

On Fuzzy Measure and Fuzzy Integral on Fuzzy Set

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Abstract: *The purpose of this paper is to study fuzzy integral of Fuzzy measure based on fuzzy set. In this paper we formulate some convergence theorems on a fuzzy σ - algebra of fuzzy sets. We provide some structural characteristics of a fuzzy measure, the auto continuity and some other concepts for a fuzzy measure are introduced. We capture some of the case of fuzzy sets, Riesz theorems and Lebesgue theorems of a sequence of measurable functions.*

Keywords:

1. Introduction

Sugeno [9] has introduced the concepts of Fuzzy integral defined on a classical σ - algebra. The concepts of fuzzy measure and fuzzy integral was defined by Ralescu and Adams [8] based on this type of realize an approach from subjective evaluation to objective evaluation for 'nonfuzzy' events [9,13], Wang [10,11] and Kruse [3] studied some structural characteristic of fuzzy measures and proved several convergence theorems for a sequence of fuzzy integrals. In this paper we consider a theory of fuzzy measure and fuzzy integral on fuzzy σ - algebra of fuzzy sets. The theory discussed in [3, 7-11] is a special case of the theory built in this paper.

In section 2, We recall some basic terms of auto continuity and some other concepts for a fuzzy measure will be introduced on fuzzy σ - algebra of fuzzy sets (the fuzzy σ - algebra here is different from that defined in [2]), and the relations between these concepts will be discussed.

In section 3, focus on the concept of fuzzy integrals. In section 4, we present some convergence theorems of a sequence of fuzzy integrals on fuzzy sets will be proved.

2. Some Structural Characteristic of Fuzzy Measure on Fuzzy σ - Algebra

We introduce in this section some definitions of auto continuity and null-subtraction and some other concepts for a fuzzy measure on a fuzzy σ - algebra of fuzzy sets, and discuss the relations between these concepts.

We can assume that X is a nonempty classical set, and $F(X) = \{\tilde{A}; \tilde{A}: X \rightarrow [0,1]\}$ is the class of all fuzzy sets of. Also we define:

$$\bigcup_{t \in \emptyset} \{\cdot\} = \emptyset, \quad \bigcap_{t \in \emptyset} \{\cdot\} = X, \quad \sup_{t \in \emptyset} \{\cdot\} = 0, \quad \inf_{t \in \emptyset} \{\cdot\} = 0, \quad 0 \cdot \infty = 0.$$

A fuzzy σ - algebra \tilde{F} is a nonempty subclass of $F(X)$ with the properties:

- (1) $\emptyset, X \in \tilde{F}$;
- (2) If $\tilde{A} \in \tilde{F}$, then $\tilde{A}^c \in \tilde{F}$;
- (3) if $\{\tilde{A}_n\} \subset \tilde{F}$, then $\bigcup_{n=1}^{\infty} \tilde{A}_n \in \tilde{F}$.

Clearly, an arbitrary classical σ - algebra must be a fuzzy σ - algebra. In this paper, \tilde{F} shall always denote a fuzzy σ - algebra.

Definition 2.1

A fuzzy set function $\tilde{\mu}: \tilde{F} \rightarrow [0, \infty]$ is called a fuzzy measure, if and only if

- 1) $\tilde{\mu}(\emptyset) = 0$;
- 2) Whenever $\tilde{A}, \tilde{B} \in \tilde{F}$, $\tilde{A} \subset \tilde{B}$, then $\tilde{\mu}(\tilde{A}) \leq \tilde{\mu}(\tilde{B})$ (monotonicity);
- 3) Whenever $\{\tilde{A}_n\} \subset \tilde{F}$, $\tilde{A}_n \subset \tilde{A}_{n+1}, n=1,2,\dots$ then $\tilde{\mu}(\bigcup_{n=1}^{\infty} \tilde{A}_n) = \lim_{n \rightarrow \infty} \tilde{\mu}(\tilde{A}_n)$ (continuity from below);
- 4) Whenever $\{\tilde{A}_n\} \subset \tilde{F}$, $\tilde{A}_n \supset \tilde{A}_{n+1}, n=1,2,\dots$, and there exists n_0 such that $\tilde{\mu}(\tilde{A}_{n_0}) < \infty$, then $\tilde{\mu}(\bigcap_{n=1}^{\infty} \tilde{A}_n) = \lim_{n \rightarrow \infty} \tilde{\mu}(\tilde{A}_n)$ (continuity from above).

