New Properties and Their Relation to type Weyl-Browder Theorems

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Abstract: In this paper we introduce and study the new spectral properties (α), (ao), (sz) and (asz). Our main goal in this paper is to study relationship between these properties and other Weyl – Browder type theorems.

1. Introduction

Let X be a Banach pace, and let B(X) be the Banach algebra of all bounded linear operators acting on X. For an operator S ∈ B(X) we denote by α(S) the dimension of the kernel of S(X), (α(S) = dim.S(X)), and β(S) the codimension of the range S(X), (β(S) = dim( X \ S(X))). If α(S) < ∞ and S(X) closed then S is said to be upper semi-Fredholm, while S ∈ B(X) is said to be lower semi-Fredholm if β(S) < ∞. Let SF(X) and SF denote the class of all upper semi-Fredholm operators and the class of all lower semi-Fredholm operators, respectively. If S ∈ B(X) is either an upper or lower semi-Fredholm operator, then S is called a semi-Fredholm operator, while if α(S) < ∞ and S(X) are finite then S is called a Fredholm operator. The index of S given by ind(S) = α(S) − β(S). For S ∈ B(X), let SF(X) = {S ∈ SF(X): ind(S) ≤ 0}. Then the Weyl essential approximate spectrum of S is defined by σWF(S) = {λ ∈ C: S − λI ∉ SF(X)}.

An operator S ∈ B(X) is said to be a Weyl operator if it is a Fredholm operator of index zero. The Weyl spectrum σw(T) of S is defined by σw(S) = {λ ∈ C: S − λI is not Weyl operator}. The ascent p = p(S) of an operator S is defined as the smallest non-negative integer p such that kerSp = kerSp+1 and the descent q = q(S) defined as the smallest non-negative integer q such that S′q(X) = S′q+1(X). If p(S) and q(S) are both finite then p(S) = q(S). Now we will define the class of all upper semi-Browder operators B+(X) = {S ∈ SF(X): p(S) < ∞}, and the class of all lower semi-Browder operators B−(X) = {S ∈ SF(X): q(S) < ∞}. The class of all Browder operators is defined by B(X) = B+(X) ∩ B−(X). Browder spectrum of S is defined by σB(S) = {λ ∈ C: S − λI ∉ B(X)}.

Recall that the operator S is said to have the single valued extension property at λq ∈ C (SVEP for short), only analytic function f:D → X which satisfies the equation (S − λI)f (λ) = 0 for all λ ∈ D is the function f ≡ 0. An operator S is said to be have SVEP if S has SVEP at every point λ ∈ C, (see, [8]).

For S ∈ B(X) and non-negative integer m, define S0 to be restriction of S to R(Sm) viewed as a map from R(Sm) into R(Sm+1). If S is a Fredholm operator, then S is called semi-Fredholm operator, if for some integer n the range space R(Sn) is closed and S is an upper (resp. lower ) semi-Fredholm operator, and in this case the index of S is defined as the index of semi-Fredholm operator S, (see, [5]). If S is a Fredholm operator, then S is said to be a B-Fredholm.

Lemma 1.1[2] Let S ∈ B(X). Then (i) S is upper semi-Fredholm and α(S) < ∞ if and only if S ∈ SF(X).

(ii) S is lower semi-Fredholm and β(S) < ∞ if and only if S ∈ SF(X).

An operator S ∈ B(X) is called a B-Weyl operator, if it is a B-Fredholm operator of index zero. The B-Weyl spectrum σBW(S) of S is defined by σBW(S) = {λ ∈ C: S − λI is not a B-Weyl operator}. The Drazin spectrum of S is defined by σD(S) = {λ ∈ C: S − λI is not a Drazin invertible}. Define also the set LD(X) by LD(X) = {S ∈ p(S) < ∞ and R(Sp(S) + 1) is closed } and σLD(S) = {λ ∈ C: S − λI ∉ LD(X)}.

