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# On Fuzzy Measure and Fuzzy Integral on Fuzzy Set

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**Abstract:** ***The purpose of this paper is to study fuzzy integral of Fuzzy measure based on fuzzy set. In this paper we formulate some convergence theorems on a fuzzy $\sigma$- algebra of fuzzy sets. We provide some structural characteristics of a fuzzy measure, the auto continuity and some other concepts for a fuzzy measure are introduced. We capture some of the case of fuzzy sets, Rieszs theorems and Lebesgue theorems of a sequence of measurable functions.***

**Keywords:**

## 1. Introduction

Sugeno [9] has introduced the concepts of Fuzzy integral defined on a classical $\sigma$- algebra. The concepts of fuzzy measure and fuzzy integral was defined by Ralescu and Adams [8] based on this type of realize an approach from subjective evaluation to objective evaluation for 'nonfuzzy' events [9,13], Wang [10,11] and Kruse [3] studied some structural characteristic of fuzzy measures and proved several convergence theorems for a sequence of fuzzy integrals. In this paper we consider a theory of fuzzy measure and fuzzy integral on fuzzy $\sigma$- algebra of fuzzy sets. The theory discussed in [3, 7-11] is a special case of the theory built in this paper.

In section 2, We recall some basic terms of auto continuity and some other concepts for a fuzzy measure will be introduced on fuzzy $\sigma$- algebra of fuzzy sets (the fuzzy $\sigma$- algebra here is different from that defined in [2]), and the relations between these concepts will be discussed.

In section 3, focus on the concept of fuzzy integrals. In section 4, we present some convergence theorems of a sequence of fuzzy integrals on fuzzy sets will be proved.

## 2. Some Structural Characteristic of Fuzzy Measure on Fuzzy σ- Algebra

We introduce in this section some definitions of auto continuity and null-subtraction and some other concepts for a fuzzy measure on a fuzzy $\sigma$- algebra of fuzzy sets, and discuss the relations between these concepts.
We can assume that $X$ is a nonempty classical set, and F($X$)= $\{\tilde{A};\tilde{A}\text{: } X\rightarrow [0.1]\}$ is the class of all fuzzy sets of. Also we define:

$$\bigcup_{t\in\emptyset}\{\cdot\}=\emptyset,\ \bigcap_{t\in\emptyset}\{\cdot\}=X,\ \sup_{t\in\emptyset}\{\cdot\}=0,\ \inf_{t\in\emptyset}\{\cdot\}=0, 0.\infty=0.$$

A fuzzy $\sigma$- algebra $\tilde{\mathrm{F}}$ is a nonempty subclass of F($X$) with the properties:
(1) $\emptyset, X\in F$;
(2) If $\tilde{A}\in\tilde{F}$,then$\tilde{A}^c\in\tilde{F}$;
(3) if $\{\tilde{A}_n\}\subset\tilde{F}$, then $\cup_{n=1}^{\infty}\tilde{A}_n\in\tilde{F}$,
Clearly, an arbitrary classical $\sigma$- algebra must be a fuzzy $\sigma$- algebra. In this paper, $\tilde{F}$ shall always denote a fuzzy $\sigma$- algebra.

**Definition 2.1**
A fuzzy set function $\tilde{\mu}\colon\tilde{F}\rightarrow[0,\infty]$ is called a fuzzy measure, if and only if
1) $\tilde{\mu}(\emptyset)=0$;
2) Whenever $\tilde{A},\tilde{B}\in\tilde{F}$, $\tilde{A}\subset\tilde{B}$, then $\tilde{\mu}(\tilde{A})\leq\tilde{\mu}(\tilde{B})$ (monotonicity);
3) Whenever $\{\tilde{A}_n\}\subset\tilde{F}$, $\widetilde{A_n}\subset\tilde{A}_{n+1}$,n=1,2……then $\tilde{\mu}(\cup_{n=1}^{\infty}\widetilde{A_n})$=$\lim_{n\rightarrow\infty}\tilde{\mu}(\tilde{A}_n)$ (continuity from below);
4) Whenever$\{\tilde{A}_n\}\subset\tilde{F}$, $\tilde{A}_n\supset\tilde{A}_{n+1}$, n $=1,2\ldots$, and there exists $n_0$ such that $\tilde{\mu}(\tilde{A}_{n0})<\infty$, *then* $\tilde{\mu}\left(\cap_{n=1}^{\infty}\tilde{A}_n\right)=\lim_{n\rightarrow\infty}\tilde{\mu}(\tilde{A}_n)$ (continuity from above).