The triple $(X, \tilde{F}, \tilde{\mu})$ is called a fuzzy measure space. $\tilde{\mu}$ is a fuzzy measure.

Definition 2.2

$\tilde{\mu}$ is called null-subtractive, if we have $\tilde{\mu}(\tilde{A} \cap \tilde{B}^c) = \tilde{\mu}(\tilde{A})$, whenever $\tilde{A}, \tilde{B} \in \tilde{F}$, $\tilde{\mu}(\tilde{B}) = 0$.

Definition 2.3

Let $\tilde{A} \in \tilde{F}$, $\tilde{\mu}(\tilde{A}) < \infty$. $\tilde{\mu}$ is called null-subtractive with respect to \tilde{A} , if for any $\tilde{E} \in \tilde{A} \cap \tilde{F}$ we have $\tilde{\mu}(\tilde{E} \cap \tilde{B}) = \tilde{\mu}(\tilde{E})$ whenever $\tilde{B} \in \tilde{F}$, $\tilde{\mu}(\tilde{A} \cap \tilde{B}) = \tilde{\mu}(\tilde{A})$. Here $\tilde{A} \cap \tilde{F} = \{\tilde{A} \cap \tilde{D}; \tilde{D} \in \tilde{F}\}$.

Definition 2.4

$\tilde{\mu}$ is called auto continuous from above (resp. auto continuous from below),

If $\tilde{\mu}(\tilde{B}_n) \rightarrow 0$ implies $\tilde{\mu}(\tilde{A} \cup \tilde{B}_n) \rightarrow \tilde{\mu}(\tilde{A})$ (resp. $\tilde{\mu}(\tilde{A} \cap \tilde{B}_n) \rightarrow \tilde{\mu}(\tilde{A})$), whenever is called autocontinuous, if it is both autocontinuous from above and autocontinuous from below.

Definition 2.5

Let $\tilde{A} \in \tilde{F}$, $\tilde{\mu}(\tilde{A}) < \infty$, is called autocontinuous from above with respect to \tilde{A} (resp. from below with respect to \tilde{A}), if for any $\{\tilde{B}_n\} \subset \tilde{F}$ when $\tilde{\mu}(\tilde{B}_n \cap \tilde{A}) \rightarrow \tilde{\mu}(\tilde{A})$, then $\tilde{\mu}((\tilde{B}_n \cap \tilde{A}) \cup \tilde{E}) \rightarrow \tilde{\mu}(\tilde{E})$ (resp. $\tilde{\mu}(\tilde{B}_n \cap \tilde{E}) \rightarrow \tilde{\mu}(\tilde{E})$) whenever $\tilde{E} \in \tilde{A} \cap \tilde{F}$, $\tilde{\mu}$ is called autocontinuous with respect to \tilde{A} , if it is both autocontinuous from above with respect to \tilde{A} and autocontinuous from below with respect to \tilde{A} .

Definition 2.6

1) $\tilde{\mu}$ is called fuzzy additive, if for any $\tilde{A}, \tilde{B} \in \tilde{F}$, we have $\tilde{\mu}(\tilde{A} \cup \tilde{B}) = \tilde{\mu}(\tilde{A}) \vee \tilde{\mu}(\tilde{B})$;

- 2) $\tilde{\mu}$ is called sub additive , if for any $\tilde{A}, \tilde{B} \in \tilde{F}$, we have $\tilde{\mu}(A \cup \tilde{B}) \leq \tilde{\mu}(A) + \tilde{\mu}(\tilde{B})$;
- 3) $\tilde{\mu}$ is called super subtractive, if for any $\tilde{A}, \tilde{B} \in \tilde{F}$, we have $\tilde{\mu}(\tilde{A} \cap \tilde{B}^c) \geq \tilde{\mu}(\tilde{A}) - \tilde{\mu}(\tilde{B})$.

