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New Properties and Their Relation to type Weyl-Browder Theorems

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Abstract: ***In this paper we introduce and study the new spectral properties (o), (ao), (sz) and (asz). Our main goal in this paper is to study relationship between these properties and other Weyl – Browder type theorems.***

1. Introduction

Let X be a Banach pace, and let $B(X)$ be the Banach algebra of all bounded linear operators acting on X. For an operator $S \in B(X)$ we denote by $\alpha(S)$ the dimension of the kernel of $S(X)$, $(\alpha(S) = dim.S(X))$, and $\beta(S)$ the codimension of the range $S(X)$, $(\beta(S) = \dim(X \setminus S(X)))$. If $\alpha(S) < \infty$ and $S(X)$ closed then S is said to be upper semi-Fredholm, while $S \in B(X)$ is said to be lower semi-Fredholm if $\beta(S) < \infty$. Let SF_+ and SF_- denote the class of all upper semi-Fredholm operators and the class of all lower semi-Fredholm operators, respectively. If $S \in B(X)$ is either an upper or lower semi-Fredholm operator, then S is called a semi-Fredholm operator, while if $\alpha(S) < \infty$ and $S(X)$ are finite then S is called a Fredholm operator. The index of S given by $ind(S) = \alpha(S) - \beta(S)$. For $S \in B(X)$, let $SF_+^-(X) = \{S \in SF_+(X): ind(S) \le 0\}$. Then the Weyl essential approximate spectrum of S is defined by $\sigma_{SF_+^-}(S) = \{\lambda \in \mathbb{C}: S - \lambda I \notin SF_+^-(X)\}$.

An operator $S \in B(X)$ is said to be a Weyl operator if it is a Fredholm operator of index zero. The Weyl spectrum $\sigma_w(T)$of S is defined by $\sigma_w(S) = \{\lambda \in \mathbb{C}: S - \lambda I \; is \; not \; Wely \; operator\}$. The ascent $p = p(S)$ of an operator S is defined as the smallest non-negative integer p such that $kerS^p = kerS^{p+1}$ and the descent $q = q(S)$ defined as the smallest non-negative integer q such that $S^q(X) = S^{q+1}(X)$. If $p(S)$ and $q(S)$ are both finite then $p(S) = q(S)$.Now we will define the class of all upper semi-Browder operators $\mathcal{B}_+(X) = \{S \in SF_+(X): p(S) < \infty\}$, and the class of all lower semi-Browder operators $\mathcal{B}_-(X) = \{S \in SF_-(X): q(S) < \infty\}$. The class of all Browder operators is defined by $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. Browder spectrum of S is defined by $\sigma_b(S) = \{\lambda \in \mathbb{C}: S - \lambda I \notin \mathcal{B}(X)\}$.

Recall that the operator S is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ ($SVEP$ for short), only analytic function $f: D \to X$ which satisfies the equation $(S - \lambda I)f(\lambda) = 0$ for all $\lambda \in D$ is the function$f \equiv 0$. An operator S is said to be have $SVEP$ if S has $SVEP$ at every point $\lambda \in \mathbb{C}$, (see, [8]).

For $S \in B(X)$ and non-negative integer$n$, define $S_{[0]}$ to be restriction of S to $R(S^n)$viewed as a map from $R(S^n)$ into $R(S^n)$ [in particular $S_{[n]} = S$]. Abounded linear operator $S$is said to be an upper (resp. lower) semi-B-Fredholm operator if for some integer n the range space $R(S^n)$is closed and $S_{[n]}$ is an upper (resp. lower) semi-Fredhlom operator, and in this case the index of S is defined as the index of semi-Fredholm operator $S_{[n]}$,(see, [5]). If $S_{[n]}$ is a Fredholm operator, then S is said to be a B-Fredholm.

Lemma 1.1[2] Let $S \in B(X)$. Then
(i) $S$is upper semi-B-Fredholm and $\alpha(S) < \infty$ if and only if $S \in SF_+(X)$.
(ii) $S$is lower semi-B-Fredholm and $\beta(S) < \infty$ if and only if $S \in SF_-(X)$.

