On Fuzzy Measure and Fuzzy Integral on Fuzzy Set

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Abstract: The purpose of this paper is to study fuzzy integral of fuzzy measure based on fuzzy set. In this paper we formulate some convergence theorems on a fuzzy σ-algebra of fuzzy sets. We provide some structural characteristics of a fuzzy measure, the auto continuity and some other concepts for a fuzzy measure are introduced. We capture some of the case of fuzzy sets, Riesz theorems and Lebesgue theorems of a sequence of measurable functions.

Keywords:

1. Introduction

Sugeno [9] has introduced the concepts of Fuzzy integral defined on a classical σ-algebra. The concepts of fuzzy measure and fuzzy integral was defined by Ralescu and Adams [8] based on this type of realize an approach from subjective evaluation to objective evaluation for ‘nonfuzzy’ events [9,13], Wang [10,11] and Kruse [3] studied some structural characteristics of fuzzy measures and proved several convergence theorems for a sequence of fuzzy integrals. In this paper we consider a theory of fuzzy measure and fuzzy integral on fuzzy σ-algebra of fuzzy sets.

In this paper, we will discuss continuity

We introduce in this section some definitions of auto continuity and null-subtraction and some other concepts for a fuzzy measure on a fuzzy σ-algebra of fuzzy sets, and discuss the relations between these concepts.

In section 3, focus on the concept of fuzzy integrals. In section 4, we present some convergence theorems of a sequence of fuzzy integrals on fuzzy sets will be proved.

2. Some Structural Characteristic of Fuzzy Measure on Fuzzy σ-Algebra

We introduce in this section some definitions of auto continuity and null-subtraction and some other concepts for a fuzzy measure on a fuzzy σ-algebra of fuzzy sets, and discuss the relations between these concepts.

We can assume that X is a nonempty classical set, and F(X)= {A; A: X → [0,1]} is the class of all fuzzy sets of. Also we define:

$$\bigcup_{\emptyset} \emptyset = \emptyset, \bigcap_{\emptyset} \emptyset = \emptyset, \sup_{\emptyset} \emptyset = 0, \inf_{\emptyset} \emptyset = 0, 0, \infty = 0.$$ 

A fuzzy σ-algebra $\mathcal{F}$ is a nonempty subclass of F(X) with the properties:

1) $\emptyset, X \in \mathcal{F}$;
2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;
3) If $\{A_n\} \subseteq \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Clearly, an arbitrary classical σ-algebra must be a fuzzy σ-algebra. In this paper, $\mathcal{F}$ shall always denote a fuzzy σ-algebra.

Definition 2.1

A fuzzy set function $\bar{\mu}: \mathcal{F} \rightarrow [0, \infty]$ is called a fuzzy measure, if and only if

1) $\bar{\mu}(\emptyset) = 0$;
2) Whenever $A, B \in \mathcal{F}$, $A \subset B$, then $\bar{\mu}(A) \leq \bar{\mu}(B)$ (monotonicity);
3) Whenever $\{A_n\} \subseteq \mathcal{F}$, $A_n \subseteq A_{n+1}$, then $\bar{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \bar{\mu}(A_n)$ (continuity from below);
4) Whenever $\{A_n\} \subseteq \mathcal{F}$, $A_n \supseteq A_{n+1}$, $n = 1, 2, \ldots$, and there exists $n_0$ such that $\bar{\mu}(A_{n_0}) < \infty$ then $\bar{\mu}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \bar{\mu}(A_n)$ (continuity from above).

The triple $(X, \mathcal{F}, \bar{\mu})$ is called a fuzzy measure space. $\bar{\mu}$ is a fuzzy measure.

Definition 2.2

$\bar{\mu}$ is called null-subtractive, if we have $\bar{\mu}\left(\bar{\text{A}} \cap \bar{\text{B}}\right) = \bar{\mu}(\bar{\text{A}})$, whenever $\bar{\text{A}}, \bar{\text{B}} \in \mathcal{F}$, $\bar{\mu}(\bar{\text{B}}) = 0$.

