)=\varphi(x-\tau),\ if\ x-\tau<a
\end{aligned}$$

where p, p', p'', q and r are given real-valued continuous functions defined on the interval $[a,b]$, p and r are positive, not both coefficients in one condition are zero, $\tau > 0$ is the time delay. φ is the initial function defined on $x\in[x_0-\tau, x_0]$.

Like the linear fourth order eigen-value problems, the problem given by equations (1.1) and (1.2) satisfies the following remarks which are given by Asmer and Bhattacharyya, [5] and [6].

Remarks 2.5

1) The linear operator:
$$L=-(p(x)\frac{d^4}{dx^4}+2p'(x)\frac{d^3}{dx^3}+p''(x)\frac{d^2}{dx^2})+A(x)q(x)$$
of the problem given by equations (1.1) and (1.2) is self-adjoint, where $A(x)$ is an operator defined by:
$A(x)y(x) = y(x-\tau)$
2) All the delay eigen-values are real.
3) The delay eigen-functions are orthogonal to the weight function r.
4) There are infinite numbers of delay eigen-values forming a monotone increasing sequence with $\lambda_j \to \infty$ as $j\to \infty$**.** Moreover, the delay eigen-functions corresponding to the delay eigen-values has exactly j roots on the interval (a,b)**.**
5) The delay eigen-functions are complete and normalized in $L^2[a,b]$.
6) Each delay eigen-value corresponds only one eigen-function in $L^2[a,b]$.

3. Collocation Method

In this section, the collocation method is modified to solve the problem given by equations (1.1) and (1.2). This method is one of the weighted residual methods that based on approximating the unknown function y as a linear combination of n linearly independent functions $\{\phi_i(x)\}^n_{i=1}$. That is:

$$y(x) = \sum c_i\, \phi_i(x), \qquad i = 1,2,\dots,n \tag{3.1}$$

But, this approximated solution must satisfy the boundary conditions given by equations (1.2) to get a new approximated solution. By substituting this approximated solution into equation (1.1) one can get:

$$R(x,\lambda,\vec{c}) = -(p(x)\sum_{i=1}^{n} c_i\phi_i''(x))'' + (q(x)-\lambda r(x))(\sum_{i=1}^{n} c_i\phi_i(x-\tau)) \tag{3.2}$$

where R is the error in the approximation of equation (3.2) and $\vec{C}$ is a vector of $n-4$ of c_i, $i\in\{1,2,\dots,n\}$.

Next, choose $n-3$ points say $\{x_k\}_{k=1}^{n-3}$, $a \le x_k \le b$ where the error R will be vanished at them.

That is,

$$-(p(x)\sum_{i=1}^{n} c_i\phi_i''(x_k))'' + (q(x_k)-\lambda r(x_k))(\sum_{i=1}^{n} c_i\phi_i(x_k-\tau)) = 0 \tag{3.3}$$

Then, by evaluating equation (3.3) at $k = 1,2,\dots, n-3$, one can obtain a system of $n-3$ nonlinear equations with $n-3$ unknowns, [7].

This system can be solved to get the values of $n-4$ of c_i and λ.

To illustrate the collocation method we shall give the following example

Example 3.1

Consider the linear delay fourth order Sturm-Liouville eigen-value problem:

$$-(x\frac{d^4}{dx^4}y(x) + 2\frac{d^3}{dx^3}y(x)) + (5-\lambda)y(x-1) = 0 \tag{3.4}$$

subject to the boundary conditions:

$$\begin{aligned} &y(1)=0, && 2y(2)-(y'''(2)+y''(2))=0, && x\in[0,1]\\ &y'(1)=0, && y'(2)-y''(2)=0, && x\in[1,2] \\ &y(x-1)=x-1 \end{aligned} \tag{3.5}$$

The collocation method is used here to solve the above Sturm-Liouville problem.