An operator $S \in B(X)$ is called a B-Weyl operator, if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{Bw}(S)$of S is defined by$\sigma_{Bw}(S) = \{\lambda \in \mathbb{C}: S - \lambda I \; is \; not \; B\text{−}Weyl \; operator$ .If $S \in B(X)$ has finite ascent and descent then S is called Drazin invertible. The Drazin spectrum of S is defined by $\sigma_D(S) = \{\lambda \in \mathbb{C}: S - \lambda I \; is \; not \; Drazin \; invertible.$ Define also the set $LD(X)$ by $LD(X) = \{S \in p(S) < \infty \; and \; R(S^{p(S)+1}) \; is \; closed\}$ and $\sigma_{LD}(S) = \{\lambda \in \mathbb{C}: S - \lambda I \notin LD(X)\}$.

From [1], $\lambda \in \mathbb{C}$ is pole of resolvent of S if and only if $0 < \max(p(S - \lambda I), q(S - \lambda I)) < \infty$. Moreover, if this is true, then $p(S - \lambda I) = q(S - \lambda I)$. While, $\lambda \in \sigma_a(S)$ is a left pole of S if $S - \lambda I \in LD(X)$ and is a left pole of S of finite rank if λ is a left pole of S and $\alpha(S - \lambda I) < \infty$. Let $SBF_+^-(X)$ be the class of all upper semi-B-Fredholm operators, $SBF_+^-(X) = \{S \in SBF_+(X): ind(S) \le 0\}$. The upper B-Weyl spectrum of S is defined by $\sigma_{SBF_+^-}(S) = \{\lambda \in \mathbb{C}: S - \lambda I \notin SBF_+^-(X)\}$.

In the following table we will provide some symbol and notations that we will use:

symbol and notations
$\sigma(S)$: spectrum of S, $\sigma_a(S)$: approximate point spectrum of S, $\sigma_w(S)$: Weyl spectrum of S, $\sigma_{Bw}(S)$: B-Weyl spectrum of S, $\sigma_{SF_+^-}(S)$: upper semi-Weyl spectrum of S, $\sigma_{SBF_+^-}(S)$:upper semi-B-Weyl spectrum of S,

$E(S)$: eigenvalues of Sthat are isolated in the spectrum $\sigma(S)$ of S,
$E^0(S)$: eigenvalues of S of finite multiplicity that are isolated in the spectrum $\sigma(S)$ of S,
$E_a(S)$: eigenvalues of S that are isolated in the approximate point spectrum $\sigma_a(S)$ of S,
$E_a^0(S)$: eigenvalues of S of finite multiplicity that are isolated in the spectrum$\sigma_a(S)$ of S,
$\Pi(S)$: poles of S,
$\Pi^0(S)$: poles of S of finite rank,
$\Pi_a(S)$: left poles of S,
$\Pi_a^0(S)$: left poles of S of finite rank,
$\Delta(S) = \sigma(S)\backslash\sigma_w(S)$,
$\Delta_a(S) = \sigma_a(S)\backslash\sigma_{SF_-^+}(S)$,
$\Delta^g(S) = \sigma(S)\backslash\sigma_{Bw}(S)$,
$\Delta_a^g(S) = \sigma_a(S)\backslash\sigma_{SBF_-^+}(S)$,
$\Delta_+(S) = \sigma(S)\backslash\sigma_{SF_+^-}(S)$,
$\Delta_+^g(S) = \sigma(S)\backslash\sigma_{SF_+^-}(S)$,
$\Delta(S) = \Pi^0(S)$: Browder's theorem holds for S,
$\Delta_+(S) = E^0(S)$:Weyl's theorem holds for S,
$\Delta^g(S)= \Pi(S)$: generalized Browder's theorem holds for S,
$\Delta^g(S)=E(S)$: generalized Weyl's theorem holds for S,
$\Delta_a(S)=\Pi_a^0(S)$: a-Browder's theorem holds for S,
$\Delta_a(S)=E_a^0(S)$: a-Weyl's theorem holds for S,
$\Delta_a^g(S)=\Pi_a(S)$: generalized a-Browder's theorem holds for S,
$\Delta_a^g(S)=E_a(S)$: generalized a-Weyl's theorem holds for S.

A new Browder and Weyl type theorems have been studied by many researchers, researchers are interested in adding and expanding some properties and their relationship with each other. In this paper, we investigate the new properties (o),(ao), (sz) and (asz), which will be defined later, in connection with Weyl-Browder type theorems.