The triple (X, $\tilde{F},\tilde{\mu}$) is called a fuzzy measure space. $\tilde{\mu}$ is a fuzzy measure.

**Definition 2.2**
$\tilde{\mu}$ is called null-subtractive, if we have $\tilde{\mu}\ (\tilde{A}\cap\widetilde{B^c})=\tilde{\mu}\ (\tilde{A})$, whenever $\tilde{A},\tilde{B}\in\tilde{F}$, $\tilde{\mu}(\tilde{B})=0$.

**Definition 2.3**
Let $\tilde{A}\in\tilde{F}$ , $\tilde{\mu}\ (\tilde{A})<\infty$.$\tilde{\mu}$ is called null-subtractive with respect to $\tilde{A}$ , if for any $\tilde{E}\in\tilde{A}\cap\tilde{F}$ we have $\tilde{\mu}(\tilde{E}\cap\tilde{B})=\tilde{\mu}(\tilde{E})$ whenever $\tilde{B}\in\tilde{F}$, $\tilde{\mu}\ (\tilde{A}\cap\tilde{B})=\tilde{\mu}(\tilde{A})$ Here $\tilde{A}\cap\tilde{F}=\{\tilde{A}\cap\tilde{D};\tilde{D}\in\tilde{F}\}$.

**Definition 2.4**
$\tilde{\mu}$ is called auto continuous from above (resp. auto continuous from below),
If $\tilde{\mu}\ (\widetilde{B_n})\rightarrow 0$ implies $\tilde{\mu}\ (\tilde{A}\cup\tilde{B}_n)\rightarrow\tilde{\mu}(\tilde{A})$ (resp.$\tilde{\mu}\ (\tilde{A}\cap\tilde{B}_n)\rightarrow\tilde{\mu}(\tilde{A})$), whenever is called autocontinous, if it is both autocontinuos from above and autocontinious from below.

**Definition 2.5**
Let $\tilde{A}\in\tilde{F}$, $\tilde{\mu}(\tilde{A})<\infty$, is called autocontinous from above with respect to $\tilde{A}$ (resp. from below with respect to $\tilde{A}$), if for any $\{\widetilde{B_n}\}\subset\tilde{F}$ when $\tilde{\mu}(\widetilde{B_n}\cap\tilde{A})\rightarrow\tilde{\mu}(\tilde{A})$, then $\tilde{\mu}((\widetilde{B_n}\cap\tilde{A})\cup\tilde{E})\rightarrow\tilde{\mu}\ (\tilde{E})$ (resp. $\tilde{\mu}(\widetilde{B_n}\cap\tilde{E})\rightarrow\tilde{\mu}(\tilde{E})$) whenever $\tilde{E}\in\tilde{A}\cap\tilde{F}$,$\tilde{\mu}$ is called autocontinous with respect to $\tilde{A}$, if it is both autocontinous from above with respect to $\tilde{A}$ and autocontinous from below with respect to $\tilde{A}$.

**Definition 2.6**
1) $\tilde{\mu}$ is called fuzzy additive, if for any $\tilde{A},\tilde{B}\in\tilde{F}$, we have $\tilde{\mu}$ $(\tilde{A}\cup\tilde{B})$= $\tilde{\mu}\ (\tilde{A})\vee\tilde{\mu}\ (\tilde{B})$;

2) $\tilde{\mu}$ is called sub additive , if for any $\tilde{A}$, $\tilde{B} \in \tilde{F}$, we have $\tilde{\mu}\left(A \cup \tilde{B}\right) \leq \tilde{\mu}(A) + \tilde{\mu}(\tilde{B})$;
3) $\tilde{\mu}$ is called super subtractive, if for any $\tilde{A}$, $\tilde{B} \in \tilde{F}$, we have
4) $\tilde{\mu}(\tilde{A} \cap \tilde{B}^c) \geq \tilde{\mu}(\tilde{A})$ - $\tilde{\mu}(\tilde{B})$.

**Lemma : 2.1**
If $\tilde{\mu}$ is sub additive (resp. super subtractive), then it is autocontinous from above (resp. from below).