First, we approximate the unknown function y as a polynomial of degree five. Thus

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6x^5, \; x\in[0,1]$$

But, the above approximated solution ought to satisfy the boundary conditions given by equation (3.5), therefore:

$$y(x) = c_3x^2 + (\frac{57}{11}x^3 + \frac{42}{11}x^4 + x^5)c_6, \quad x\in[0,1].$$

That is,

$$y(x-1) = c_3(x-1)^2 + (\frac{57}{11}(x-1)^3 + \frac{42}{11}(x-1)^4 + (x-1)^5)c_6, \quad x\in[1,2].$$

In this case, the error function takes the form:

$$R(x,\lambda,c_3,c_6) = -x(-\frac{1008}{11}c_6 + 120c_6x) - \frac{684}{11}c_6 + \frac{2016}{11}c_6x - 120c_6x^2 + (5-\lambda)c_3(x-1)^2 + (\frac{57}{11}(x-1)^3 + \frac{42}{11}(x-1)^4 + (x-1)^5)c_6$$

Next, to find c_3, c_6 and λ we choose any three points in the interval [1,2], say $x = 1$, $x=\frac{3}{2}$, $x=2$ at which the error between the exact and approximated solution defined in the above equation will be vanish since y is a function, to get the following system of nonlinear equations:

$$\begin{aligned} &-\frac{300}{11}c_6 = 0\\ &-\frac{66393}{352}c_6 + \frac{1}{4}(5-\lambda)c_3 = 0\\ &-\frac{5086}{11}c_6 + (5-\lambda)c_3 = 0 \end{aligned}$$

The solution of this system is $\lambda = 5$, $c_6 = 0$ and $c_3 \neq 0$ because $y(x)$ is a non-trivial solution for y is an eigen-function.

Hence $(5, c_3(x-1)^2)$ is the eigen-pair of the problem given by equations [2.4]-[2.5] where $x \in [1,2]$.

Moreover, if we choose another three points in the domain of y represented by the closed interval [1,2], say $x=\frac{5}{4}$, $x=\frac{3}{2}$, $x=\frac{7}{4}$ to get the following system of nonlinear equations:

$$\frac{-1052605}{1264}\cdot c_6 + \frac{1}{16}\cdot(5-\lambda)\cdot c_3 = 0$$

$$\frac{-66493}{352}\cdot c_6 + \frac{1}{4}\cdot(5-\lambda)\cdot c_3 = 0$$

$$\frac{-3519543}{1264}\cdot c_6 + \frac{1}{16}\cdot(5-\lambda)\cdot c_3 = 0$$

Then, solving the above system, the same solution can be obtained.

Example 3.2

Consider the linear delay fourth order Sturm-Liouville eigen-value problem:

$$-(e^{-x}\frac{d^4}{dx^4}y(x-1) - 2e^{-x}\frac{d^3}{dx^3}y(x) - e^{-x}\frac{d^2}{dx^2}y(x-1)) + (3\sin x - \lambda\sin x)y(x-1) = 0 \tag{3.6}$$

together with the homogenous boundary conditions

$$(e^{-x}y''(1))' = 0, \quad (e^{-x}y(2))' = 0 \quad , x \in [0,1]$$
$$y'(1) = 0, \quad y'(2) = 0 \quad , x \in [1,2] \qquad (3.7)$$
$$y(x-1) = x-1$$

The collocation method is used here to solve the above Sturm-Liouville problem.

First, we approximate the unknown function y as a polynomial of degree zero.
i.e; $y(x-1) = C_1$, then this solution is automatically satisfy the above boundary

$$R(x, \lambda, c_1) = -(e^{-1}\, c_1)'' + (3\sin x - \lambda \sin x)c_1$$

are the solutions of equations (2.6) and (2.7) using the same argument as in example (3.1) the eigen-pair(3,c_1) is the solution of the problem given by equations (2.6) and (2.7).

Hint
The above problems has been solved with the help of Math-Cad software package, and the programs are so easy that omitted.

4. Real Live Problem

In this section a real live problem namely the elastic vibrations of beams has been discussed.

Problem (4.1): [5]

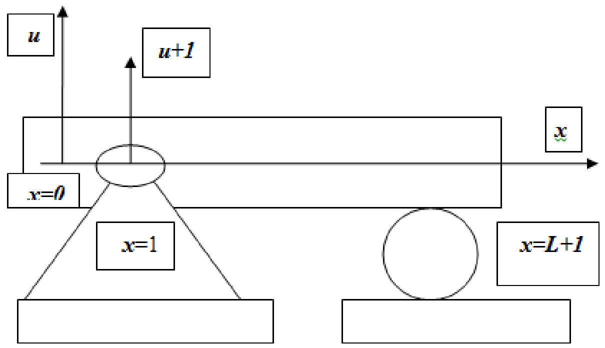

Figure 4.1: A simply supported beam is pinned at one end and roller supported in the other. The ends can rotate freely but do not move vertically

The application that we consider in this problem lead to partial differential equation which is fourth order in the space of variables. We consider the transverse vibrations of an elastic homogeneous beam with various supports at its ends.