In the following definition we well mention most of the properties presented by researchers in, [3], [4], [9], [10] and [11], which are used in this paper.

Definition 1.2A bounded linearoperator $T \in L(X)$ is said to have:
1- Property (z) [10] if $\Delta_+(S) = E_a^0(S)$.
2- Property (az) [10] if $\Delta_+(S) = \Pi_a^0(S)$.
3- Property (gz) [10] if $\Delta_+^g(S) = E_a(S)$.
4- Property (gaz) [10] if $\Delta_+^g(S) = \Pi_a(S)$.
5- Property (h) [11] if $\Delta_+(S) = E^0(S)$.
6- Property (ah) [11] if $\Delta_+(S) = \Pi^0(S)$.
7- Property (gh) [11] if $\Delta_+^g(S) = E(S)$.
8- Property (gah) [11] if $\Delta_+^g(S) = \Pi(S)$.
9- Property (UW_E) [3] if $\Delta_a(S) = E(S)$.
10- Property (UW_{Ea}) [3] if $\Delta_a(S) = E_a(S)$.
11- Property (UW_{Π_a}) [4] if $\Delta_a(S) = \Pi_a(S)$.
12- Property (UW_Π) [4] if $\Delta_a(S) = \Pi(S)$.
13- Property (Saw) [9] if $\Delta_+^g(S) = E_a^0(S)$.
14- Property (sab) [9] if $\Delta_+^g(S) = \Pi_a^0(S)$.

2. Properties (o) and (sz)

Definition 2.1 A bounded linear operator $S \in B(X)$ is said to have property (o) if $\Delta_+(S) = E_a(S)$, and property (sz) if $\Delta_+(S) = E(S)$.

Example 2.2 Let S be defined on the Banach space $\ell^2(\mathbb{N})\oplus\ell^2(\mathbb{N})$ by $S = R\oplus 0$, where R is the unilateral right shift operator. We have $\sigma(S) = D(0,1)$, the closed unit disc in $\mathbb{C}$, and $\sigma_a(S) = \sigma_{SF_+^-}(S) = C(0,1) \cup \{0\}$, where $C(0,1)$ is unit circle of $\mathbb{C}$. Moreover, $E_a(S) = \{0\}$, and $E(S) = \emptyset$.Thus S satisfies property (o) but S does not satisfy property (sz).

Example 2.3 Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and let U be defined on $\ell^2(\mathbb{N})$by $U(x_1, x_2, \ldots.) = (0, x_2, x_3, \ldots..)$, $(x_n) \in \ell^2(\mathbb{N})$. If $S = R\oplus U$, then $\sigma(S) = D(0,1)$, the closed unit disc in $\mathbb{C}$, $E_a(S) = \{0\}$. Moreover, We have $\sigma_{SF_+^-}(S) = C(0,1)$, where $C(0,1)$ is unit circle of $\mathbb{C}$. Consequently,S does not satisfy property (o).

Example 2.4 Let L be the unilateral left shift operator defined on $\ell^2(\mathbb{N})$. It is well know that $\sigma(S) = D(0,1)$ is the closed unit disc in $\mathbb{C}$, $\sigma_{SF_+^-}(S) = D(0,1)$, and $E(L) = \emptyset$. Hence L satisfies property (sz).

Proposition 2.5 Let $S \in B(X)$. Then S satisfies property (o) if and only if S satisfies property (UW_{Ea}) and $\sigma(S) = \sigma_a(S)$.

*Proof.*We assume that S satisfies property (o). If $\lambda \in \Delta_a(S)$, then $\lambda \in \Delta_+(S)$. Therefor $\lambda \in E_a(S)$ and $\Delta_a(S) \subset E_a(S)$.
To show the opposite inclusion $E_a(S) \subset \Delta_a(S)$, let $\lambda \in E_a(S)$ then $\lambda \in \sigma_a(S)$ and since S satisfies property (o) , it follows that $\lambda \notin \sigma_{SF_+^-}(S)$ and so, $\lambda \in \Delta_a(S)$. Hence, $E_a(S) \subset \Delta_a(S)$ and S satisfies property(UW_{Ea}). We then have $\sigma(S)\backslash \sigma_{SF_+^-}(S) = E_a(S)$and $\sigma_a(S)\backslash\sigma_{SF_+^-}(S) = E_a(S)$. Therefore, $\sigma(S)\backslash \sigma_{SF_+^-}(S) = \sigma_a(S)\backslash\sigma_{SF_+^-}(S)and\sigma(S) = \sigma_a(S)$.
Conversely, assume that S satisfies property (UW_{Ea}) and $\sigma(S) = \sigma_a(S)$. $\sigma(S)\backslash \sigma_{SF_+^-}(S) = \sigma_a(S)\backslash\sigma_{SF_+^-}(S) = E_a(S)$.
Thus, $\sigma(S)\backslash \sigma_{SF_+^-}(S) = E_a(S)$ and S satisfies property (o).