From [1], λ ∈ C is pole of resolvent of S if and only if 0 < max(q(S − λI), q(S − λI)) < ∞. Moreover, if this is true, then [p(S − λI) = q(S − λI)]. While, λ ∈ σp(S) is a left pole of S if S − λI ∈ LD(X) and is a left pole of S of finite rank if λ is a left pole of S and α(S − λI) < ∞. Let SBF−(X) be the class of all upper semi-B-Fredholm operators, SBF−(X) = {S ∈ SBF−(X): ind(S) ≤ 0}. The upper B-Weyl spectrum of S is defined by σSBF−(S) = {λ ∈ C: S − λI ∉ SBF−(X)}.

In the following table we will provide some symbol and notations that we will use:

<table>
<thead>
<tr>
<th>symbol and notations</th>
<th>definition</th>
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<tr>
<td>σ(S): spectrum of S</td>
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<td>σα(S): approximate point spectrum of S</td>
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<td>σBW(S): Weyl spectrum of S</td>
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<td>σB−(S): B-Weyl spectrum of S</td>
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<td>σWF−(S): upper semi-Weyl spectrum of S</td>
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<tr>
<td>σSBF−(S): upper semi-B-Weyl spectrum of S</td>
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A new Browder and Weyl type theorems have been studied by many researchers. Researchers are interested in adding and expanding some properties and their relationship with each other. In this paper, we investigate the new properties (o), (ao), (sz) and (asz), which will be defined later, in connection with Weyl-Browder type theorems.

In the following definition we will mention most of the properties presented by researchers in, [3], [4], [9], [10] and [11], which are used in this paper.

Definition 1.2A bounded linear operator \( T \in L(X) \) is said to have:

1- Property (z) [10] if \( \Delta_+(S) = \Pi_0^0(S) \).
2- Property (az) [10] if \( \Delta_+(S) = \Pi_0^0(S) \).
3- Property (gz) [10] if \( \Delta_+(S) = \Pi_0^0(S) \).
4- Property (gaz) [10] if \( \Delta_+(S) = \Pi_0^0(S) \).
5- Property (b) [11] if \( \Delta_+(S) = E(S) \).
6- Property (ab) [11] if \( \Delta_+(S) = \Pi_0^0(S) \).
7- Property (gh) [11] if \( \Delta_+(S) = E(S) \).
8- Property (gah) [11] if \( \Delta_+(S) = \Pi_0^0(S) \).
9- Property \( (UW_E) \) [3] if \( \Delta_+(S) = E(S) \).
10- Property \( (UW_{E_0}) \) [3] if \( \Delta_+(S) = E_0(S) \).
11- Property \( (UW_{E_0}) \) [4] if \( \Delta_+(S) = \Pi_0^0(S) \).
12- Property \( (UW_{E_0}) \) [4] if \( \Delta_+(S) = \Pi_0^0(S) \).
13- Property \( (Saw) \) [9] if \( \Delta_+(S) = E_2(S) \).
14- Property \( (sab) \) [9] if \( \Delta_+(S) = \Pi_0^0(S) \).

2. Properties (o) and (sz)

Definition 2.1 A bounded linear operator \( S \in B(X) \) is said to have property (o) if \( \Delta_+(S) = E_0(S) \), and property (sz) if \( \Delta_+(S) = E(S) \).

Example 2.2 Let \( S \) be defined on the Banach space \( \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \) by \( S = R \oplus 0 \), where \( R \) is the unilateral right shift operator. We have \( \sigma_+(S) = D(0,1) \), the closed unit disc in \( \mathbb{C} \), and \( \sigma_+(S) = \sigma_+(S) = C(0,1) \cup \{0\} \), where \( C(0,1) \) is the unit circle of \( \mathbb{C} \). Moreover, \( E_0(S) = \{0\} \) and \( E(S) = \emptyset \). Thus \( S \) satisfies property (o) but \( S \) does not satisfy property (sz).