**Proof:**
Suppose that $\tilde{\mu}$ is super subtractive $\tilde{A} \in \tilde{F}$, and $\{\tilde{B}_n\} \subset \tilde{F}$ $\varepsilon > 0$. For any given, take $= \varepsilon; if\ \tilde{\mu}\left(\widetilde{B_n}\right) < \delta$,

Then $\tilde{\mu}\left(\tilde{A}\right)$ -$\varepsilon < \tilde{\mu}\left(\widetilde{B_n}\right) \leq \tilde{\mu}(\tilde{A} \cap \widetilde{B_n}) \leq \tilde{\mu}\left(\tilde{A}\right) + \varepsilon$.

That is $\tilde{\mu}$ is auto continuous from below.

Suppose $\tilde{\mu}$ is auto continuous, $\tilde{\mu}\ (\widetilde{B_n}) \rightarrow 0$ ie. $\tilde{\mu}\ (\tilde{A} \cup \tilde{B}_n) \rightarrow \tilde{\mu}\left(\tilde{A}\right)$ (resp.$\tilde{\mu}\ (\tilde{A} \cap \tilde{B}_n) \rightarrow\ \tilde{\mu}(\tilde{A})$).

For any given, take $= \varepsilon; if\ \tilde{\mu}\left(\widetilde{B_n}\right) < \delta$,

Then $\tilde{\mu}\left(\tilde{A}\right)$ -$\varepsilon < \tilde{\mu}\left(\widetilde{B_n}\right) \leq \tilde{\mu}(\tilde{A} \cup \tilde{B}_n) \leq \tilde{\mu}\left(\tilde{A}\right) + \varepsilon$.
$\tilde{\mu}$ is super sub additive $\tilde{A} \in \tilde{F}$, and $\{\tilde{B}_n\} \subset \tilde{F}$ $\varepsilon > 0$.

That is $\tilde{\mu}$ is super subtractive .
Hence the proof.

## 3. Fuzzy Integrals On Fuzzy Sets and Their Properties

In this section we introduce a new sequence of $\lambda$-cuts and the concepts of fuzzy integrals on fuzzy sets, and derive its properties. We assume $\alpha \geq 0$.

**Definition 3.1**
Let $\tilde{A} \in \tilde{F}$ and $f_\lambda \in \tilde{M}^+$. The fuzzy integral of $f_\lambda$ on $\tilde{A}$ with respect to $\tilde{\mu}$ is defined by

$$\int_{\tilde{A}} (f_\lambda)\, \mathrm{d}\tilde{\mu} \triangleq \sup_{\alpha \in [0.\infty]}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cap F_\alpha\right)\right]$$

Where $F_\alpha = \{x; f_\lambda(x) \geq \alpha\}$, $\alpha \in [0,\infty]$, $\lambda \in [0,\infty]$

**Remark: 3.1**

$$\int_{\tilde{A}} f_\lambda\, \mathrm{d}\tilde{\mu} = \sup_{\alpha \in [0.\infty]}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cap F_\alpha\right)\right] = \sup_{\alpha \in (0,\infty)}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cap F_\alpha\right)\right]$$

$$\int_{\tilde{A}} f_\lambda\, \mathrm{d}\tilde{\mu} = \inf_{\alpha \in [0.\infty]}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cup F_\alpha\right)\right] = \inf_{\alpha \in (0,\infty)}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cup F_\alpha\right)\right]$$

**Theorem: 3.1**

$$\int_{\tilde{A}} f_\lambda\, \mathrm{d}\tilde{\mu} = \sup_{\alpha[0.\infty]}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cap F_{\overline{\alpha}}\right)\right] = \sup_{\alpha \in (0,\infty)}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cap F_{\overline{\alpha}}\right)\right]$$

Where $F_{\overline{\alpha}} = \{x; f_\lambda(x) > \alpha\}, \alpha \in [0,\infty]$

**Proof:**
We only prove the first equation.

By using the monotonicity of $\tilde{\mu}$, we have
$\tilde{\mu}\left(\tilde{A} \cap F_\alpha\right) \geq \tilde{\mu}\left(\tilde{A} \cap F_\alpha\right)$ for any $\alpha \in [0.\infty]$,
And therefore

$$\int_{\tilde{A}} f_\lambda\, \mathrm{d}\tilde{\mu} \geq \sup_{\alpha[0.\infty]}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cap F_{\overline{\alpha}}\right)\right]$$