Proposition 2.6 Let $S \in B(X)$. Then S satisfies property (sz) if and only if S satisfies property (UW_E) and $\sigma(S) = \sigma_a(S)$.

Proof .We assume that S satisfies property (sz). If $\lambda \in \Delta_a(S) = \sigma_a(S)\backslash\sigma_{SF_+^-}(S)$, then $\lambda \in \Delta_+(S)$. Therefor $\lambda \in E(S)$ and $\Delta_a(S) \subset E_a(S)$.

Now, if $\in E(S)$, then $\lambda \in \sigma_a(S)$ and since S satisfies property (sz) , it follows that $\lambda \notin \sigma_{SF_+^-}(S)$ and so, $\lambda \in \Delta_a(S)$. Hence, $E(S) \subset \Delta_a(S)$ and S satisfies property(UW_E). We then have $\sigma(S)\backslash \sigma_{SF_+^-}(S) = E(S)$and $\sigma_a(S)\backslash\sigma_{SF_+^-}(S) = E(S)$. Therefore, $\sigma(S)\backslash \sigma_{SF_+^-}(S) = \sigma_a(S)\backslash\sigma_{SF_+^-}(S)$and $\sigma(S) = \sigma_a(S)$.

Conversely, assume that S satisfies property (UW_E) and $\sigma(S) = \sigma_a(S)$. $\sigma(S)\backslash \sigma_{SF_+^-}(S) = \sigma_a(S)\backslash\sigma_{SF_+^-}(S) = E(S)$. Thus, $\sigma(S)\backslash \sigma_{SF_+^-}(S) = E(S)$ and S satisfies property (sz).

Remark 2.7 From Proposition 2.5 and Proposition 2.6,if $S \in B(X)$satisfies property (o) [resp. property (sz)] if and only if S satisfies property (UW_{Ea}) and $\sigma(S) = \sigma_a(S)$ [resp. property $(UW_E)$and $\sigma(S) = \sigma_a(S)$.]. However, the converses do not satisfies in general when the condition $\sigma(S) = \sigma_a(S)$is not verified as seen by the following examples.

Example 2.8 Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and let U be defined on $\ell^2(\mathbb{N})$ by $U(x_1, x_2, \ldots.) = (0, x_2, x_3, \ldots..)$, $(x_n) \in \ell^2(\mathbb{N})$. If $S = R \oplus U$, then $\sigma(S) = D(0,1)$, the closed unit disc in $\mathbb{C}$.Moreover, we have$\sigma_a(S) = C(0,1) \cup \{0\}$, where $C(0,1)$ is unit circle of $\mathbb{C}$,$\sigma_{SF_+^-}(S) = C(0,1)$. This implies that $\sigma_a(S)\backslash\sigma_{SF_+^-}(S) = \{0\}$ and$E_a(S) = \{0\}$.consequently,S does not satisfy property (UW_{Ea}) but S does not satisfy property (o).

Example 2.9 Let S be defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $S = R \oplus 0$, where R is the unilateral right shift operator. We have $(S) = D(0,1)$, , the closed unit disc in $\mathbb{C}$, and $\sigma_a(S) = \sigma_{SF_+^-}(S) = C(0,1) \cup \{0\}$, where $C(0,1)$ is unit circle of $\mathbb{C}$. Moreover, $E(S) = \emptyset$. So S satisfies property (UW_E) but S does not satisfy property (sz).
The following theorem links the property (o) and property (sz)

Theorem 2.10 Let $S \in B(X)$. Then S satisfies property (o) if and only if S satisfies property (sz).
*Proof.*Assume that S satisfies property (o). Then from proposition 2.5 we have$\sigma(S) = \sigma_a(S)$.And so $E(S) = E_a(S)$. Therefor $\sigma(S)\backslash\sigma_{SF_+^-}(S) = E(S)$.
Necessity: Assume that S satisfies property (sz). Then from proposition 2.6 we have $\sigma(S) = \sigma_a(S)$.And so $E(S) = E_a(S)$. Therefor $\sigma(S)\backslash\sigma_{SF_+^-}(S) = E_a(S)$.