Definition 2.3

Let $\bar{\text{A}} \in \mathcal{F}$, $\bar{\mu}(\bar{\text{A}}) < \infty$. $\bar{\mu}$ is called null-subtractive with respect to $\bar{\text{A}}$, if for any $\bar{E} \in \mathcal{F}$ and $\bar{F}$ we have $\bar{\mu}(\bar{E} \cap \bar{B}) = \bar{\mu}(\bar{E})$ whenever $\bar{B} \in \mathcal{F}$, $\bar{\mu}(\bar{A} \cap \bar{B}) = \bar{\mu}(\bar{A})$. Here $\bar{\text{A}} \cap \mathcal{F} = (\bar{\text{A}} \cap \bar{\text{B}})$.

Definition 2.4

$\bar{\mu}$ is called auto continuous from above (resp. auto continuous from below),

If $\bar{\mu}(\bar{B}_n) \rightarrow 0$ implies $\bar{\mu}(\bar{A} \cup \bar{B}_n) \rightarrow \bar{\mu}(\bar{A})$ (resp. $\bar{\mu}(\bar{B}_n \cap \bar{A}) \rightarrow \bar{\mu}(\bar{A})$), whenever is called autocontinuous, if it is both autocontinuous from above and autocontinuous from below.

Definition 2.5

Let $\bar{\text{A}} \in \mathcal{F}$, $\bar{\mu}(\bar{\text{A}}) < \infty$, is called autocontinuous from above with respect to $\bar{\text{A}}$ (resp. from below with respect to $\bar{\text{A}}$), if for any $\{\bar{B}_n\} \subseteq \mathcal{F}$ when $\bar{\mu}(\bar{B}_n \cap \bar{A}) \rightarrow \bar{\mu}(\bar{A})$, then $\bar{\mu}(\bar{B}_n \cap \bar{A}) \cap \bar{E} \rightarrow \bar{\mu}(\bar{E})$ (resp. $\bar{\mu}(\bar{B}_n \cap \bar{A}) \cup \bar{E} \rightarrow \bar{\mu}(\bar{E})$) whenever $\bar{E} \in \mathcal{F}$.

Definition 2.6

1) $\bar{\mu}$ is called fuzzy additive, if for any $\bar{A}, \bar{B} \in \mathcal{F}$, we have $\bar{\mu}(\bar{A} \cup \bar{B}) = \bar{\mu}(\bar{A}) \cup \bar{\mu}(\bar{B})$;
2) \( \bar{\mu} \) is called sub additive, if for any \( \bar{A}, \bar{B} \in \hat{F} \), we have 
\[ \bar{\mu}(A \cup B) \leq \bar{\mu}(A) + \bar{\mu}(B); \]
3) \( \bar{\mu} \) is called super sub additive, if for any \( \bar{A}, \bar{B} \in \hat{F} \), we have 
\[ \bar{\mu}(A \cap B) \geq \bar{\mu}(A) \cdot \bar{\mu}(B). \]

**Lemma 2.1**
If \( \bar{\mu} \) is sub additive (resp. super sub additive), then it is autocontinuous from above (resp. from below).

**Proof:**
Suppose that \( \bar{\mu} \) is super sub additive \( \bar{A} \in \hat{F} \), and \( (B_n) \in \hat{F} \varepsilon > 0 \). For any given, take \( \varepsilon = \delta \); if \( \bar{\mu}(B_n) < \delta \),
Then \( \bar{\mu}(\bar{A}) - \varepsilon < \bar{\mu}(B_n) \leq \bar{\mu}(\bar{A} \cup B_n) \leq \bar{\mu}(\bar{A}) + \varepsilon. \)
That is \( \bar{\mu} \) is auto continuous from below.

Suppose \( \bar{\mu} \) is auto continuous, \( \bar{\mu}(B_n) \to 0 \) ie. \( \bar{\mu}(A \cup B_n) \to \bar{\mu}(A) \) (resp. \( \bar{\mu}(A \cap B_n) \to \bar{\mu}(A) \)).
For any given, take \( \varepsilon = \delta \); if \( \bar{\mu}(B_n) < \delta \),
Then \( \bar{\mu}(\bar{A}) - \varepsilon < \bar{\mu}(B_n) \leq \bar{\mu}(\bar{A} \cup \bar{B}_n) \leq \bar{\mu}(\bar{A}) + \varepsilon. \)
That is \( \bar{\mu} \) is super sub additive.
Hence the proof.

3. Fuzzy Integrals On Fuzzy Sets and Their Properties

In this section we introduce a new sequence of \( \lambda \)-cuts and the concepts of fuzzy integrals on fuzzy sets, and derive its properties. We assume \( \lambda \geq 0 \).