If we assume that

$$\int_{\tilde{A}} f_\lambda\, \mathrm{d}\tilde{\mu} > \sup_{\alpha \in [0.\infty]}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cap F_{\overline{\alpha}}\right)\right] = b,$$

Then there exists $\varepsilon > 0$ such that $\sup_{\alpha[0.\infty]}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cap F_{\overline{\alpha}}\right)\right] > b + \varepsilon$.
Thus there exists $\alpha_0$ such that $\alpha_0 \wedge \tilde{\mu}\left(\tilde{A} \cap F_{\alpha 0}\right) > b + \varepsilon$, by which $\alpha_0\ > b + \varepsilon$ and $\tilde{\mu}\left(\tilde{A} \cap F_{\alpha 0}\right) > b + \varepsilon$ and

It follows that $\tilde{\mu}\left(\tilde{A} \cap F_{\overline{b+c}}\right) \geq \tilde{\mu}\left(\tilde{A} \cap F_{\alpha 0}\right) > b + \varepsilon$

therefore,

$$\sup_{\alpha[0.\infty]}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cap F_{\overline{\alpha}}\right)\right] \geq (b+\varepsilon) \wedge \tilde{\mu}\left(\tilde{A} \cap F_{\overline{b+c}}\right) = b + \varepsilon > b.$$

This is a contradiction.

**Remark: 3.2**
Let $F_\alpha = \{x; f_\lambda(x) \geq \alpha\}$ and $F_{\overline{\alpha}} = \{x; f_\lambda(x) > \alpha\}$. Then we have

$$\lim_{\beta \to \alpha - 0} F_\beta = \lim_{\beta \to \alpha - 0} F_\beta = F_\alpha \supset F_{\overline{\alpha}} = \lim_{\beta \to \alpha - 0} F_\beta = \lim_{\beta \to \alpha - 0} F_\beta$$

**Definition :3.2**
$f$ is called fuzzy integrable on $\tilde{A}$, if $\int_{\tilde{A}} f\, \mathrm{d}\tilde{\mu}\ < \infty$.

**Theorem: 3.2**
$\int_{\tilde{A}} f_\lambda\, \mathrm{d}\tilde{\mu}\ < \infty$, if and only if there exists $\alpha_0 \in [0,\infty)$ such that $\tilde{\mu}\left(\tilde{A} \cap F_{\alpha 0}\right) < \infty$.

**Proof :**
If there exists $\alpha_0 \in [0,\infty)$ such that $\tilde{\mu}\left(\tilde{A} \cap F_{\alpha 0}\right)$ then $\tilde{\mu}\left(\tilde{A} \cup F_\alpha\right) \geq \tilde{\mu}\left(\tilde{A} \cup F_{\alpha 0}\right) = \alpha$ for any $\alpha > \alpha_0$ consequently

$$\int_{\tilde{A}} f_\lambda\ \ \mathrm{d}\tilde{\mu} = \sup_{\alpha \in [0,\alpha_0]}\left[\alpha \wedge \tilde{\mu}(\tilde{A} \cup F_\alpha\right] \ \ \mathrm{v} \ \sup_{\alpha \in [\alpha_{0,}\infty]}\left[\alpha \wedge\right.$$

$\mu A \cup F\alpha \geq \alpha 0 \vee \alpha < \infty$.

Conversely if for any $\alpha \in [0,\infty)$, $\tilde{\mu}\left(\tilde{A} \cup F_\alpha\right) = \infty$, then

$$\int_{\tilde{A}} f_\lambda\, \mathrm{d}\tilde{\mu} = \sup_{\alpha \in [0,\infty]}\left[\alpha \wedge \tilde{\mu}\left(\tilde{A} \cup F_\alpha\right)\right] = \sup_{\alpha \in [0,\infty]} \alpha = \infty.$$

**Theorem: 3.3**
Let $\tilde{A} \in \tilde{F}$, $\alpha \in [0,\infty)$, Then

1. $\int_{\tilde{A}} f_\lambda \mathrm{d}\tilde{\mu} > \alpha \Leftrightarrow \exists \beta \in [0,\alpha)$ such that $\tilde{\mu}(\tilde{A} \cup F\beta > \alpha$;hence $\int A\, f \mathrm{d}\mu => \alpha \Rightarrow \mu\, A \cup F\alpha\, \mu A \cup F\alpha > \alpha$.