Theorem 2.11 Let $S \in B(X)$. Then S satisfies property (gz) if and only if S satisfies property (o) and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$.
*Proof.*Suppose that S satisfies property (gz) that is $\sigma(S)\backslash \sigma_{SBF_+^-}(S) = E_a(S)$. Let$\lambda \in \sigma(S)\backslash\sigma_{SF_+^-}(S)$. Since$\sigma(S)\backslash \sigma_{SF_+^-}(S) \subset \sigma(S)\backslash\sigma_{SBF_+^-}(S)$, we have $\lambda \in \sigma(S)\backslash \sigma_{SBF_+^-}(S)$.As$S$ satisfies property (gz), then $\lambda \in E_a(S)$, so $\sigma(S)\backslash\sigma_{SBF_+^-}(S) \subset E_a(S)$.

If $\lambda \in E_a(S)$ then $\lambda \in iso\ \sigma_a(S)$ and $\alpha(S - \lambda I) < \infty$. Since S satisfies property (gz) we have $\lambda \in \sigma(S)$ and$\lambda \notin \sigma_{SBF_+^-}(S)$ i.e. $(S - \lambda I)$is an upper semi-B-Weyl operator of S. Consequently, $(S - \lambda I)$is an upper semi-B-Fredholm operator of S and$\alpha(S - \lambda I) < \infty$. By ([2], Lemma (2.4)),$(S - \lambda I)$ is an upper semi-Fredholm operator and so $(S - \lambda I)$ is an upper semi-Weyl operator of S. that means $\lambda \in \sigma(S)\backslash\sigma_{SF_+^-}(S)$, thus, S satisfies property (o).

Since S satisfies properties (gz) and (o) then $\sigma(S)\backslash \sigma_{SF_+^-}(S) = E_a(S)$and $\sigma(S)\backslash\sigma_{SBF_+^-}(S) = E_a(S)$. And so $\sigma(S)\backslash\sigma_{SF_+^-}(S) = \sigma(S)\backslash\sigma_{SBF_+^-}(S)$ and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$.
Conversely, suppose that S satisfies property (o) and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}S$, then $\sigma(S)\backslash\sigma_{SBF_+^-}(S) = \sigma(S)\backslash\sigma_{SF_+^-}(S) = E_a(S)$. Then S satisfies property (gz).

Theorem 2.12 Let $S \in B(X)$. Then S satisfies property (gh) if and only if S satisfies property (sz) and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$.
*Proof.*Suppose that S satisfies property (gh) that is $\sigma(S)\backslash \sigma_{SBF_+^-}(S) = E(S)$. Let $\lambda \in \sigma(S)\backslash\sigma_{SF_+^-}(S)$. Since $\sigma(S)\backslash \sigma_{SF_+^-}(S) \subset \sigma(S)\backslash\sigma_{SBF_+^-}(S)$, we have $\lambda \in \sigma(S)\backslash\sigma_{SBF_+^-}(S)$. As S satisfies property (gh), then $\lambda \in E(S)$.

Now, let $\lambda \in E(S)$. Since $E(S) \subset E_a(S)$ then $\lambda \in E_a(S)$ from the previous theorem we get $\lambda \in \sigma(S)\backslash \sigma_{SBF_+^-}(S)$. Consequently, $E(S) = \sigma(S)\backslash\sigma_{SBF_+^-}(S)$. Since S satisfies property (gh) and property (sz) then $\sigma(S)\backslash \sigma_{SF_+^-}(S) = E(S)$and $\sigma(S)\backslash\sigma_{SBF_+^-}(S) = E(S)$. And so $\sigma(S)\backslash\sigma_{SF_+^-}(S) = \sigma(S)\backslash\sigma_{SBF_+^-}(S)$ and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$.

Conversely, suppose that if S satisfies property (sz) and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$, then $\sigma(S)\backslash\sigma_{SBF_+^-}(S) = \sigma(S)\backslash \sigma_{SF_+^-}(S) = E(S)$. Then S satisfies property (gh).