**Definition 3.1**
Let \( \bar{A} \in \hat{F} \) and \( f_\lambda \in \hat{M}^+ \). The fuzzy integral of \( f_\lambda \) on \( \bar{A} \) with respect to \( \bar{\mu} \) by \( \bar{\mu} \) is defined by
\[ \int_A f_\lambda d\bar{\mu} = \sup_{\lambda \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
Where \( F_\lambda = \{ x; f_\lambda(x) \geq \alpha \}, \alpha \in [0, \infty], \lambda \in [0, \infty] \)

**Remark 3.1**
\[ \int_A f_\lambda d\bar{\mu} = \sup_{\lambda \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
\[ = \sup_{\alpha \in (0, \infty)} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
\[ \int_A f_\lambda d\bar{\mu} = \inf_{\alpha \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
\[ = \inf_{\alpha \in (0, \infty)} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
**Theorem 3.1**
\[ \int_A f_\lambda d\bar{\mu} = \sup_{\alpha \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
\[ = \sup_{\alpha \in (0, \infty)} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
Where \( F_\lambda = \{ x; f_\lambda(x) > \alpha \}, \alpha \in [0, \infty] \)

**Proof:**
We only prove the first equation.

By using the monotonicity of \( \bar{\mu} \), we have
\[ \bar{\mu}(\bar{A} \cap F_\lambda) \geq \bar{\mu}(\bar{A} \cap F_\alpha) \] for any \( \alpha \in [0, \infty] \).
And therefore
\[ \int_A f_\lambda d\bar{\mu} \geq \sup_{\alpha \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
If we assume that
\[ \int_A f_\lambda d\bar{\mu} > \sup_{\alpha \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] = b, \]
Then there exists \( \varepsilon > 0 \) such that
\[ \int_A f_\lambda d\bar{\mu} > \sup_{\alpha \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] = b + \varepsilon. \]
Thus there exists \( \alpha_0 \) such that \(\alpha_0 \wedge \bar{\mu}(\bar{A} \cap F_{\alpha_0}) > b + \varepsilon, \) by which \( \alpha_0 > b + \varepsilon \) and \( \bar{\mu}(\bar{A} \cap F_{\alpha_0}) > b + \varepsilon \) and
It follows that
\[ \bar{\mu}(\bar{A} \cap F_{\alpha_0}) > b + \varepsilon \] therefore,
\[ \sup_{\alpha \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \geq (b + \varepsilon) \wedge \bar{\mu}(\bar{A} \cap F_{\alpha_0}) = b + \varepsilon \]
This is a contradiction.

**Remark 3.2**
\( f \) is called fuzzy integrable on \( \bar{A} \), if \( \int_A f d\bar{\mu} < \infty \).

**Theorem 3.2**
\( \int_A f d\bar{\mu} < \infty \), if and only if there exists \( \alpha_0 \in [0, \infty] \) such that
\( \bar{\mu}(\bar{A} \cap F_{\alpha_0}) < \infty \).

**Proof:**
If there exists \( \alpha_0 \in [0, \infty] \) such that \( \bar{\mu}(\bar{A} \cap F_{\alpha_0}) \) then
\( \bar{\mu}(\bar{A} \cap F_\lambda) \geq \bar{\mu}(\bar{A} \cap F_{\alpha_0}) = a \) for any \( \lambda > a \) consequently
\[ \int_A f d\bar{\mu} = \sup_{\alpha \in [0, \alpha_0]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
\[ \sup_{\alpha \in [0, \alpha_0]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] \]
Conversely if for any \( \alpha \in [0, \infty], \bar{\mu}(\bar{A} \cap F_\lambda) = \infty \), then
\[ \int_A f d\bar{\mu} = \sup_{\alpha \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] = \sup_{\alpha \in [0, \infty]} \left[ a \wedge \bar{\mu}(\bar{A} \cap F_\lambda) \right] = \infty \]