Lemma : 2.1

If $\tilde{\mu}$ is sub additive (resp. super subtractive), then it is autocontinuous from above (resp. from below).

Proof:

Suppose that $\tilde{\mu}$ is super subtractive $\tilde{A} \in \tilde{F}$, and $\{\tilde{B}_n\} \subset \tilde{F} \ \varepsilon > 0$. For any given, take $= \varepsilon$; if $\tilde{\mu}(\tilde{B}_n) < \delta$,

$$\text{Then } \tilde{\mu}(\tilde{A}) - \varepsilon < \tilde{\mu}(\tilde{B}_n) \leq \tilde{\mu}(\tilde{A} \cap \tilde{B}_n) \leq \tilde{\mu}(\tilde{A}) + \varepsilon.$$

That is $\tilde{\mu}$ is auto continuous from below.

Suppose $\tilde{\mu}$ is auto continuous, $\tilde{\mu}(\tilde{B}_n) \rightarrow 0$ ie. $\tilde{\mu}(\tilde{A} \cup \tilde{B}_n) \rightarrow \tilde{\mu}(\tilde{A})$ (resp. $\tilde{\mu}(\tilde{A} \cap \tilde{B}_n) \rightarrow \tilde{\mu}(\tilde{A})$).

For any given, take $= \varepsilon$; if $\tilde{\mu}(\tilde{B}_n) < \delta$,

$$\text{Then } \tilde{\mu}(\tilde{A}) - \varepsilon < \tilde{\mu}(\tilde{B}_n) \leq \tilde{\mu}(\tilde{A} \cup \tilde{B}_n) \leq \tilde{\mu}(\tilde{A}) + \varepsilon.$$

$\tilde{\mu}$ is super sub additive $\tilde{A} \in \tilde{F}$, and $\{\tilde{B}_n\} \subset \tilde{F} \ \varepsilon > 0$.

That is $\tilde{\mu}$ is super subtractive .
Hence the proof.

3. Fuzzy Integrals On Fuzzy Sets and Their Properties

In this section we introduce a new sequence of λ -cuts and the concepts of fuzzy integrals on fuzzy sets, and derive its properties. We assume $\alpha \geq 0$.

Definition 3.1

Let $\tilde{A} \in \tilde{F}$ and $f_\lambda \in \tilde{M}^+$. The fuzzy integral of f_λ on \tilde{A} with respect to $\tilde{\mu}$ is defined by

$$\int_{\tilde{A}} (f_\lambda) d\tilde{\mu} \triangleq \sup_{\alpha \in [0, \infty]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cap F_\alpha)]$$

Where $F_\alpha = \{x; f_\lambda(x) \geq \alpha\}$, $\alpha \in [0, \infty]$, $\lambda \in [0, \infty]$

Remark: 3.1

$$\begin{aligned} \int_{\tilde{A}} f_\lambda d\tilde{\mu} &= \sup_{\alpha \in [0, \infty]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cap F_\alpha)] \\ &= \sup_{\alpha \in (0, \infty)} [\alpha \wedge \tilde{\mu}(\tilde{A} \cap F_\alpha)] \\ \int_{\tilde{A}} f_\lambda d\tilde{\mu} &= \inf_{\alpha \in [0, \infty]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cup F_\alpha)] \\ &= \inf_{\alpha \in (0, \infty)} [\alpha \wedge \tilde{\mu}(\tilde{A} \cup F_\alpha)] \end{aligned}$$

Theorem: 3.1

$$\begin{aligned} \int_{\tilde{A}} f_\lambda d\tilde{\mu} &= \sup_{\alpha \in [0, \infty]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cap F_\alpha)] \\ &= \sup_{\alpha \in (0, \infty)} [\alpha \wedge \tilde{\mu}(\tilde{A} \cap F_\alpha)] \end{aligned}$$

Where $F_\alpha = \{x; f_\lambda(x) > \alpha\}$, $\alpha \in [0, \infty]$

Proof:

We only prove the first equation.