3. Properties (ao) and (asz)

Now, we define another property of spectrum of an operator

Definition 3.1 A bounded linear operator $S \in B(X)$ is said to have property (ao) if $\Delta_+(S) = \Pi_a(S)$, and property (asz) if $\Delta_+(S) = \Pi(S)$.

Example 3.2 Let S be defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $S = R \oplus 0$, where R is the unilateral right shift operator. Then we have $(T) = D(0,1)$, , the closed unit disc in $\mathbb{C}$, and $\sigma_a(S) = \sigma_{SF_+^-}(S) = C(0,1) \cup \{0\}$, where $C(0,1)$ is unit circle of $\mathbb{C}$. Moreover, $\Pi_a(S) = \{0\}$,$\Pi(S) = \phi$and $\sigma(S)\backslash\sigma_{SF_+^-}(S) = \{0\}$. So S satisfies property (ao) but does not satisfy property (asz).

Example 3.3 The Volterra operator V on $L^2([0,1])$ is defined by $Vf(t) = \int_0^t f(s)ds$,for $f \in L^2([0,1])$. It is well known that $\sigma(V) = \{0\}$, and $\sigma_{SF_+^-}(V) = \{0\}$. Moreover, $\Pi(V) = \phi$.

Hence the Volterra operator is an example satisfying the property (asz).

Example 3.4 Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$ and $U \in (\ell^2(\mathbb{N}))$ be defined by $U(x_1, x_2, \ldots.) = (0, x_2, x_3, \ldots..)$. Define an operator S on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$by $S = R \oplus U$. Then we have $\sigma(S) = D(0,1)$, the closed unit disc in $\mathbb{C}$, and $\sigma_{SF_+^-}(S) = C(0,1)$, where $C(0,1)$ is unit circle of $\mathbb{C}$. then$\Delta_+(S) = \phi$ and , $\Pi_a(S) = \{0\}$. Then$S$ does not satisfy property (ao).

Proposition 3.5 Let $S \in B(X)$. Then S satisfies property (ao) if and only if S satisfies property $(UW_{\Pi a})$ and $\sigma(S) = \sigma_a(S)$.

proof. Assume that S has satisfies property (ao). If $\lambda \in \Delta_a(S)$, then $\lambda \in \Delta(S)$. Hence $\lambda \in \Pi_a(S)$, and $\Delta_a(S) \subset \Pi_a(S)$.

Closure of the other side, if $\lambda \in \Pi_a(S)$, then $\lambda \in \sigma_a(S)$. And since S satisfies property (ao), then $\lambda \notin \sigma_{SF^+}(S)$, and $\Pi_a(S) \subset \Delta_a(S)$.Thus, S satisfies property $(UW_{\Pi a})$. We then have $\sigma(S) \setminus \sigma_{SF_+^-}(S) = \Pi_a(S)$and $\sigma_a(S) \setminus \sigma_{SF_+^-}(S) = \Pi_a(S)$. Therefore, $\sigma(S) \setminus \sigma_{SF_+^-}(S) = \sigma_a(S) \setminus \sigma_{SF_+^-}(S)$ and$\sigma(S) = \sigma_a(S)$.

Conversely, assume that S satisfies property $(UW_{\Pi a})$ and $\sigma(S) = \sigma_a(S)$. $\sigma(S) \setminus \sigma_{SF_+^-}(S) = \sigma_a(S) \setminus \sigma_{SF_+^-}(S) = \Pi_a(S)$. Thus, $\sigma(S) \setminus \sigma_{SF_+^-}(S) = \Pi_a(S)$ and S satisfies property (ao).

Proposition 3.6 Let $S \in B(X)$. Then S satisfies property (asz) if and only if S satisfies property (UW_Π) and $\sigma(S) = \sigma_a(S)$.

*proof.*Assume that S satisfies property (asz). If $\lambda \in \Delta_a(S)$then $\lambda \in \Delta_+(S)$. Since S satisfies property (asz) then $\lambda \in \Pi(S)$.