Example 2.3 Let \( R \in \ell^2(\mathbb{N}) \) be the unilateral right shift and let \( U \) be defined on \( \ell^2(\mathbb{N}) \) by \( U(x_1, x_2, \ldots) = (x_2, x_3, \ldots) \), \( x_n \in \ell^2(\mathbb{N}) \). If \( S = R \oplus U \), then \( \sigma(S) = D(0,1) \), the closed unit disc in \( \mathbb{C} \). \( E_0(S) = \{0\} \). Moreover, We have \( \sigma_+(S) = C(0,1) \), where \( C(0,1) \) is unit circle of \( \mathbb{C} \). Consequently, \( S \) does not satisfy property (o).

Example 2.4 Let \( L \) be the unilateral left shift operator defined on \( \ell^2(\mathbb{N}) \). It is well known that \( \sigma(S) = D(0,1) \) is the closed unit disc in \( \mathbb{C} \), \( \sigma_+(S) = C(0,1) \), and \( E(L) = \emptyset \). Hence \( L \) satisfies property (sz).

Proposition 2.5 Let \( S \in B(X) \). Then \( S \) satisfies property (o) if and only if \( S \) satisfies property \( (UW_{E_0}) \) and \( \sigma(S) = \sigma_+(S) \).

Proof: We assume that \( S \) satisfies property (o). If \( \lambda \in \Delta_+(S) \), then \( \lambda \in \Delta_+(S) \). Therefore \( \lambda \in E_0(S) \) and \( \Delta_+(S) \subseteq E_0(S) \).

To show the opposite inclusion \( E_0(S) \subseteq \Delta_+(S) \) and \( S \) satisfies property \( (UW_{E_0}) \), it follows that \( \lambda \in \sigma_+(S) \) and therefore, \( \lambda \in \Delta_+(S) \). Hence, \( E_0(S) \subseteq \Delta_+(S) \) and \( S \) satisfies property \( (UW_{E_0}) \). We then have \( \sigma(S) \subseteq \sigma_+(S) \) and \( \sigma_+(S) \). Conversely, assume that \( S \) satisfies property \( (UW_{E_0}) \) and \( \sigma(S) = \sigma_+(S) \). Therefore, \( \sigma(S) \subseteq \sigma_+(S) \) and \( \sigma_+(S) \).

Thus, \( \sigma(S) \subseteq \sigma_+(S) \) and \( S \) satisfies property (o).

Proposition 2.6 Let \( S \in B(X) \). Then \( S \) satisfies property (sz) if and only if \( S \) satisfies property \( (UW_\ell) \) and \( \sigma(S) = \sigma_+(S) \).
Proof. We assume that \( S \) satisfies property (sz). If \( \lambda \in \Delta_+(S) = \sigma_a(S) \setminus \sigma_{SF^+}(S) \), then \( \lambda \in \Delta_+(S) \). Therefore \( \lambda \in E(S) \) and \( \Delta_+(S) \subseteq E_a(S) \).

Now, if \( \lambda \in E(S) \) and \( \lambda \in \sigma_a(S) \) and \( S \) satisfies property (sz), it follows that \( \lambda \not\in \sigma_{SF^+}(S) \) and so \( \lambda \in \Delta_+(S) \). Hence, \( E(S) \subseteq \Delta_+(S) \) and \( S \) satisfies property \( (UW_{E_a}) \). We then have \( \sigma(S) \setminus \sigma_{SF^+}(S) = E(S) \) and \( \sigma_a(S) \setminus \sigma_{SF^+}(S) = E(S) \). Therefore, \( \sigma(S) \setminus \sigma_{SF^+}(S) = \sigma_a(S) \setminus \sigma_{SF^+}(S) \) (sz).

Conversely, assume that \( S \) satisfies property \( (UW_{E_a}) \) and \( \sigma(S) \setminus \sigma_{SF^+}(S) = \sigma_a(S) \setminus \sigma_{SF^+}(S) = E(S) \). Thus, \( \sigma(S) \setminus \sigma_{SF^+}(S) = E(S) \) and \( S \) satisfies property (sz).