2. $\int_{\tilde{A}} f_\lambda \mathrm{d}\tilde{\mu} < \alpha \Leftrightarrow \tilde{\mu}\left(\tilde{A} \cup F_\alpha\right) < \alpha$; therefore $\int_{\tilde{A}} f \mathrm{d}\,\tilde{\mu} <\ \alpha \Rightarrow\ \tilde{\mu}\left(\tilde{A} \cup F_{\tilde{\alpha}}\right) < \alpha$.
3. $\int_{\tilde{A}} f_\lambda \mathrm{d}\tilde{\mu} = \alpha \Leftrightarrow \forall \beta \in [0,\alpha), \tilde{\mu}\left(\tilde{A} \cup F_\alpha\right) \leq \alpha \leq \tilde{\mu}\left(\tilde{A} \cup F_\alpha\right)$. particulary if $\tilde{\mu}\left(\tilde{A}\right) < \infty$, then $\int_{\tilde{A}} f_\lambda \mathrm{d}\tilde{\mu} = \alpha \Leftrightarrow \tilde{\mu}\left(\tilde{A} \cup F_\alpha\right) \leq \alpha \leq \tilde{\mu}\left(\tilde{A} \cup F_{\tilde{\alpha}}\right)$.

**Proof:**

1. Let $\alpha \in (0,\infty)$. If $\tilde{\mu}\left(\tilde{A} \cup F_\beta\right) \leq \alpha$ for any $\beta > \alpha$, then

$\int_{\tilde{A}} f_\lambda d\tilde{\mu} \leq \inf_{\beta \in [0,\alpha]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cup F_\alpha)] \leq \inf_{\beta \in [0,\alpha]} [\beta \wedge$ $\alpha$=infβ∈0,α β=α.

On the other hand if there exists $\beta > \alpha$ such that $\tilde{\mu}(\tilde{A} \cup F\beta > \alpha$ then whenever $\mu A \cup Fr \geq \mu A \cup F\beta$ whenever $r \leq \beta$. Thus we have

$\int_{\tilde{A}} f_\lambda \quad d\tilde{\mu} = \inf_{r \in [0,\beta]} [r \wedge \tilde{\mu}(\tilde{A} \cup F_r)] \vee \inf_{r \in [\beta,\infty]} [r \wedge \mu A \cup Fr$

$\geq \beta \vee \tilde{\mu}(\tilde{A} \cup F_\beta) > \alpha.$

The equivalence relation is proved.

2. If $\int_{\tilde{A}} f_\lambda \, d\tilde{\mu} < \alpha$,

Then there exists $\alpha_0 < \alpha$ such that $\int_{\tilde{A}} f_\lambda \quad d\tilde{\mu} \leq \alpha_0$ it follows by using (1), that $\tilde{\mu}(\tilde{A} \cup F_\beta) \leq \alpha_0$ whenever $\beta > \alpha_0$

Now, If we take $\beta = \frac{1}{2}(\alpha_0 + \alpha)$, then $\alpha > \beta > \alpha_0$, and therefore, $\tilde{\mu}(\tilde{A} \cup F_{\tilde{\alpha}}) \leq \tilde{\mu}(\tilde{A} \cup F_\beta) \leq \alpha_0 < \alpha$

On the other hand let, $\tilde{\mu}(\tilde{A} \cup F_{\tilde{\alpha}}) < \alpha$ If $\beta_n \nearrow \alpha$,

By remark 3.2 that $\tilde{A} \cup F_{\beta_n} \searrow \tilde{A} \cup F_{\tilde{\alpha}}$.

By the continuity from above of $\tilde{\mu}$ we have $\tilde{\mu}(\tilde{A} \cup F_{\beta_n}) \rightarrow \tilde{\mu}(\tilde{A} \cup F_{\tilde{\alpha}})$ and therefore there exists $\alpha_0 < \alpha$ such that $\tilde{\mu}(\tilde{A} \cup F_{\alpha_0}) < \alpha$

Thus we have $\int_{\tilde{A}} f_\lambda \, d\tilde{\mu} \geq \alpha_0 \wedge \tilde{\mu}(\tilde{A} \cup F_{\alpha_0}) > \alpha$.