Now, if $\lambda \in \Pi(S)$ then $\lambda \in \Pi_a(S)$, Since $(\Pi(S) \subset \Pi_a(S))$, and $\lambda \in \sigma_a(S)$. And since S satisfies property (asz) then $\lambda \notin \sigma_{SF\pm}(S)$. It follows that $\Pi(S) \subset \Delta_a(S)$.Then$S$ satisfies property (UW_Π). Thus, S has property (asz) and property (UW_Π). Then $\Delta_a(S) = \Delta_+(S) = \Pi(S)$, and so $\sigma(S) = \sigma_a(S)$.

Conversely, assume that S satisfies property $(UW_\Pi)$and $(S) = \sigma_a(S)$. $\Delta_a(S) = \Delta_+(S) = \Pi(S)$. Thus, $\sigma(S) \setminus \sigma_{SF_+^-}(S) = \Pi(S)$. Then S satisfies property (asz).

But the convers of this inclusion does not satisfy in general, as we can see in examples 3.7, and 3.8.

Example 3.7 Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$ and $U \in \ell^2(\mathbb{N})$be defined by $U(x_1, x_2, \ldots.) = (0, x_2, x_3, \ldots..)$. Define an operator S on $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$by $S = R \oplus U$. Then we have $\sigma(S) = D(0,1)$, the closed unit disc in $\mathbb{C}$, $\sigma_a(S) = C(0,1) \cup \{0\}$, and $\sigma_{SF_+^-}(S) = C(0,1)$, where $C(0,1)$ is unit circle of $\mathbb{C}$. then$\Delta_+(S) = \phi, \Delta_a(S) = \{0\}$, and $\Pi_a(S) = \{0\}$. Then S satisfy property $(UW_{\Pi a})$ but does not satisfy property (ao).

Example 3.8 Let Sbe operator define on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $S = R \oplus 0$, where R is the unilateral right shift operator. We have $(S) = D(0,1)$, , the closed unit disc in $\mathbb{C}$, and $\sigma_a(S) = \sigma_{SF_+^-}(S) = C(0,1) \cup \{0\}$, where $C(0,1)$ is unit circle of $\mathbb{C}$. Moreover, $\Pi(S) = \emptyset$. So S satisfies property (UW_Π) but S does not satisfy property (asz).

Theorem 3.9 Let $S \in B(X)$. Then S has property (Sab) if and only if Shas property (ao).

proof. Suppose that S has property (Sab) . If $\lambda \in \Delta_+(S)$, then $\lambda \in \Pi_a^0(S)$.Hence $\lambda \in \sigma_a(S)$, is called left pole of S of finite rank if λ left pole and$\alpha(S - \lambda I) < \infty$, consequently,$\lambda \in \Pi_a(S)$. Then $\Delta_+(S) \subset \Pi_a(S)$.

On other hand, if $\lambda \in \Pi_a(S)$, then $\lambda \in \sigma_a(S)$ and so $\lambda \in \sigma(S)$. Now we must prove that $\lambda \notin \sigma_{SF\pm}(S)$.Since S satisfies property (Sab) we have $\lambda \notin \sigma_{SBF\pm}(S)$ i.e. $S - \lambda I$ is an upper semi-B-Weyl operator of S.Then $(S - \lambda I)$ is an upper semi-B-Fredholm operator of S and $\alpha(S - \lambda I) < \infty$. By ([2], Lemma (2.4)), then $(S - \lambda I)$ is an upper semi-Fredholm operator and so $(S - \lambda I)$ is an upper semi-Weyl operator of S. that means $\lambda \in \sigma(S) \setminus \sigma_{SF_+^-}(S)$, thus, S satisfies property (ao).

Conversely, suppose that Ssatisfies property (ao). By proposition (3.5) we have S has property $(UW_{\Pi_a})$and $\sigma(S) = \sigma_a(S)$. And by ([4], theorem 2.6), S satisfies property (SBab), and from ([9], theorem 2.7), S has property (Sab).

Theorem 3.10 Let $S \in B(X)$. Then S satisfies property (ao) if and only if S satisfies property (asz).

proof. We assume that S satisfies property (ao)**.** If $\lambda \in \Delta_+(S)$then $\lambda \in \Pi_a(S)$, and S is a left Drizan invertible. From ([7], theorem 2.8), then $ind\ S \leq 0$ and $0 < p(S) = q(S) < \infty$. Therefore, $\lambda$is a pole of resolvent. On other hand, if $\lambda \in \Pi(S)$, by ([1], theorem 3.81), then $\lambda \in \sigma(S)$.Since$S$ satisfies property (ao), then $\lambda \notin \sigma_{SF_+^-}(T)$ and $\lambda \in \Delta_+(S)$. Then S satisfies property (asz).