By using the monotonicity of $\tilde{\mu}$, we have $\tilde{\mu}(\tilde{A} \cap F_\alpha) \geq \tilde{\mu}(\tilde{A} \cap F_\alpha)$ for any $\alpha \in [0, \infty]$.
And therefore

$$\int_{\tilde{A}} f_\lambda d\tilde{\mu} \geq \sup_{\alpha \in [0, \infty]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cap F_\alpha)]$$

If we assume that

$$\int_{\tilde{A}} f_\lambda d\tilde{\mu} > \sup_{\alpha \in [0, \infty]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cap F_\alpha)] = b,$$

Then there exists $\varepsilon > 0$ such that $\sup_{\alpha \in [0, \infty]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cap F_\alpha)] > b + \varepsilon$.

Thus there exists α_0 such that $\alpha_0 \wedge \tilde{\mu}(\tilde{A} \cap F_{\alpha_0}) > b + \varepsilon$, by which $\alpha_0 > b + \varepsilon$ and $\tilde{\mu}(\tilde{A} \cap F_{\alpha_0}) > b + \varepsilon$ and

$$\text{It follows that } \tilde{\mu}(\tilde{A} \cap F_{\overline{b+\varepsilon}}) \geq \tilde{\mu}(\tilde{A} \cap F_{\alpha_0}) > b + \varepsilon$$

therefore,

$$\sup_{\alpha \in [0, \infty]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cap F_\alpha)] \geq (b + \varepsilon) \wedge \tilde{\mu}(\tilde{A} \cap F_{\overline{b+\varepsilon}}) = b + \varepsilon > b.$$

This is a contradiction.

Remark: 3.2

Let $F_\alpha = \{x; f_\lambda(x) \geq \alpha\}$ and $F_\alpha = \{x; f_\lambda(x) > \alpha\}$. Then we have

$$\lim_{\beta \rightarrow \alpha - 0} F_\beta = \lim_{\beta \rightarrow \alpha - 0} F_\beta = F_\alpha \supset F_\alpha = \lim_{\beta \rightarrow \alpha - 0} F_\beta = \lim_{\beta \rightarrow \alpha - 0} F_\beta$$

Definition :3.2

f is called fuzzy integrable on \tilde{A} , if $\int_{\tilde{A}} f d\tilde{\mu} < \infty$.

Theorem: 3.2

$\int_{\tilde{A}} f_\lambda d\tilde{\mu} < \infty$, if and only if there exists $\alpha_0 \in [0, \infty)$ such that $\tilde{\mu}(\tilde{A} \cap F_{\alpha_0}) < \infty$.

Proof :

If there exists $\alpha_0 \in [0, \infty)$ such that $\tilde{\mu}(\tilde{A} \cap F_{\alpha_0}) < \infty$ then $\tilde{\mu}(\tilde{A} \cup F_\alpha) \geq \tilde{\mu}(\tilde{A} \cap F_{\alpha_0}) = a$ for any $\alpha > \alpha_0$ consequently

$$\int_{\tilde{A}} f_\lambda d\tilde{\mu} = \sup_{\alpha \in [0, \alpha_0]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cup F_\alpha)] \vee \sup_{\alpha \in [\alpha_0, \infty]} [\alpha \wedge \mu_{A \cup F_\alpha} \geq \alpha_0 \vee a < \infty].$$

Conversely if for any $\alpha \in [0, \infty)$, $\tilde{\mu}(\tilde{A} \cup F_\alpha) = \infty$, then

$$\int_{\tilde{A}} f_\lambda d\tilde{\mu} = \sup_{\alpha \in [0, \infty]} [\alpha \wedge \tilde{\mu}(\tilde{A} \cup F_\alpha)] = \sup_{\alpha \in [0, \infty]} \alpha = \infty.$$