Remark 2.7 From Proposition 2.5 and Proposition 2.6, if \( S \in \mathbb{B}(X) \) satisfies property (o) (resp. property (sz)) if and only if \( S \) satisfies property \( (UW_{E_a}) \) and \( \sigma(S) = \sigma_a(S) \) (resp. property \( (UW_{E_a}) \) and \( \sigma(S) = \sigma_a(S) \)). However, the converses do not satisfy in general when the condition \( \sigma_a(S) \) is not verified as seen by the following examples.

Example 2.8 Let \( R \in \ell^2(\mathbb{N}) \) be the unilateral right shift and let \( U \) be defined on \( \ell^2(\mathbb{N}) \) by \( U(x_1, x_2, \ldots) = (0, x_2, x_3, \ldots) \). If \( S = R \oplus U \), then \( \sigma(S) = \{ 0, 1 \} \), the closed unit disc in \( \mathbb{C} \). We have \( \sigma_a(S) = \sigma_a(U) = C(0,1) \cup \{ 0 \} \), where \( C(0,1) \) is the closed unit disc in \( \mathbb{C} \). Moreover, \( E_a(S) = 0 \). So \( S \) satisfies property \( (UW_{E_a}) \) but \( S \) does not satisfy property (sz).

The following theorem links the property (o) and property (sz).

Theorem 2.10 Let \( S \in \mathbb{B}(X) \). Then \( S \) satisfies property (o) if and only if \( S \) satisfies property (sz).

Proof. Assume that \( S \) satisfies property (o). Then from Proposition 2.5 we have \( \sigma_a(S) = \sigma_a(U) \) and so \( E(S) = E_a(S) \). Therefore, \( \sigma(S) \setminus \sigma_{SF^+}(S) = E(S) \).

Necessity: Assume that \( S \) satisfies property (sz). Then from Proposition 2.6 we have \( \sigma(S) = \sigma_a(S) \) and so \( E(S) = E_a(S) \). Therefore, \( \sigma(S) \setminus \sigma_{SF^+}(S) = E_a(S) \).

Theorem 2.11 Let \( S \in \mathbb{B}(X) \). Then \( S \) satisfies property (gz) if and only if \( S \) satisfies property (o) and \( \sigma_{SF^+}(S) = \sigma_{SBF^+}(S) \).

Proof. Suppose that \( S \) satisfies property (gz) and \( \sigma_a(S) \setminus \sigma_{SF^+}(S) = E_a(S) \). Let \( \lambda \in \sigma(S) \setminus \sigma_{SF^+}(S) \). Since \( \sigma(S) \setminus \sigma_{SF^+}(S) \subseteq \sigma(S) \setminus \sigma_{SBF^+}(S) \), we have \( \lambda \in \sigma(S) \setminus \sigma_{SBF^+}(S) \).

3. Properties (ao) and (asz)

Now, we define another property of spectrum of an operator

Definition 3.1 A bounded linear operator \( S \in \mathbb{B}(X) \) is said to have property (ao) if \( \Delta_+(S) = \Pi_a(S) \), and property (asz) if \( \Delta_+(S) = \Pi(S) \).

Example 3.2 Let \( S \) be defined on the Banach space \( \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \) by \( S = R \oplus 0 \), where \( R \) is the unilateral right shift operator on \( \ell^2(\mathbb{N}) \). Then we have \( \Pi_a(S) = \{ 0 \} \) and \( \Pi(S) = \{ 0 \} \). So \( S \) satisfies property (ao) and does not satisfy property (asz).

Example 3.3 The Volterra operator \( V \) on \( L^2([0,1]) \) is defined by \( Vf = \int_0^1 f(s)ds \), for \( f \in L^2([0,1]) \). It is well known that \( \sigma(V) = \{ 0 \} \), and \( \sigma_{SF^+}(V) = \{ 0 \} \). Moreover, \( \Pi(V) = \phi \).
Hence the Volterra operator is an example satisfying the property (asz).