In the opposite direction, if $\lambda \in \Delta_+(S)$ and since S satisfies property $\Delta_+(S) = \Pi(S)$, then $\lambda \in \Pi(S)$. But $\Pi(S) \subset \Pi_a(S)$, then $\lambda \in \Pi_a(S)$. Therefore, $\Delta_+(S) \subset \Pi_a(S)$. Now, if $\lambda \in \Pi_a(S)$, then $\lambda \in \sigma_a(S)$ and so $\lambda \in \sigma(S)$. Since S satisfies property $\Delta_+(S) = \Pi(S)$, then $\Pi_a(S) \subset \Delta_+(S)$. Thus, S satisfies property (ao).

Theorem 3.11 Let $S \in B(X)$. Then S satisfies property (gah) if and only if S satisfies property (asz) and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$.

*Proof.*suppose that S satisfies property (gah) that is $\sigma(S) \setminus \sigma_{SBF_+^-}(S) = \Pi(S)$. Let $\lambda \in \sigma(S) \setminus \sigma_{SF_+^-}(S)$. Since $\sigma(S) \setminus \sigma_{SF_+^-}(S) \subset \sigma(S) \setminus \sigma_{SBF_+^-}(S)$, we have $\lambda \in \sigma(S) \setminus \sigma_{SBF_+^-}(S)$. As S satisfies property (gah), then $\lambda \in \Pi(S)$.

If $\lambda \in \Pi(S)$ then $\lambda \in \Pi_a(S)$, Since $(\Pi(S) \subset \Pi_a(S))$. But since $\Pi_a(S) = \sigma_a(S) \setminus \sigma_{LD}(S)$, this means that, S is left

Drizn invertible. From ([7], theorem 2.8), Then S is an upper semi-B-Fredholm operator equal zero. And by ([2], Lemma (2.4)), $\lambda \in \sigma(S)\setminus\sigma_{SF_+^-}(S)$. Thus, S satisfies property (asz).

Now we will prove that $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$. Since S satisfies property (gah) and property (asz) , we get $\sigma(S)\setminus\sigma_{SF_+^-}(S) = \sigma(S)\setminus\sigma_{SBF_+^-}(S) = \Pi(S)$. Thus, $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$.

On other hand, suppose that S satisfies property (asz) and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S), \sigma(S)\setminus\sigma_{SBF_+^-}(S) = \sigma(S)\setminus\sigma_{SF_+^-}(S) = \Pi(S)$. Then S satisfies property (gah).

Theorem 3.12 Let $S \in B(X)$. Then S satisfies property (gaz) if and only if S satisfies property (ao) and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$.

Proof. Suppose that S satisfies property (gaz) that is $\sigma(S)\setminus\sigma_{SBF_+^-}(S) = \Pi_a(S)$. Let $\lambda \in \sigma(S)\setminus\sigma_{SF_+^-}(S)$. Since $\sigma(S)\setminus\sigma_{SF_+^-}(S) \subset \sigma(S)\setminus\sigma_{SBF_+^-}(S)$, we have $\lambda \in \sigma(S)\setminus\sigma_{SBF_+^-}(S)$. As S satisfies property (gaz), then $\lambda \in \Pi_a(T)$.Now, if $\lambda \in \Pi_a(T)$, then by previous theorem we get $\lambda \in \sigma(S)\setminus\sigma_{SF_+^-}(S)$. Thus, S satisfies property (ao).

Let us also prove $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$. Since S satisfies property (gaz) and property (ao) , we get $\sigma(S)\setminus\sigma_{SF_+^-}(S) = \sigma(S)\setminus\sigma_{SBF_+^-}(S) = \Pi_a(S)$. Thus, $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$.

On other hand, suppose that if S satisfies property (ao) and $\sigma_{SF_+^-}(S) = \sigma_{SBF_+^-}(S)$, then $\sigma(S)\setminus\sigma_{SBF_+^-}(S) = \sigma(S)\setminus\sigma_{SF_+^-}(S) = \Pi_a(S)$. Then S satisfies property (gaz).

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