Theorem: 3.3

Let $\tilde{A} \in \tilde{F}$, $\alpha \in [0, \infty)$, Then

1. $\int_{\tilde{A}} f_\lambda d\tilde{\mu} > \alpha \Leftrightarrow \exists \beta \in [0, \alpha)$ such that $\tilde{\mu}(\tilde{A} \cup F_\beta) > \alpha$; hence $\int_{\tilde{A}} f d\mu > \alpha \Rightarrow \mu_{A \cup F_\alpha} \mu_{A \cup F_\alpha} > \alpha$.
2. $\int_{\tilde{A}} f_\lambda d\tilde{\mu} < \alpha \Leftrightarrow \tilde{\mu}(\tilde{A} \cup F_\alpha) < \alpha$; therefore $\int_{\tilde{A}} f d\tilde{\mu} < \alpha \Rightarrow \tilde{\mu}(\tilde{A} \cup F_\alpha) < \alpha$.
3. $\int_{\tilde{A}} f_\lambda d\tilde{\mu} = \alpha \Leftrightarrow \forall \beta \in [0, \alpha)$, $\tilde{\mu}(\tilde{A} \cup F_\beta) \leq \alpha \leq \tilde{\mu}(\tilde{A} \cup F_\alpha)$. particularly if $\tilde{\mu}(\tilde{A}) < \infty$, then $\int_{\tilde{A}} f_\lambda d\tilde{\mu} = \alpha \Leftrightarrow \tilde{\mu}(\tilde{A} \cup F_\alpha) \leq \alpha \leq \tilde{\mu}(\tilde{A} \cup F_\alpha)$.

Proof:

1. Let $\alpha \in (0, \infty)$. If $\tilde{\mu}(\tilde{A} \cup F_\beta) \leq \alpha$ for any $\beta > \alpha$, then

$$\int_{\tilde{A}} f_{\lambda} d\tilde{\mu} \leq \beta \in [0, \alpha] [\alpha \wedge \tilde{\mu}(\tilde{A} \cup F_{\alpha})] \leq \beta \in [0, \alpha] [\beta \wedge \alpha = \inf \beta \in [0, \alpha] \beta = \alpha]$$

On the other hand if there exists $\beta > \alpha$ such that $\tilde{\mu}(\tilde{A} \cup F_{\beta}) > \alpha$ then whenever $\mu_{AUFr} \geq \mu_{AUF\beta}$ whenever $r \leq \beta$. Thus we have

$$\int_{\tilde{A}} f_{\lambda} d\tilde{\mu} = \inf_{r \in [0, \beta]} [r \wedge \tilde{\mu}(\tilde{A} \cup F_r)] \vee \inf_{r \in [\beta, \infty]} [r \wedge \mu_{AUFr} \geq \beta \vee \tilde{\mu}(\tilde{A} \cup F_{\beta}) > \alpha]$$

The equivalence relation is proved.

2. If $\int_{\tilde{A}} f_{\lambda} d\tilde{\mu} < \alpha$,

Then there exists $\alpha_0 < \alpha$ such that $\int_{\tilde{A}} f_{\lambda} d\tilde{\mu} \leq \alpha_0$ it follows by using (1), that $\tilde{\mu}(\tilde{A} \cup F_{\beta}) \leq \alpha_0$ whenever $\beta > \alpha_0$

Now, If we take $\beta = \frac{1}{2}(\alpha_0 + \alpha)$, then $\alpha > \beta > \alpha_0$, and therefore, $\tilde{\mu}(\tilde{A} \cup F_{\beta}) \leq \tilde{\mu}(\tilde{A} \cup F_{\alpha_0}) \leq \alpha_0 < \alpha$

On the other hand let, $\tilde{\mu}(\tilde{A} \cup F_{\alpha}) < \alpha$ If $\beta_n \nearrow \alpha$,

By remark 3.2 that $\tilde{A} \cup F_{\beta_n} \searrow \tilde{A} \cup F_{\alpha}$.

By the continuity from above of $\tilde{\mu}$ we have $\tilde{\mu}(\tilde{A} \cup F_{\beta_n}) \rightarrow \tilde{\mu}(\tilde{A} \cup F_{\alpha})$ and therefore there exists $\alpha_0 < \alpha$ such that $\tilde{\mu}(\tilde{A} \cup F_{\alpha_0}) < \alpha$

Thus we have $\int_{\tilde{A}} f_{\lambda} d\tilde{\mu} \geq \alpha_0 \wedge \tilde{\mu}(\tilde{A} \cup F_{\alpha_0}) > \alpha$.