**Theorem 3.3**
Let \( \bar{A} \in \hat{F}, \alpha \in [0, \infty], \) Then
1. \( \int_A f_\lambda d\bar{\mu} > \alpha \Leftrightarrow \exists \beta \in [0, \alpha] \) such that \( \bar{\mu}(\bar{A} \cap F_{\beta}) > \alpha \) hence \( f_{\lambda \mu}(\bar{A} \cap F_{\beta}) > \alpha \) therefore \( \int_A f_{\lambda \mu} d\bar{\mu} > \alpha \).
2. \( \int_A f_\lambda d\bar{\mu} < \alpha \Leftrightarrow \bar{\mu}(\bar{A} \cap F_\lambda) < \alpha \) therefore \( \int_A f d\bar{\mu} < \alpha \).
3. \( \int_A f_{\lambda \mu} d\bar{\mu} \leq \alpha \Leftrightarrow \forall \beta \in [0, \alpha], \bar{\mu}(\bar{A} \cap F_\lambda) \leq \alpha \Leftrightarrow \bar{\mu}(\bar{A} \cap F_\lambda) \leq a \bar{\mu}(\bar{A} \cap F_\lambda) \) consequently if \( \bar{\mu}(\bar{A} < \infty \) then
\[ \int_A f_{\lambda \mu} d\bar{\mu} \leq \alpha \Leftrightarrow \bar{\mu}(\bar{A} \cap F_\lambda) \leq \alpha \Leftrightarrow \bar{\mu}(\bar{A} \cap F_\lambda) \]

**Proof:**
1. Let \( \alpha \in (0, \infty) \). If \( \bar{\mu}(\bar{A} \cap F_{\beta}) \leq \alpha \) for any \( \beta > \alpha \), then
\[
\int_A f \, d\mu \leq \inf_{\beta \in [0,\alpha]} \left[ \alpha \wedge \bar{\mu}(A \cup F_{\beta}) \right] = \inf_{\alpha \in \mathbb{R}} \left[ \inf_{\beta \in [0,\alpha]} [\beta \wedge \alpha] \right] = \alpha.
\]

On the other hand if there exist \( \beta > \alpha \) such that \( \bar{\mu}(A \cup F_{\beta}) \) then whenever \( \mu(A \cup F_{\beta}) \geq \alpha \) \( \beta \leq \alpha \).

Moreover, we have
\[
\int_A f \, d\mu = \inf_{\beta \in [0,\alpha]} \left[ \beta \wedge \bar{\mu}(A \cup F_{\beta}) \right] \leq \alpha.
\]

The equivalence relation is proved.

2. If \( \int_A f \, d\mu < \alpha \), then there exists \( \alpha > \alpha \) such that \( \int_A f \, d\mu = \alpha \) if and only if \( \bar{\mu}(A \cup F_{\beta}) = \alpha \) whenever \( \beta > \alpha \).

Now, if we take \( \beta = \frac{1}{2}(\alpha + \alpha) \), then \( \alpha > \beta > \alpha \), and therefore, \( \bar{\mu}(A \cup F_{\beta}) = \alpha \) whenever \( \beta > \alpha \).

On the other hand, let \( \bar{\mu}(A \cup F_{\beta}) < \alpha \) if \( \beta < \alpha \), By remark 3.2 that \( A \cup F_{\beta} \) is \( \bar{\mu} \).

By the continuity from above of \( \bar{\mu} \), we have \( \bar{\mu}(A \cup F_{\beta}) < \alpha \) if \( \beta < \alpha \), and therefore, there exists \( \alpha < \alpha \) such that \( \bar{\mu}(A \cup F_{\beta}) < \alpha \).

Thus we have \( \int_A f \, d\mu = \alpha \).

The equivalence relation is proved.

3. First of all, by using (1) and (2), it follows immediately that
\[
\int_A f \, d\mu = \alpha \Leftrightarrow \forall \beta \in [0,\alpha], \quad \bar{\mu}(A \cup F_{\beta}) = \alpha \\
\quad \text{and therefore, there exists } \alpha < \alpha \text{ such that } \bar{\mu}(A \cup F_{\beta}) < \alpha.
\]

Thus we have \( \int_A f \, d\mu = \alpha \).

Hence the theorem.

4. Convergence Theorems of a Sequence of Fuzzy Integrals on Fuzzy Sets

This section presents some convergence theorems of a sequence of fuzzy integrals on fuzzy sets and we now provide certain definitions which are useful in the sequel.