Example 3.4 Let $R$ be the unilateral right shift operator on $\ell^2(\mathbb{N})$ and $U \in (\ell^2(\mathbb{N}))$ be defined by $U(x_1, x_2, \ldots) = (0, x_2, x_3,\ldots)$. Define an operator $S$ on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $S = R \oplus U$. Then we have $\sigma(S) = D(0,1)$, the closed unit disc in $\mathbb{C}$, and $\sigma_{SF^*}(S) = C(0,1) \cup \{0\}$, where $C(0,1)$ is unit circle of $\mathbb{C}$. Moreover, $\Pi(S) = \Phi$. So $S$ satisfies property $(UW_{\Pi\alpha})$ but $S$ does not satisfy property (asz).

Proposition 3.5 Let $S \in B(X)$. Then $S$ satisfies property (ao) if and only if $S$ satisfies property $(UW_{\Pi\alpha})$ and $\sigma(S) = \sigma_{\alpha}(S)$.

Example 3.8 Let $S$ be the unilateral right shift operator on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $S = R \oplus U$, where $R$ is the unilateral right shift operator. We have $\sigma(S) = D(0,1)$, the closed unit disc in $\mathbb{C}$, and $\sigma_{SF^*}(S) = C(0,1) \cup \{0\}$, where $C(0,1)$ is unit circle of $\mathbb{C}$. Moreover, $\Pi(S) = \Phi$. So $S$ satisfies property $(UW_{\Pi\alpha})$ but $S$ does not satisfy property (asz).

Theorem 3.9 Let $S \in B(X)$. Then $S$ has property (Sab) if and only if $S$ has property (ao).

proof. Suppose that $S$ has property (Sab). If $\lambda \in \Pi_0(S)$, then $\lambda \in \sigma(S)$. Hence $\lambda \in \sigma(S)$. By (12), Lemma (2.4)), then $(S - \lambda I)$ is an upper semi-B-Fredholm operator of $S$. Then $(S - \lambda I)$ is an upper semi-B-Fredholm operator of $S$. That means $\lambda \in \sigma(S)$, thus $S$ satisfies property (ao).

Conversely, suppose that $S$ satisfies property (ao). By proposition (3.5) we have $S$ has property $(UW_{\Pi\alpha})$ and $\sigma(S) = \sigma_{\alpha}(S)$. And by (14), theorem 2.6), $S$ satisfies property (SBab), and from (9), theorem 2.7), $S$ has property (Sab).

Theorem 3.10 Let $S \in B(X)$. Then $S$ satisfies property (ao) if and only if $S$ satisfies property (asz).

proof. We assume that $S$ satisfies property (ao). If $\lambda \in \Delta_+(S)$, then $\lambda \in \Pi_0(S)$, and $S$ is a left Drizan invertible. From (17), theorem 2.8), then $\inf d \leq 0$ and $\alpha = q(S) < \infty$. Therefore, $\alpha$ is a pole of resolvent. On other hand, if $\lambda \in \Pi(S)$, by (11), theorem 3.81, then $\lambda \in \sigma(S)$. Since $S$ satisfies property (ao), then $\lambda \in \sigma_{SF^*}(T)$ and $\lambda \in \Delta_+(S)$.

In the opposite direction, if $\lambda \in \Delta_+(S)$ and since $S$ satisfies property $\Delta_+(S) = \Pi(S)$, then $\Pi(S) \subset \Pi_0(S)$, and $S$ is a left Drizan invertible. From (17), theorem 2.8), then $\inf d \leq 0$ and $\alpha = q(S) < \infty$. Therefore, $\alpha$ is a pole of resolvent. On other hand, if $\lambda \in \Pi_0(S)$, then $\lambda \in \sigma(S)$ and so $\lambda \in \sigma(S)$. Since $S$ satisfies property $\Delta_+(S) = \Pi(S)$, then $\Pi_0(S) \subset \Delta_+(S)$.