The equivalence relation is proved.

3. First of all, by using (1) and (2), it follows immediately that $\int_{\tilde{A}} f_{\lambda} d\tilde{\mu} = \alpha \Leftrightarrow \forall \beta \in [0, \alpha), \tilde{\mu}(\tilde{A} \cup F_{\beta}) \leq \alpha \leq \tilde{\mu}(\tilde{A} \cup F_{\alpha})$.

When $\tilde{\mu}(\tilde{A}) < \infty$, we take a sequence $\{r_n\} \subset [0, \alpha)$ such that $r_n \nearrow \alpha$.

We have $\tilde{A} \cup F_{r_n} \nearrow \tilde{A} \cup F_{\alpha}$, and therefore, it follows from the continuity from below of $\tilde{\mu}$ that $\tilde{\mu}(\tilde{A} \cup F_{r_n}) \rightarrow \tilde{\mu}(\tilde{A} \cup F_{\alpha})$.

By the above results,

We obtain $\int_{\tilde{A}} f_{\lambda} d\tilde{\mu} = \alpha \Leftrightarrow \tilde{\mu}(\tilde{A} \cup F_{\alpha}) \leq \alpha \leq \tilde{\mu}(\tilde{A} \cup F_{\alpha})$.

Hence the theorem.

4. Convergence Theorems of a Sequence of Fuzzy Integrals on Fuzzy Sets

This section presents some convergence theorems of a sequence of fuzzy integrals on fuzzy sets and we now provide certain definitions which are useful in the sequel.

$$F_{\alpha}^n = \{x; f_n(x) \geq \alpha\}, F_{\alpha}^{n-} = \{x; f_n(x) > \alpha\}$$

$$g_n = \inf_{i \geq n} f_i, h_n = \sup_{i \geq n} f_i$$

Definition: 4.1

Let $\tilde{f} \in \tilde{F}(x), A \in \mathcal{A}, \tilde{\mu} \in \tilde{M}(x)$. Then the fuzzy integral of \tilde{f} and A with respect to $\tilde{\mu}$ is defined as

$$\int_A \tilde{f} d\tilde{\mu}(r) = \sup\{\lambda \in (0,1]: r \in \int_A f_{\lambda} d\mu_{\lambda}\}$$

Where $f_{\lambda}(x) = \{r \in (0,1]: f(x)(r) > \lambda\}$ and μ_{λ} is similar.

Theorem: 4.1

Let $\{f_n, f\} \subset \tilde{M}^+$ and $\tilde{A} \in \tilde{F}$. if $f_n \xrightarrow{e.} f$ on \tilde{A} and there exists n_0 and a constant $c \leq \int_{\tilde{A}} f d\tilde{\mu}$ ($0 \leq c$) such that $\tilde{\mu}(\tilde{A} \cap \{h_{n_0} > c\}) < \infty$, then $\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n d\tilde{\mu}$ is existent, and

$$\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n d\tilde{\mu} = \int_{\tilde{A}} f d\tilde{\mu}$$

Proof:

For a fixed $\lambda \in (0,1]$ let $\lambda_n = (1 - 1/n+1) \lambda$ then $\lambda_n \uparrow \lambda$.

Since $g_{\lambda n} \leq f_{\lambda n} \leq h_{\lambda n}, n = 1, 2, \dots$. By using theorem 3.3, $\int_{\tilde{A}} g_{\lambda n} d\tilde{\mu} \leq \int_{\tilde{A}} f_{\lambda n} d\tilde{\mu} \leq \int_{\tilde{A}} h_{\lambda n} d\tilde{\mu}, n = 1, 2, \dots$

Moreover, when $f_{\lambda n} \xrightarrow{e.} f_{\lambda}$ on \tilde{A} we have

$$g_{\lambda n} \nearrow f_{\lambda} \text{ on } \tilde{A}, h_{\lambda n} \searrow f_{\lambda} \text{ on } \tilde{A}$$

It follows $\tilde{\mu}(\tilde{A} \cap \{h_{\lambda n_0} > c\}) < \infty$, that

$$\lim_{n \rightarrow \infty} \int_{\tilde{A}} g_{\lambda n} d\tilde{\mu} = \lim_{n \rightarrow \infty} \int_{\tilde{A}} h_{\lambda n} d\tilde{\mu} = \int_{\tilde{A}} f_{\lambda} d\tilde{\mu}$$

Hence, we know that $\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n d\tilde{\mu}$ exists, and equals $\int_{\tilde{A}} f d\tilde{\mu}$

which concludes the proof.