Definition: 4.1
Let \( f \in L^1(A) \), \( \alpha \in \mathbb{R} \), \( \mu \in M(A) \). Then the fuzzy integral of \( f \) and \( A \) with respect to \( \mu \) is defined as
\[
\int_A f \, d\mu = \sup_{\lambda \in [0,1]} \int f(x) \, d\mu = \int_A f(x) \, d\mu(A).
\]

Where \( f(x) = \{ r \in (0,1) : f(x) > r \} \) and \( \mu(A) \) is similar.

Theorem: 4.1

Let \( f_n \rightarrow f \) in \( L^1(A) \) and \( A \in F \), then \( \mu(A) \) is continuous. Suppose that \( \mu \) is continuous and \( f_n \rightarrow f \) on \( X \).

Then for any \( A \in F \), \( \varepsilon > 0 \),
\[
0 \leq \mu(A \cap (|f_n - f| > \varepsilon)) \leq \mu \left( |f_n - f| > \varepsilon \right) \leq 0.
\]

That is \( \mu(A \cap (|f_n - f| > \varepsilon)) \leq 0 \) as \( n \rightarrow \infty \).
Take $c = \int_{A} f_{n} \, d\mu$.

(1) If $c < \infty$ then, by using $\mu(A \cap F_{c^{-}}) \geq c \geq \mu(A \cap F_{c^{+}})$ for every given $\varepsilon > 0$.

We easily prove $F_{n}^{c+2\varepsilon} \subset F_{c^{+}} \cup \{ |f_{n} - f| \geq \varepsilon \}$ and therefore,

$A \cap F_{n}^{c+2\varepsilon} \subset (A \cap F_{c^{+}}) \cup \{ |f_{n} - f| \geq \varepsilon \}$.

Since $\mu$ is autocontinuous from above, we have

$\mu((A \cap F_{c^{+}}) \cup (A \cap \{ |f_{n} - f| \geq \varepsilon \})) \rightarrow \mu(A \cap F_{c^{+}})$.

And hence there exists $n_{0}$ such that

$\mu(A \cap F_{c^{+}}) + \varepsilon \leq c + \varepsilon \leq c + 2\varepsilon$ as $n \geq n_{0}$.

We have

$\int_{A} f_{n} \, d\mu \leq c + 2\varepsilon$ as $n \geq n_{0}$.

On the other hand, in order to show that there exists $\tilde{n}_{0}$ such that $c-2\varepsilon \leq \int_{A} f_{\tilde{n}_{0}} \, d\mu$ as $n \geq \tilde{n}_{0}$ It is sufficient to consider the case $c > 0$.

Clearly, for $\varepsilon \in \left(0, \frac{1}{2}c\right)$.

We can prove $F_{n}^{c+2\varepsilon} \subset F_{c^{+}} \cap \{ |f_{n} - f| \geq \varepsilon \}$, and therefore

$A \cap F_{n}^{c+2\varepsilon} \subset A \cap F_{c^{+}} \cap \{ |f_{n} - f| \geq \varepsilon \}$.

By the auto continuity from below of $\mu$, we have

$\mu(A \cap F_{c^{+}} \cap \{ |f_{n} - f| \geq \varepsilon \}) \rightarrow \mu(A \cap F_{c^{+}})$.

And thus there exists $\tilde{n}_{0}$ such that

$\mu(A \cap F_{c^{+}}) \geq \mu(A \cap F_{c^{+}}) - \varepsilon \geq c-2\varepsilon$ as $n \geq \tilde{n}_{0}$.

It follows from theorem 4.9(1) that $c-2\varepsilon \leq \int_{A} f_{\tilde{n}_{0}} \, d\mu$ as $n \geq \tilde{n}_{0}$.

Consequently, $\lim_{n \to \infty} \int_{A} f_{n} \, d\mu$ exists, and equals $c$.

(2) If $c = \infty$, then for any $\alpha \in [0, \infty)$,

$\mu(A \cap F_{\alpha}) = \infty$, for any $N > 0$, since

$A \cap F_{n} \supseteq A \cap F_{n+1} \cap \{ |f_{n} - f| \geq \varepsilon \}$.

By using the auto continuity from below of $\mu$, there exists $n_{0}$ such that

$\mu(A \cap F_{n}) \geq \mu(A \cap F_{n+1} \cap \{ |f_{n} - f| \geq \varepsilon \}) \geq \alpha$ as $n \geq n_{0}$.

Therefore, we conclude that $\int_{A} f_{n} \, d\mu \leq \alpha$ as $n \geq n_{0}$.

Which concludes the proof.

References