The vertical vibrations of a uniform beam of length $L+1$ are described by the equation

$$u_{tt} = -\left(\frac{EI}{pA}\right)^2 u_{xxxx} \qquad (4.1)$$

where E is Young's constant determined by the constitutive material of the beam, I is the moment of intertia of cross section of the beam with respect to an axis through its center of mass and perpendicular to the (x,u)-plane, p is the density (mass per unit volume), and A is an area of cross section.

It is assumed that the beam is of uniform density throughout, that the cross sections are constant, and that is its equilibrium position the centers of mass of the cross sections lie on the x-axis (see Fig.(4.1)).

The variable $u(x-1,t)$ represents the displacement of the point on the beam corresponding to position $x-1$ at time t.

The boundary conditions that accompany equation (3.1) depend on the end supports of the beam. The case under consideration, that of simply supported ends, described by

$$u(1,t) = 0, \; u(L+1,t) = 0, \; u_{xx}(1,t) = 0, \; u_{xx}(L+1,t) = 0. \qquad (4.2)$$

The fact that the beam is prevented form translating vertically at its supports is expressed by the first two equations in equation (4.2). To understand the remaining equations in equation (4.2), recall that u_{xx} represents the curvature or concavity of the beam at $x-1$. It is a fact from the theory of strength of materials that curvature is proportional to bending moment for an elastic beam. The last two equations in equation (4.2) state that the moments at the supports are zero, as the case for a simply supported beam, since the beam can rotate at its ends.

To complete the description of the initial value problem, we specify the initial conditions

$$u(x-1,0) = f(x-1), \; u_t(x-1,0) = g(x-1). \qquad (4.3)$$

where x belongs to the open interval $(1, L+1)$, which give the initial displacement and velocity of the beam.

To solve equations (4.1) and (4.2), we use the method of separation of variables. This leads to the equations

$$-\frac{y^{(4)}(x-1)}{y(x-1)} = \frac{T''}{\left(\frac{EI}{pA}\right)^2 T} = \alpha^4 \qquad (3.4)$$

We have chosen the separation constant to be positive because we expect periodic behavior in t, and for reasons that will become apparent momentarily it is convenient to denote this constant by α^4. For the sake of completeness, we note that it can be shown that the boundary value problem $-y^{(4)}(x-1) - \lambda y(x-1) = 0$ together with the boundary conditions implied by eq.(3.3) has nontrivial solutions only for positive choices of the separation constant λ.

We consider the boundary value problem for y

$$-y^{(4)}(x-1) - \alpha^4 y(x-1) = 0$$
$$y(1) = 0, \quad y(L+1) = 0, \quad x \in [0, L]$$
$$y''(1) = 0, \quad y''(L+1) = 0, \quad x \in [L, L+1]$$
$$y(x-1) = x-1$$

note that this is a fourth order delay eigen-value Sturm-Liouville problem, with $\lambda = \alpha^4$.

References

[1] Baker C., Paul, C. and Wille, D., “Revise for Publication in Adv. Com. Math. ,” Numerical Analysis Report. No.248, 1994.
[2] Rama M. and Rao M., “Ordinary Differential Equations-Theory and Applications,” Affiliated East-West Private Limited, London, 1980.
[3] Brouer F. and Nohel J., “Ordinary Differential Equations,”W. Benjamin, Inc., London, 1973.
[4] Bellman, R. and Cooke, K., “Differential Difference Equations,” Academic Press. Inc., New York, 1963.
[5] Asmer N., “Partial Differential Equations and Boundary Value Problems,” Prentice-Hall, London, 2000.
[6] Mathematical Analysis and Applications, Vol. 264, 560-576.
[7] Burden R., “Numerical Analysis,” PWS Publisher, New York, 1985.