Theorem 4.2

Let $\{f_n, f\} \subset \tilde{M}^+$ and $\tilde{A} \in \tilde{F}$ and $\tilde{\mu}$ be null-subtractive if $f_n \xrightarrow{a.e.} f$ on \tilde{A} and there exist n_0 and a constant $c \leq \int_{\tilde{A}} f d\tilde{\mu}$ ($0 \leq c$) such that $\tilde{\mu}(\tilde{A} \cap \{h_{n_0} > c\}) < \infty$, then $\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n d\tilde{\mu}$ is existent, and

$$\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n d\tilde{\mu} = \int_{\tilde{A}} f d\tilde{\mu}$$

Proof:

For a fixed $\lambda \in (0,1]$ let $\lambda_n = (1 - 1/n+1) \lambda$ then $\lambda_n \uparrow \lambda$.

Suppose that $f_{\lambda n} \xrightarrow{a.e.} f_{\lambda}$ on \tilde{A} .

Namely there exists $\tilde{E} \in \tilde{F}$ with $\tilde{\mu}(\tilde{E}) = 0$, such that $f_{\lambda n} \xrightarrow{e.} f_{\lambda}$ on $\tilde{A} \cap \tilde{E}^c$.

We observing that the null - subtraction of $\tilde{\mu}$ and the condition

$$\tilde{\mu}(\tilde{A} \cap \tilde{E}^c \cap \{h_{\lambda n_0} > c\}) \leq \tilde{\mu}(\tilde{A} \cap \{h_{\lambda n_0} > c\}) < \infty,$$

It follows that,

$$\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_{\lambda n} d\tilde{\mu} = \lim_{n \rightarrow \infty} \int_{\tilde{A} \cap \tilde{E}^c} f_{\lambda} d\tilde{\mu} = \int_{\tilde{A} \cap \tilde{E}^c} f d\tilde{\mu} = \int_{\tilde{A}} f d\tilde{\mu}$$

Hence the proof.

Theorem 4.3

Let $\{f_n, f\} \subset \tilde{M}^+$ and $\tilde{\mu}$ be autocontinuous. If $f_n \xrightarrow{\tilde{\mu}} f$ then for any

$\tilde{A} \in \tilde{F}$,

$$\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n d\tilde{\mu} = \int_{\tilde{A}} f d\tilde{\mu}$$

Proof

For a fixed $\lambda \in (0,1]$ let $\lambda_n = (1 - 1/n+1) \lambda$ then $\lambda_n \uparrow \lambda$.

Suppose that $\tilde{\mu}$ is autocontinuous and $f_{n\lambda} \xrightarrow{\tilde{\mu}} f_{\lambda}$ on X.

Then for any $\tilde{A} \in \tilde{F}, \varepsilon > 0$,

$$0 \leq \tilde{\mu}(\tilde{A} \cap \{|f_{\lambda n} - f_{\lambda}| \geq \varepsilon\}) \leq \tilde{\mu}(\{|f_{\lambda n} - f_{\lambda}| \geq \varepsilon\}) \rightarrow 0,$$

That is $\tilde{\mu}(\tilde{A} \cap \{|f_{n\lambda} - f_{\lambda}| \geq \varepsilon\}) \rightarrow 0$. as $n \rightarrow \infty$ too.

Take $c = \int_{\tilde{A}} f_{\lambda} d\tilde{\mu}$.

(1) If $c < \infty$ then, by using $\tilde{\mu}(\tilde{A} \cap F_{c-\varepsilon}) \geq c - \varepsilon$ for every given $\varepsilon > 0$.