The equivalence relation is proved.

3. First of all, by using (1) and (2), it follows immediately that $\int_{\tilde{A}} f_\lambda \quad d\tilde{\mu} = \alpha \Leftrightarrow \forall \beta \in [0,\alpha), \quad \tilde{\mu}(\tilde{A} \cup F_\beta) \leq \alpha \leq \tilde{\mu}(\tilde{A} \cup F_{\tilde{\alpha}})$.

When $\tilde{\mu}(\tilde{A}) < \infty$, we take a sequence $\{r_n\} \subset [0,\alpha)$ such that $r_n \swarrow \alpha$.

We have $\tilde{A} \cup F_{r_n} \nearrow \tilde{A} \cup F_\alpha$, and therefore, it follows from the continuity from below of $\tilde{\mu}$ that $\tilde{\mu}(\tilde{A} \cup F_{r_n}) \rightarrow \tilde{\mu}(\tilde{A} \cup F_\alpha)$.

By the above results,

We obtain $\int_{\tilde{A}} f_\lambda d\tilde{\mu} = \alpha \Leftrightarrow \tilde{\mu}(\tilde{A} \cup F_\alpha) \leq \alpha \leq \tilde{\mu}(\tilde{A} \cup F_{\tilde{\alpha}})$.

Hence the theorem.

## 4. Convergence Theorems of a Sequence of Fuzzy Integrals on Fuzzy Sets

This section presents some convergence theorems of a sequence of fuzzy integrals on fuzzy sets and we now provide certain definitions which are useful in the sequel.

$F^n{}_\alpha = \{x; f_n(x) \geq \alpha\}, F^n{}_{\tilde{\alpha}} = (x; f_n(x) > \alpha)$

$g_n = \inf_{i \geq n} f_i, \; h_n = \sup_{i \geq n} f_i$

**Definition: 4.1**

Let $\bar{f} \in \bar{F}(x), A \in \mathcal{A}, \bar{\mu} \in \bar{M}(x)$. Then the fuzzy integral of $\bar{f}$ and A with respect to μ is defined as

$$\int_A f d\mu(r) = \sup\{\lambda \in (0,1]: r \in \int_A f_\lambda d\mu_\lambda\}$$

Where $f_\lambda(x) = \{r \in (0,1]: f(x)(r) > \lambda\}$ and $\mu_\lambda$ is similar.

**Theorem: 4.1**

Let $\{f_n, f\} \subset \tilde{M}^+$ and $\tilde{A} \in \tilde{F}$. if $f_n \overset{e.}{\rightarrow} f$ on $\tilde{A}$ and there exists $n_0$ and a constant $c \leq \int_{\tilde{A}} f \, d\tilde{\mu} \; (0 \leq c)$ such that $\tilde{\mu}(\tilde{A} \cap hn0>c<\infty$, then $\lim_{n\rightarrow\infty} \int A \, fn \, d\mu$ is existent, and

$\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n \, d\tilde{\mu} = \int_{\tilde{A}} f \, d\tilde{\mu}$.

**Proof:**

For a fixed $\lambda \in (0,1]$ let $\lambda_n = (1- 1/n+1) \lambda$ then $\lambda_n \uparrow \lambda$.

Since $g_{\lambda n} \leq f_{\lambda n} \leq h_{\lambda n}, n = 1,2,....$ By using theorem 3.3, $\int_{\tilde{A}} g_{\lambda n} d\tilde{\mu} \leq \int_{\tilde{A}} f_{\lambda n} \, d\tilde{\mu} \leq \int_{\tilde{A}} h_{\lambda n} \, d\tilde{\mu}$, n= 1,2, ....

Moreover, when $f_{\lambda n} \overset{e.}{\rightarrow} f_\lambda$ on $\tilde{A}$ we have

$g_{\lambda n} \nearrow f_\lambda$ on $\tilde{A}$, $h_{\lambda n} \searrow f_\lambda$ on $\tilde{A}$.