Thus $S$ satisfies property (ao).

Theorem 3.11 Let $S \in B(X)$. Then $S$ satisfies property (gah) if and only if $S$ satisfies property (asz) and $\sigma_{SF^*}(S) = \sigma_{SF^*}(S)$.

proof. Suppose that $S$ satisfies property (gah) that is $\sigma(S) \setminus \sigma_{SF^*}(S) = \Pi(S)$. Let $\lambda \in \sigma(S) \setminus \sigma_{SF^*}(S)$. Since $\sigma(S) \setminus \sigma_{SF^*}(S) \subset \sigma(S) \setminus \sigma_{SF^*}(S)$, we have $\lambda \in \sigma(S) \setminus \sigma_{SF^*}(S)$. As $S$ satisfies property (gah), then $\lambda \in \Pi(S)$.

If $\lambda \in \Pi(S)$ then $\lambda \in \Pi_0(S)$. Since $\Pi(S) \subset \Pi_0(S)$, but since $\Pi_0(S) = \sigma(S) \setminus \sigma_{SD}(S)$, this means that, $S$ is left
Drizn invertible. From ([7], theorem 2.8), Then $S$ is an upper semi-B-Fredholm operator equal zero. And by ([2], Lemma (2.4)), $\lambda \in \sigma(S)\backslash \sigma_{SBF}^-(S)$. Thus, $S$ satisfies property (asz).

Now we will prove that $\sigma_{SF}^-(S) = \sigma_{SBF}^-(S)$. Since $S$ satisfies property (gah) and property (asz), we get $\sigma(S) \backslash \sigma_{SF}^-(S) = \sigma(S) \backslash \sigma_{SBF}^-(S) = \Pi(S)$. Thus, $\sigma_{SF}^-(S) = \sigma_{SBF}^-(S)$.

On other hand, suppose that $S$ satisfies property (asz) and $\sigma_{SF}^-(S) = \sigma_{SBF}^-(S)$, we get $\sigma(S) \backslash \sigma_{SF}^-(S) = \sigma(S) \backslash \sigma_{SBF}^-(S) = \Pi(S)$. Then $S$ satisfies property (gah).

**Theorem 3.12** Let $S \in B(X)$. Then $S$ satisfies property (gaz) if and only if $S$ satisfies property (ao) and $\sigma_{SF}^-(S) = \sigma_{SBF}^-(S)$.

**Proof.** Suppose that $S$ satisfies property (gaz) that is $\sigma(S) \backslash \sigma_{SBF}^-(S) = \Pi_a(S)$. Let $\lambda \in \sigma(S) \backslash \sigma_{SF}^-(S)$. Since $\sigma(S) \backslash \sigma_{SF}^-(S) \subset \sigma(S) \backslash \sigma_{SBF}^-(S)$, we have $\lambda \in \sigma(S) \backslash \sigma_{SBF}^-(S)$. As $S$ satisfies property (gaz), then $\lambda \in \Pi_a(T)$. Now, if $\lambda \in \Pi_a(T)$, then by previous theorem we get $\lambda \in \sigma(S) \backslash \sigma_{SF}^-(S)$. Thus, $S$ satisfies property (ao).

Let us also prove $\sigma_{SF}^-(S) = \sigma_{SBF}^-(S)$. Since $S$ satisfies property (gaz) and property (ao), we get $\sigma(S) \backslash \sigma_{SF}^-(S) = \sigma(S) \backslash \sigma_{SBF}^-(S) = \Pi_a(S)$. Thus, $\sigma_{SF}^-(S) = \sigma_{SBF}^-(S)$.

On other hand, suppose that if $S$ satisfies property (ao) and $\sigma_{SF}^-(S) = \sigma_{SBF}^-(S)$, then $\sigma(S) \backslash \sigma_{SBF}^-(S) = \sigma(S) \backslash \sigma_{SF}^-(S) = \Pi_a(S)$. Then $S$ satisfies property (gaz).

**References**