We easily prove $F_{c+2\varepsilon} \subset F_{c+\varepsilon} \cup (\{|f_{n\lambda} - f_{\lambda}| \geq \varepsilon\})$ and therefore,

$$\tilde{A} \cap F_{c+2\varepsilon} \subset (\tilde{A} \cap F_{c+\varepsilon}) \cup (\tilde{A} \cap \{|f_{n\lambda} - f_{\lambda}| \geq \varepsilon\}).$$

Since $\tilde{\mu}$ is autocontinuous from above, we have $\tilde{\mu}((\tilde{A} \cap F_{c+\varepsilon}) \cup (\tilde{A} \cap \{|f_n - f| \geq \varepsilon\})) \rightarrow \tilde{\mu}(\tilde{A} \cap F_{c+\varepsilon})$.

And hence there exists n_0 such that $\tilde{\mu}(\tilde{A} \cap F_{c+2\varepsilon}) \leq \tilde{\mu}(\tilde{A} \cap F_{c+\varepsilon}) + \varepsilon \leq c + \varepsilon \leq c + 2\varepsilon$ as $n \geq n_0$.

We have $\int_{\tilde{A}} f_{\lambda} d\tilde{\mu} \leq c + 2\varepsilon$ as $n \geq n_0$.

On the other hand, in order to show that there exists \bar{n}_0 such that $c - 2\varepsilon \leq \int_{\tilde{A}} f_{\lambda n} d\tilde{\mu}$ as $n \geq \bar{n}_0$. It is sufficient to consider the case $c > 0$.

Clearly, for $\varepsilon \in (0, \frac{1}{2}c)$,

We can prove $F_{c+2\varepsilon} \supset F_{c+\varepsilon} \cap \{|f_{\lambda n} - f_{\lambda}| \geq \varepsilon\}^c$, and therefore

$$\tilde{A} \cap F_{c+2\varepsilon} \supset \tilde{A} \cap F_{c+\varepsilon} \cap \{|f_{n\lambda} - f_{\lambda}| \geq \varepsilon\}^c$$

By the auto continuity from below of $\tilde{\mu}$, we have $\tilde{\mu}(\tilde{A} \cap F_{c+\varepsilon} \cap \{|f_{n\lambda} - f_{\lambda}| \geq \varepsilon\}^c) \rightarrow \tilde{\mu}(\tilde{A} \cap F_{c+\varepsilon})$.

And thus there exists \bar{n}_0 such that $\tilde{\mu}(\tilde{A} \cap F_{c+2\varepsilon}) \geq \tilde{\mu}(\tilde{A} \cap F_{c+\varepsilon}) - \varepsilon \geq c - 2\varepsilon$ as $n \geq \bar{n}_0$.

It follows from theorem 4.9(1) that $c - 2\varepsilon \leq \int_{\tilde{A}} f_{n\lambda} d\tilde{\mu}$ as $n \geq \bar{n}_0$. Consequently, $\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_{n\lambda} d\tilde{\mu}$ exists, and equals c .

(2) If $c = \infty$, then for any $\alpha \in [0, \infty)$, $\tilde{\mu}(\tilde{A} \cap F_{\alpha}) = \infty$, for any $N > 0$, since $\tilde{A} \cap F_N \supset \tilde{A} \cap F_{N+1} \cap \{|f_{n\lambda} - f_{\lambda}| \geq \varepsilon\}^c$, By using the auto continuity from below of $\tilde{\mu}$, there exists n_0 such that

$$\tilde{\mu}(\tilde{A} \cap F_N) \geq \tilde{\mu}(\tilde{A} \cap F_{N+1} \cap \{|f_n - f| \geq \varepsilon\}^c) \geq N$$
 as $n \geq n_0$.

Therefore, we conclude that $\int_{\tilde{A}} f_n d\tilde{\mu} \geq N$ as $n \geq n_0$. consequently $\lim_{n \rightarrow \infty} \int_{\tilde{A}} f_n d\tilde{\mu} = \infty = c$.

Which concludes the proof.

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