It follows $\tilde{\mu}(\tilde{A} \cap \{h_{\lambda n0} > c\}) < \infty$, that

$\lim_{n \rightarrow \infty} \int_{\tilde{A}} g_{\lambda n} \, d\tilde{\mu} = \lim_{n \rightarrow \infty} \int_{\tilde{A}} h_{\lambda n} \, d\tilde{\mu} = \int_{\tilde{A}} f_\lambda \, d\tilde{\mu}$.

Hence, we know that $\lim_{n\rightarrow\infty} \int_{\tilde{A}} f_n \, d\tilde{\mu}$ exists, and equals $\int_{\tilde{A}} f \, d\tilde{\mu}$

which concludes the proof.

**Theorem 4.2**

Let $\{f_n, f\} \subset \tilde{M}^+$and $\tilde{A} \in \tilde{F}$ and $\tilde{\mu}$ be- null-subtractive if $f_n \overset{a.e.}{\longrightarrow} f$ on $\tilde{A}$ and there exist $n_0$ and a constant $c \leq \int_{\tilde{A}} f \, d\tilde{\mu} \; (0 \leq c)$ such that $\tilde{\mu}(\tilde{A} \cap \{h_{n0} > c\}) < \infty$, then $\lim_{n\rightarrow\infty} \int_{\tilde{A}} f_n \, d\tilde{\mu}$ is existent, and

$\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n \, d\tilde{\mu} = \int_{\tilde{A}} f \, d\tilde{\mu}$

**Proof:**

For a fixed $\lambda \in (0,1]$ let $\lambda_n = (1- 1/n+1) \lambda$ then $\lambda_n \uparrow \lambda$.

Suppose that $f_{\lambda n} \overset{a.e.}{\longrightarrow} f_\lambda$ on $\tilde{A}$.

Namely there exists $\tilde{E} \in \tilde{F}$ with $\tilde{\mu}(\tilde{E}) = 0$, such that $f_{\lambda n} \overset{.e.}{\rightarrow} f_\lambda$ on $\tilde{A} \cap \tilde{E}^c$.

We observing that the null – subtraction of $\tilde{\mu}$ and the condition

$\tilde{\mu}(\tilde{A} \cap \tilde{E}^c \cap \{h_{\lambda n_0} > c\}) \leq \tilde{\mu}(\tilde{A} \cap \{h_{\lambda n_0} > c\}) < \infty$,

It follows that,

$\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_{\lambda n} \, d\tilde{\mu} = \lim_{n \rightarrow \infty} \int_{\tilde{A} \cap \tilde{E}^c} f_\lambda \, d\tilde{\mu} = \int_{\tilde{A} \cap \tilde{E}^c} f \, d\tilde{\mu} = \int_{\tilde{A}} f \, d\tilde{\mu}$.

Hence the proof.

**Theorem 4.3**

Let $\{f_n, f\} \subset \tilde{M}^+$ and $\tilde{\mu}$ be autocontinuous. If $f_n \overset{\tilde{\mu}}{\rightarrow} f$ then for any $\tilde{A} \in \tilde{F}$,

$\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n \, d\tilde{\mu} = \int_{\tilde{A}} f \, d\tilde{\mu}$

**Proof**

For a fixed $\lambda \in (0,1]$ let $\lambda_n = (1- 1/n+1) \lambda$ then $\lambda_n \uparrow \lambda$.

Suppose that $\tilde{\mu}$ is autocontinuous and $f_{n\lambda} \overset{\tilde{\mu}}{\rightarrow} f_\lambda$ on X.

Then for any $\tilde{A} \in \tilde{F}, \varepsilon > 0$,

$0 \leq \tilde{\mu}(\tilde{A} \cap \{|f_{\lambda n} - f_\lambda| \geq \varepsilon\}) \leq \tilde{\mu}(\{|f_{\lambda n} - f_\lambda| \geq \varepsilon\}) \rightarrow 0$,

That is $\tilde{\mu}(\tilde{A} \cap \{|f_{n\lambda} - f_\lambda| \geq \varepsilon\}) \rightarrow 0$ .as $n \rightarrow \infty$ too.

Take c = $\int_{\tilde{A}} f_\lambda \, d\tilde{\mu}$.

(1) If $c < \infty$ then, by using $\tilde{\mu}\left(\tilde{A} \cap F_{c-\varepsilon}\right) \geq c \geq \tilde{\mu}\left(\tilde{A} \cap F_{c-\varepsilon}\right)$ for every given $\varepsilon > 0$.

We easily prove $F^n{}_{c+2\varepsilon} \subset F_{c+\varepsilon} \cup (\{|f_{n\lambda} - f_\lambda| \geq \varepsilon\})$ and therefore,

$\tilde{A} \cap F^n{}_{c+2\varepsilon} \subset (\tilde{A} \cap F_{c+\varepsilon}) \cup \left(\tilde{A} \cap \{|f_{n\lambda} - f_\lambda| \geq \varepsilon\}\right)$.

Since $\tilde{\mu}$ is autocontinuous from above, we have

$\tilde{\mu}\left(\left(\tilde{A} \cap F_{c+\varepsilon}\right) \cup \left(\tilde{A} \cap \{|f_n - f| \geq \varepsilon\}\right)\right) \to \tilde{\mu}\left(\tilde{A} \cap F_{c+\varepsilon}\right)$.

And hence there exists $n_0$ such that

$\tilde{\mu}\left(\tilde{A} \cap F^n{}_{c+2\varepsilon}\right) \leq \tilde{\mu}\left(\tilde{A} \cap F_{c+\varepsilon}\right) + \varepsilon \leq c + \varepsilon \leq c + 2\varepsilon$ as n $\geq n_0$.

We have

$\int_{\tilde{A}} f_\lambda \, d\tilde{\mu} \leq c + 2\varepsilon$ as n $\geq$ $n_0$.

On the other hand, in order to show that there exists $\bar{n}_0$ such that c$-2\varepsilon \leq \int_{\tilde{A}} f_{\lambda n} \, d\tilde{\mu}$ as n$\geq \bar{n}_0$ It is sufficient to consider the case c > 0.

Clearly, for $\varepsilon \in \left(0, \frac{1}{2}c\right)$,

We can prove $F^n{}_{c+2\varepsilon} \supset F_{c+\varepsilon} \cap \{|f_{\lambda n} - f_\lambda| \geq \varepsilon\}^c$, and therefore

$$\tilde{A} \cap F^n{}_{c+2\varepsilon} \supset \tilde{A} \cap F_{c+\varepsilon} \cap \{|f_{n\lambda} - f_\lambda| \geq \varepsilon\}^c$$

By the auto continuity from below of $\tilde{\mu}$, we have

$\tilde{\mu}\left(\tilde{A} \cap F_{c+\varepsilon} \cap \{|f_{n\lambda} - f_\lambda| \geq \varepsilon\}^c\right) \to \tilde{\mu}\left(\tilde{A} \cap F_{c+\varepsilon}\right)$.

And thus there exists $\bar{n}_0$ such that

$\tilde{\mu}\left(\tilde{A} \cap F^n{}_{c+2\varepsilon}\right) \geq \tilde{\mu}\left(\tilde{A} \cap F_{c+\varepsilon}\right) - \varepsilon \geq$ c-2$\varepsilon$ as $n \geq \bar{n}_0$.

It follows from theorem 4.9(1) that c-2$\varepsilon \leq \int_{\tilde{A}} f_{n\lambda} \, d\tilde{\mu}$ as $n \geq \bar{n}_0$ . Consequently, $\lim_{n\to\infty} \int_{\tilde{A}} f_{n\lambda} \, d\tilde{\mu}$ exists, and equals c.

(2) If c $= \infty$, then for any $\alpha \in [0, \infty)$,

$\tilde{\mu}\left(\tilde{A} \cap F_\alpha\right) = \infty$, for any N> 0, since

$\tilde{A} \cap F^n{}_N \supset \tilde{A} \cap F_{N+1} \cap \{|f_{n\lambda} - f_n| \geq \varepsilon\}^c$,

By using the auto continuity from below of $\tilde{\mu}$, there exists $n_0$ such that

$\tilde{\mu}(\tilde{A} \cap F^n{}_N) \geq \tilde{\mu} \quad (\tilde{A} \cap F_{N+1} \cap \{|f_n - f| \geq \varepsilon\}^c) \geq N$ as $n \geq n_0$.

Therefore, we conclude that $\int_{\tilde{A}} f_n \, d\tilde{\mu} \geq N \; as \; n \geq n_0$.

consequently $\lim_{n\to\infty} \int_{\tilde{A}} f_n \, d\tilde{\mu} = \infty = c$.

Which concludes the proof.

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